A Roadmap to Search and Matching Models

Sergei Balakin

The Ohio State University

Roadmap

MATCHING, no frictions, NTU

Gale & Shapley (1962)
⋮
Roth & Sotomayor (1990)

MATCHING, no frictions, TU

Koopmans & Beckmann (1957)
Shapley & Shubik (1972)
Becker (1973)
⋮

MATCHING WITH FRICTIONS

$>$ **90s**

Stigler (1961)
McCall (1965 & 1970)
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Smith

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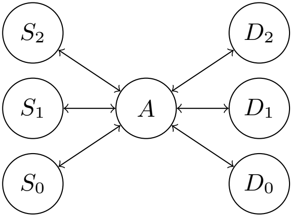
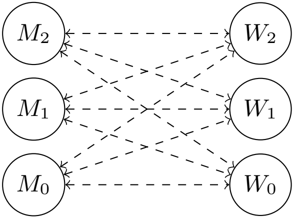
(De)centralization



Walrasian



Edgeworthian

- Walrasian setting: the issue is to find the *true* price for reaching the equilibrium (= optimum)
- Edgeworthian setting: the issue is to find the *path* leading to this equilibrium

Glossary

- A matching allocation/assignment is called *stable* if there exist no two agents who would prefer matching with each other to their current match

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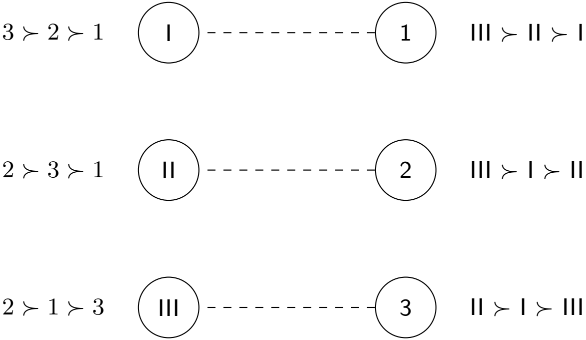

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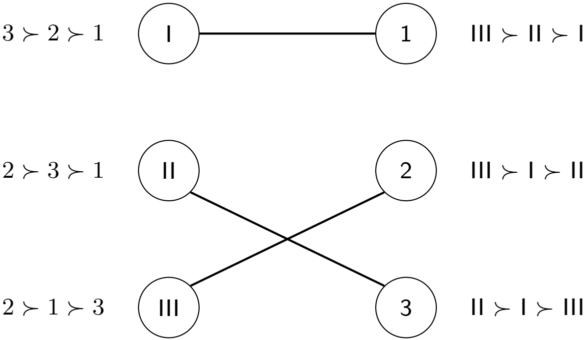

Glossary

- A matching allocation/assignment is called *stable* if there exist no two agents who would prefer matching with each other to their current match
- A real-valued function f on a lattice $X \subseteq \mathbb{R}$ is *supermodular* if

$$f(\max\{x', x''\}) + f(\min\{x', x''\}) \geqslant f(x') + f(x'')$$

  for all $x', x'' \in X$. If f is twice continuously differentiable, then this is equivalent to $\partial^2 f(x)/\partial x_i \partial x_j \geqslant 0$ for all $i \neq j$.

NO Frictions, Nontransferable Utility

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Deferred Acceptance Algorithm (Gale & Shapley 1962)

n men and n women with a complete order of preferences

1. each unengaged man proposes to the woman he prefers most
2. each woman replies "maybe" if she prefers this man over all other suitors and over her current provisional partner (in this case, she rejects her current provisional partner who becomes unengaged); she replies "no" to all other suitors
3. repeat 1 until everyone is engaged

Features:

- everyone gets engaged
- the matching is stable
- this matching is the best (optimal) for all men and worst for all women among all stable matchings
- this mechanism is truthful for men

Top Trading Cycle (Shapley & Scarf 1974, developed by Gale)

n students with a complete order of preferences over n rooms

1. initial allocation rooms to students
2. students rank the rooms indicating what is the room preferred by each of them (the "Top")
3. the algorithm looks for cycles in order to match students with the room which is preferred: for example, if A prefers room 1 to his room 2, whereas B prefers room 2 to his room 1, there is a cycle of length 2
4. **note**: cycles of length 1 are allowed!
5. students in cycles exchange their rooms and get removed from the list
6. repeat from 2 until no new cycles are observed

Features:

- the solution is unique for any initial allocation if students have strict preferences
- the matching is Pareto-efficient and stable (even *core-stable*)
- this mechanism is truthful (Roth 1982)

Applications

- Matching Principals and Agents (Legros & Newman 2010)

