Generalized Groves-Ledyard Mechanisms

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What John Taught Me

1. Think Hard
2. Do Good Science

Groves & Ledyard 1977

Groves, T. and J. Ledyard (1977). "Optimal Allocation of Public Goods: A Solution to the 'Free Rider' Problem," *Econometrica* 45(4), 783–809.

How I read it:

- Section 1: Long-winded intro about welfare theorems.
- Section 2: Excessively complex GE setup w/ PGs
- Section 3: Some general stuff about Groves mechanisms yielding some 'optimal' mechanisms in theory, but nothing concrete at all. Skip this.
- Section 4: The Famous* Groves-Ledyard Mechanism!
- Section 5: Crusty lit review (Milleron, Malinvaud, Dreze & Vallee Poussin)

Section 3

Ritesh read the paper. Here's what it really says:

- Section 3:
  - Start with a Groves demand-revealing mechanism
  - Possible announcements = convex quasilinear preferences
  - What if people don't have QL prefs?
  - Nash equilibrium: announce QL prefs that have same MRS as true prefs at the realized allocation. ('approximation')
  - Groves mech will pick PO for this 'fake' QL economy, which is the same as the real economy in NE!
- Section 4:
  - Problem: Groves mechs not budget balanced
  - Solution: Force them to announce quadratic QL prefs!
    - ★ Groves & Loeb: quadratic prefs balance budget
    - ★ Laffont & Maskin: quadratic prefs 'probably' only ones that do
  - Quadratic prefs have only one parameter (θ_i). Just announce that.
  - Groves mech announcing $\theta_i \Rightarrow$ G-L mechanism!

Tian (1996) Preferences

- Laffont & Maskin were wrong.
- Tian found parameterized higher-order polynomial functions for which Groves mechanisms are balanced.
- Still single parameter θ_i
- Quadratic is a special case.

Our paper:

- Have people announce more general Tian preferences, rather than just quadratic.
- Do the algebra and see what the resulting mechanisms look like
- Quadratic $\Rightarrow$ Groves-Ledyard mech
- Higher-order polynomials $\Rightarrow$ Generalized GL mechs

Why Do This?

- I needed an idea for Ledyard's birthday conference

OK, how to sell the paper?

- Are the resulting mechanisms nice/useful?
  - Not At All.
- Another sense in which G-L is the 'simplest' mechanism
  - See Buzz Brock and G-L "IC Since '72"
- Better understanding of how mechanisms are built
  - Samuelson condition: easy. BB & existence: hard.
- Invent a cute trick for guaranteeing existence of NE.

The Setup

- 1 pvt good x_i, 1 pub good y, I consumers, F firms
- PG price p.
- Consumer i: $(\mathcal{X}_i, \succeq_i, \omega_i, \theta_i)$
  - $\succeq_i$ rep. by u_i that's (1) diffb'l, (2) Q-C, (3) inc. in x_i.
- Firm f: $\mathcal{Z}_f$
- Set of admissible economies $\mathcal{E}$.
- Feasible allocation:
  - $(x_i, y) \in \mathcal{X}_i$
  - $z_f \in \mathcal{Z}_f$
  - $(\sum_i (x_i - \omega_i), y) \leq (\sum_f z_f^x, \sum_f z_f^y)$
- Usual definition of PO.
- Mechanism: $((M_i)_i, y(m), (t_i(m))_i)$
- Consumer chooses (x_i, m_i) s.t.
  - x_i, $y(m)$ and $t_i(m)$ are jointly feasible, given m_{-i}.

Two Simplifications

1. Linear production cost $\kappa > 0$
   - Zero profit (ignore θ_i)
   - $p = \kappa$ in any equilibrium
2. $\mathcal{X}_i = \mathbb{R}^2$
   - Optimal x_i is just $x_i = \omega_i - t_i(m)$
   - Can focus only on choice of m_i
   - Best (feasible) response: $\beta_i(m_{-i})$

- Budget balance: $\sum_i t_i(m) = \kappa y(m)$ and $\sum_f z_f^y = y(m)$
  - But firms are indifferent over z_f^y.
- Nash (competitive) equilibrium:
  - $m_i^* \in \beta_i(m_{-i}^*)$ (best response)
  - $\sum_i t_i(m^*) \geq \kappa y(m^*)$ (market clearing with free disposal)

Pareto Optimality

Define

$$MRS_i(x_i, y) = \frac{\partial u_i(x_i, y)/\partial y}{\partial u_i(x_i, y)/\partial x_i}.$$

Definition

Mechanism is **conditionally Nash efficient** if, $\forall e \in \mathcal{E}$ and every NE m^* in e,

$$\sum_i MRS_i(\omega_i - t_i(m^*), y(m^*)) = \kappa.$$

(Fully) Nash efficient if (1) NE exists and (2) every NE is budget balanced.

Conditionally Efficient Mechanisms

Proportional Tax Mechanism:

- $M_i^P = \mathbb{R}^1$
- $y^P(m) = \sum_i m_i$
- $t_i^P(m) = \alpha_i \kappa y^P(m)$ (where $\sum_i \alpha_i = 1$)
- NE requires all i 'buy' same $y^P(m)$ at price $\alpha_i \kappa$. Generically not true.

Groves Mechanism

- $M_i^G = \{$all strictly concave diffb'l functions m_i : $\arg\max_y \{\sum_i m_i(y) - \kappa y\} \neq \emptyset\}$
- $y^G(m) \in \arg\max_y \{\sum_i m_i(y) - \kappa y\}$
- $t_i^G(m) = \kappa y^G(m) - \sum_{j \neq i} m_j(y(m)) + h_i(m_{-i})$
- Equilibria may not be budget balanced. (or even market clearing.)
- Note: Any i can 'buy' any $\hat{y}_i$ via $m_i'(\hat{y}_i) = \kappa - \sum_{j \neq i} m_j'(\hat{y}_i)$

Definition

A mechanism is **quasi-direct** if

1. $y(\cdot, m_{-i})$ is surjective $\forall i, m_{-i}$ (i can 'buy' any $\hat{y}$),
2. $t_i(\cdot)$ is measurable in $\{(y(m), m_{-i})\}$ (meaning $t_i(y(m), m_{-i})$).

Agent simply chooses $\hat{y}_i$. Picks m_i to get $\hat{y}_i$.

Optimal $\hat{y}_i$:

$$\frac{\partial t_i(\hat{y}_i, m_{-i})}{\partial \hat{y}_i} = MRS_i(\omega_i - t_i(\hat{y}_i, m_{-i}), \hat{y}_i).$$

Summing gives:

Lemma

A quasi-direct mechanism is conditionally Nash efficient if and only if, for every NE m^*,

$$\sum_i \frac{\partial t_i(y(m^*), m^*_{-i})}{\partial \hat{y}_i} = \kappa.$$

Building Mechanisms

How to make a Nash efficient mechanism?

1. Use a quasi-direct mechanism.
2. Set $\sum_i \partial t_i / \partial \hat{y}_i = \kappa$ for all m.
3. Make sure equilibrium exists.
4. Make sure balanced equilibrium exists.

Again, conditional Nash efficiency is easy.
Existence and balance are hard.

Groves Mechanisms

Let's use the Groves quasi-direct mechanism:

$t_i(\hat{y}_i, m_{-i}) = \kappa\hat{y}_i - \sum_{j\neq i} m_j(\hat{y}_i) + h_i(m_{-i})$ $\qquad m_i'(\hat{y}_i) = \kappa - \sum_{j\neq i} m_j'(\hat{y}_i)$

So

$$\begin{aligned}\sum_i \partial t_i / \partial \hat{y}_i &= \sum_i \left[\kappa - \sum_{j\neq i} m_j'(\hat{y}_i)\right] \\ &= \sum_i \left[m_i'(\hat{y}_i)\right] \\ &= \kappa.\end{aligned}$$

Thus, conditionally Nash efficient. But...