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  - the principal hires an agent to perform a task
  - since the agent's actions are unobservable, the contract is based on a stochastic signal, such as output, that is correlated with those actions
  - PAM: agents with high initial wealth (and, thus, low risk aversion) match with principals with safer output distributions

Applications

- Matching Principals and Agents (Legros & Newman 2010)
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- Matching in Large Firms (Kelso & Crawford 1982)
  - many-to-one matching model of firms to any number of workers
  - no stability due to worker complementarities: if a firm hires worker 2 when it already employs worker 1, then worker 2's productivity rises, and she can thus command a higher wage
  - the *gross substitutes* condition solves this problem: if wages increase for some workers, the firm will not drop from its labor force any worker whose wage did not increase (for example, additively separable production functions)
  - the algorithm that finds the equilibrium allocation and wages is provided

Applications

- Matching Principals and Agents (Legros & Newman 2010)
- Matching in Large Firms (Kelso & Crawford 1982)
- One-Sided Matching
  - existence of stable matchings may be problematic (Roth & Sotomayor 1990)
  - sometimes it's possible to divide agents into two sides and match them accordingly (Kremer & Maskin 1996)
- Matching with Externalities
  - The value of the match to a pair also depends on the entire matching
  - does it have impact on the optimal and equilibrium matching patterns?
  - Sasaki & Toda (1996)
  - Pycia & Yenmez (2015)
  - Chade & Eeckhout (2015)

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The Basic Model (Shapley & Shubik 1972)

- N women and N men: each woman i and each man j have types $x_i \in [0,1]$ and $y_i \in [0,1]$ respectively
- Assume $x_1 < x_2 < \ldots < x_n$, $y_1 < y_2 < \ldots < y_n$
- If woman x_i marries man y_j, they produce output $f(x_i, y_j) > 0$
- Single agents produce zero outputs

Questions:

- What is the optimal matching for men and women?
- Under what conditions does this assignment exhibit positive or negative assortative matching (PAM or NAM)?

Optimization & Positive Assortative Matching

Maximization problem:

$$\max_{\pi} \sum_{i=1}^{N} f(x_i, y_{\pi(i)}),$$

where the maximization is taken over all possible permutations $\pi : \{1, 2, \ldots, N\} \to \{1, 2, \ldots, N\}$

Proposition (Becker 1973): *PAM ($\pi(i) = i$) is optimal iff f is supermodular.*

- If not PAM, there are two women i and i' with $i' > i$, respectively matched with two men j and j' with $j > j'$
- By supermodularity,

$$f(x_i, y_j) + f(x_{i'}, y_{j'}) < f(x_i, y_{j'}) + f(x_{i'}, y_j)$$

- Thus, the total output may be increased by rematching them

The Large Market Case (Gretsky, Ostroy & Zame 1992 & 1999)

- Equal unit mass continuum of men and women
- Each woman (man) has a type $x(y) \in [0,1]$ drawn from strictly increasing and continuously differentiable cdf G(H).
- A matching is a function $\mu : [0,1] \to [0,1]$ that is *measure preserving* (matching equal measures of men and women). For instance, PAM requires that $G(x) = H(\mu(x))$ for all x
- The match output of x and y is $f(x,y)$ – twice continuously differentiable

Proposition: *PAM is optimal iff f is supermodular.*

Applications

- The O-Ring Production Function (Kremer 1993)
  - positive correlation among wages of workers in different occupations withing a firm
- CEO-Firm Assignment Model (Gabaix & Landier 2008 and Tervio 2008)
  - "Superstar" property: small differences in talent can have a drastic impact in pay at the top
- Matching Principals and Agents (Serfes 2005)
  - heterogeneous principals and agents under moral hazard
  - linear contracts and CARA utility (like Holmstrom & Milgrom 1987)
  - thus, the model becomes a matching problem with TU
  - main result: negative relationship between risk and incentives
  - NAM is optimal: principals with high variance of their output are matched with less risk averse agents
  - the data exhibits either a positive or an insignificant relationship though

Search

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Why search frictions?

Downsides of the matching paradigm:

- it predicts no unmatched agents
- it says nothing about mismatch among those that do match
- **volatility** and **discontinuity**: slight imbalances can sometimes have dramatic effects (comparing to a standard Walrasian model)
  - number of agents
  - preferences
  - utility function

$$f(x,y) = 1 + \varepsilon xy$$

Question: **how search frictions distort equilibrium market outcomes?**

Search History

- Stigler (1961) – simultaneous search
- McCall (1965 & 1970) – sequential search
- tons of authors – sequential search
- Chade & Smith (2006) – simultaneous search

Simultaneous Search (Stigler 1961)