1. Budget balanced equilibria?
2. Equilibrium existence?

Budget Balanced Equilibria

Groves mechanism with quadratic preferences:

- $M_i^Q = \left\{ v_i(\cdot|\theta_i) \;:\; v_i(\cdot|\theta_i) = (\gamma\theta_i + \alpha_i\kappa)y - \frac{\gamma}{2I}y^2, \theta_i \in \mathbb{R} \right\}$.
- Mechanism parameters: $\gamma > 0$, $\sum_i \alpha_i = 1$.
- Let message be θ_i.
- Lots of tedious algebra:
- $y^Q(\theta) = \sum_i \theta_i$
- $t_i^Q(\theta) = \alpha_i \kappa y^Q(m) + \frac{\gamma}{2}\left[\frac{I-1}{I}\left(\theta_i - \bar{\theta}_{-i}\right)^2 - \sigma^2(\theta_{-i}) \right]$

Tian Preferences

Definition (Tian Preferences)

Fix any $r \in \{2, 3, , \ldots, I-1\}$ and set

$$M_i^{Tr} = \left\{ v_i(\cdot|\theta_i) : v_i(y|\theta_i) = \alpha_i \kappa y + f(y)\psi_i(\theta_i) - \frac{r-1}{rc}\left(cf(y)+d\right)^{\frac{r}{r-1}} \right\},$$

were

- $\sum_i \alpha_i = 1$, $c > 0$, $d \geq 0$
- $f(y)$ and $(\psi_i(\theta_i))_{i=1}^{I}$ continuously differentiable
- $f(y) \geq 0 \ \forall y$, $f'(y) \neq 0 \ \forall y$, $\psi'(\theta_i) \neq 0 \ \forall \theta_i$

Quadratic: $r = 2$, $f(y) = y$, $\psi_i(\theta_i) = \gamma\theta_i$, $c = \gamma/I$, and $d = 0$.

Balancing the Budget

Proposition

For any $r \in \{2, \ldots, I-1\}$, *there exists a balanced Groves mechanism with message space* M^{Tr}.

(Had to re-prove Laffont & Maskin and Tian with $\kappa > 0$)

And now we have the generalized G-L mechanism...

The Generalized G-L Mechanism

Recall: $v_i(y|\theta_i) = \alpha_i \kappa y + f(y)\psi_i(\theta_i) - \frac{r-1}{rc}\left(cf(y)+d\right)^{\frac{r}{r-1}}$
Resulting mechanism:

$$
\begin{aligned}
M_i^* &= \mathbb{R}^1 \\
y^*(\theta) &= f^{-1}\left(\frac{1}{c}\left(\bar{\psi}(\theta)^{r-1} - d\right)\right) \\
t_i^*(\theta) &= \alpha_i \kappa y^*(\theta) - \frac{I-1}{c}\left[\left(\bar{\psi}(\theta)^{r-1} - d\right)\bar{\psi}_{-i}(\theta_{-i}) + \frac{r-1}{r}\bar{\psi}(\theta)^r\right] \\
&\quad + h_i(\theta_{-i})
\end{aligned}
$$

where

- $\bar{\psi}(\theta) = \frac{1}{I}\sum_i \psi_i(\theta_i)$
- $\bar{\psi}_{-i}(\theta_{-i}) = \frac{1}{I-1}\sum_{j\neq i} \psi_j(\theta_j)$

The Balancing Term

For budget balance, we need $h_i(\theta_{-i})$ to satisfy:

$$\sum_i h_i(\theta_{-i}) = \sum_i \left[\text{blah blah blah} \left(\sum_j \theta_j \right)^r \right]$$

Need to make that look more like

$$\sum_i h_i(\theta_{-i}) = \sum_i \left[\text{blah blah blah } g_i(\theta_i) + \sum_{j \neq i} g(\theta_j) \right]$$

Re-engineer the multinomial formula...