- A distribution of prices is given by a non-degenerate distribution $F(p)$on $[0,1]$
- A consumer chooses a fixed sample size n to minimize the expected total cost (expected purchase cost plus search cost) of purchasing it
- Cost of one search is c
- With n *independent* draws, the distribution of the lowest price is

$$F_n(p) = 1 - (1 - F(p))^n$$

- If one purchase K units, the expected total outlay is

$$P(n) = K \int_0^1 p dF_n(p) = K \int_0^1 (1 - F(p))^n dp$$

- Discrete FOC for optimal n^*:

$$P(n^* - 1) - P(n^*) \geqslant c > P(n^*) - P(n^* + 1)$$

- Larger K raises the marginal benefit of sampling and thus unduces weakly more searches n^*

Simultaneous Search (Chade & Smith 2006)

- Heterogeneous options: the decision maker chooses not only the number of options to sample but also the sample composition
- Each option generates a stochastic reward
- After observing the rewards of each option, the decision maker chooses the largest one
- Specifically, imagine a set of colleges $\{1, 2, \ldots, N\}$ with payoffs $v_1 > v_2 > \ldots > v_N$ and inversely ranked admission chances $\alpha_1 < \alpha_2 < \ldots < \alpha_N$
- the optimal portfolio of any given size $n < N$ when all college application costs are $c > 0$ is deduced

Simultaneous Search (Chade & Smith 2006)

- *Marginal improvement algorithm* (MIA)
  - At stage 1, one selects the school with greatest expected value
  - If that value exceeds c, put college i in the tentative portfolio

    $\vdots$
  - At any stage $n + 1$, one finds school i_{n+1} yielding the greatest marginal benefit on the portfolio constructed so far
  - Add that school to the tentative portfolio if the incremental value is at least c. Otherwise, stop.
- MIA is a member of a class of "greedy algorithms", in which a sequence of locally optimal choices leads to a global optimum
- Rational students should not blindly apply to their best expected options

Sequential Search (McCall 1965 & 1970)

- A worker samples a wage from a distribution in each period and decides whether to continue the search, or stop and work at that wage
- The worker's optimal strategy is fully summarized by a reservation wage $\bar{w}$ above which the worker stops searching and below which she continues:

$$\bar{w} = \int_0^\infty \max\{\bar{w}, w\} dF(w) - c \qquad \Rightarrow \qquad c = \int_{\bar{w}}^\infty (1 - F(w)) dw$$

- Weitzman (1979) extended McCall's model to ex ante heterogeneous options:
  - A finite number of options, each represented by cdf $F_k(w)$ over prizes
  - Opening box k costs c_k and incurs a time discounting factor $\delta_i \in (0, 1]$ due to delay
  - Only one prize may ultimately be accepted
  - An optimal strategy requires specifying the order to explore options and a stopping rule

Sequential Search. Extensions

- Learning the distribution while searching
  - Rothschild (1974)
  - Adam (2001)
  - Gershkov & Moldovanu (2012)
- Search with hidden and known components
  - Choi & Smith (2016)
- Search by committee
  - Albrecht, Anderson & Vroman (2010)
  - Compte & Jehiel (2010)

Matching with Frictions

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Search and Matching Frictions

- Block & Strict Assortative Matching under NTU
  - McNamara & Collins (1990)
  - Burdett & Coles (1997)
  - Shimer & Smith (2000)
  - Smith (2006)
  - Chade (2001)
- Random Search under TU
  - Shimer & Smith (2000)
  - Noldeke and Troger (2009)
  - Manea (2017)
  - Eeckhout and Kircher (2011)

Search and Matching Frictions

- Informational Frictions (Liu et al. 2014)
    - matching with one-sided incomplete information and TU
    - stability notion in the spirit of rationalizability in game theory
    - mild strengthening of supermodularity yields PAM under incomplete information
- Dynamic Frictions and Participation
    - stability notion for dynamic markets (Doval 2021, Ho 2021)
    - new agents replace the ones who have left the market (Chade 2006)
    - agents arrive (and perish) independently of a matching process (Akbarpour, Li & Oveis Gharan 2020, Baccara, Lee & Yariv 2020)

Search and Matching Frictions

- Sorting with Evolving Types or Reputations
  - Anderson & Smith (2010)
  - Anderson (2015)
  - Jovanovic (2014)
- Sorting with Signaling Costs
  - Hoppe, Moldovanu & Sela (2009)
- Sorting with Application Costs
  - Chade, Lewis & Smith (2014)
  - Nguyen, Peters & Poitevin (2017)
  - Arnosti, Johari & Kanoria (2019)

Search and Matching Frictions

- Student to School Choice
  - Early Admissions (Avery & Levin 2010)
  - Job Market Signaling (Coles, Kushnir & Niederle 2013)
  - Exploding Offers (Niederle & Roth 2009, Pan 2018)

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