The balancing term:

$$h_i(\theta_{-i}) = \sum_{q=1}^{r} \sum_{(t_1,\ldots,t_q)\in\mathcal{T}_{q,r}} \left[\frac{I-1}{I^{r-1}} \frac{2r-1}{cr(I-q)} \times \binom{r}{t_1, t_2, \ldots, t_q} \left(\sum_{i_1 \neq i} \sum_{i_2 \notin \{i,i_1\}} \cdots \sum_{i_q \notin \{i,i_1,\ldots,i_{q-1}\}} \prod_{k=1}^{q} \psi_{i_k}(\theta_{i_k})^{t_k} \right) \right] - \frac{d}{c} \sum_{j\neq i} \psi_j(\theta_j)$$

where

$$\mathcal{T}_{q,r} = \left\{ (t_1, \ldots, t_n) \in \mathbb{N}^q : \sum_{k=1}^{q} t_k = r \text{ and } (\forall k > 1)\ t_{k-1} > t_k \right\}$$

$$\binom{r}{t_1, t_2, \ldots, t_q} = \frac{r!}{t_1! t_2! \cdots t_q!}$$

Groves-Ledyard Parameters, general r

Fix $f(y) = y$, $\psi_i(\theta_i) = \gamma\theta_i$, $c = \gamma/I$, $d = 0$, as in G-L:

$$f(y^*(\theta)) = (\sum_i \theta_i)^{r-1}$$

$$t_i^*(\theta) = \alpha_i \kappa (\sum_{i=1}^{I} \theta_i)^{r-1} + \frac{\gamma(r-1)}{r} \left(\frac{I-1}{I} \left(\theta_i - \bar{\theta}_{-i}\right)^r + \sum_{j=1}^{r-1} \theta_i^{r-j} b_j I^j \bar{\theta}_{-i}^j \right) + \gamma \frac{I-1}{I} \frac{(r-1)}{r} b_r I^r \bar{\theta}_{-i}^r + h_i(\theta_{-i}),$$

where

$$b_i = -\Big(\frac{(-1)^i \binom{r}{i}}{(I-1)^i} + \frac{\frac{r}{r-1} I \binom{r-1}{i-1} - (I-1)\binom{r}{i}}{I-1} \Big).$$

Existence

Budget balance comes from Tian preferences.

How to get existence?

Highly non-linear game

Ritesh thought hard, found examples of non-existence.

The Cheap Trick

A cheap trick to get existence:

Suppose subjects announce $v_i(\cdot|\theta_i) + \beta_i$

β_i has no effect on public good or t_i, but does shift t_j.

For any $(y, t_1, \ldots, t_n)$ there is announcement that hits it

Two-Dimensional Groves Mechanisms

$$M_i^{Sr} = \left\{ v_i(\cdot|\theta_i, \beta_i) = v_i(\cdot|\theta_i) + \beta_i : v_i(\cdot|\theta_i) \in M_i^{Tr}, \beta_i \in \mathbb{R}_i \right\}$$

$y^S(\theta, \beta) = y^G(\theta)$

$$t_i^S(\theta, \beta) = t_i^G(\theta) - \sum_{j \neq i} \beta_j$$

This creates unbalance of $\sum_i \sum_{j \neq i} \beta_j$.

Fix: Add this to $\sum_i h_i(\theta_{-i}, \beta_{-i})$

How? Add $(I-1)\beta_{i+1}$ to each h_i.

$$t_i^S(\theta, \beta) = t_i^G(\theta) - \sum_{j \neq i} \beta_j + (I-1)\beta_{i+1}.$$

Equilibrium Existence

Proposition

For any generalized Groves-Ledyard mechanism, the two-dimensional 'shifting' mechanism defined above

1. *satisfies the Sanuelson condition at any equilibrium*
2. *is budget balanced, and*
3. *has an equilibrium in every $e \in \mathcal{E}$.*

Idea: Allows us to shift transfer aribtrarily to get 'right' allocation.

Summary

1. Can use Tian preferences to construct new budget-balanced 'generalized' Groves-Ledyard mechanisms.
2. Conditional Nash efficiency is easy. Existence and balance are hard.
3. Balance: Tian preferences
4. Existence: second dimension
   - Equilibrium requires coordination on specific vector of transfers. Unrealistic.
5. Is it useful? No.