GENERALIZED GROVES-LEDYARD MECHANISMS

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ABSTRACT. Groves and Ledyard (1977) construct a mechanism for public goods procurement that can be viewed as a direct-revelation Groves mechanism in which agents announce a parameter of a quadratic approximation of their true preferences. The mechanism's Nash equilibrium outcomes are efficient. The budget is balanced because Groves mechanisms are balanced for the announced quadratic preferences. Tian (1996) subsequently discovered a richer set of budget-balancing preferences. We replicate the Groves-Ledyard construction using this expanded set of preferences, and uncover a new set of complex mechanisms that generalize the original Groves-Ledyard mechanism. The original mechanism, however, remains the most appealing in terms of both simplicity and stability.

Keywords: Nash implementation; public goods; demand revelation.

JEL Classification: D02, D47, D61, D62, D71.

I. INTRODUCTION

Groves and Ledyard (1977) provided one of the first decentralized economic mechanisms to solve the free rider problem for general economies. Specifically, they devised a government (or, mechanism) such that self-interested equilibrium behavior by all parties always leads to a Pareto optimal allocation. This work has been cited widely, with many researchers building off its original insights.

What has not been appreciated, however, is the manner in which Groves and Ledyard (1977) actually construct their mechanism. The final mechanism looks simple: players announce a single number, and taxes are based on a proportional share of the cost and a quadratic penalty. In fact, the mechanism has a more complex foundation: Groves and Ledyard (1977) present it as being derived from a Groves mechanism (Groves, 1973) in which agents announce entire quasilinear utility functions and are taxed (or rewarded) based on the Marshallian surplus calculated from the announced preferences of all other

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agents in the economy. But agents may not actually have quasilinear preferences, so these announcements represents approximations of their true preferences. The space of admissible announcements is parameterized with a single parameter, which greatly simplifies agents' messages. In equilibrium, agents with non-quasilinear preferences announce quasilinear preferences that best approximate their true willingness to pay at the Pareto optimal allocation, and that allocation is selected by the mechanism.[1]

What is crucial to the Groves-cum-Groves-Ledyard construction is that the set of approximating preferences that the agents are allowed to announce always generate a budget-balanced allocation in the Groves mechanism. Budget balance does not always obtain with general quasilinear preferences; Groves and Loeb (1975) showed that balance is achieved if we further restrict agents to quadratic valuation functions. Following this insight, Groves and Ledyard (1977) only allow agents to announce quadratic (approximate) preferences, guaranteeing that the final allocation will balance the budget. Agents need only to announce the intercept of their (approximate) valuation function, and are charged according to the balanced Groves mechanism. Algebraic manipulation reveals that these outcome and tax functions reduce to the familiar Groves-Ledyard mechanism with its quadratic form.

For two decades it was believed that quadratic preferences are the only ones for which a Groves mechanism could be budget balanced.[2] This belief was overturned when Tian (1996) discovered a much richer class of preferences for which Groves mechanisms can be balanced. And, like the quadratic preferences, these are parameterized by a single parameter.

In this paper we replicate the Groves-Ledyard procedure of turning a Groves mechanism into a one-dimensional, budget-balanced, Pareto efficient mechanism, but we do so on Tian (1996)'s larger domain of preferences. In doing so, we uncover a broad family of complex mechanisms that, in theory, could also be used to solve the free-rider problem. We also gain a deeper understanding of the incentive properties of Groves mechanisms applied to general preferences. We see that getting the Samuelson condition (equating the sum of marginal rates of substitution to marginal costs) is a relatively trivial matter, so that the real difficulties in the design problem lie in achieving budget balance and equilibrium existence. We lean on Tian (1996) to accomplish the former; for the latter,

[1]Groves (1973) mechanisms are usually only discussed in settings where all agents actually have quasilinear preferences, and therefore have a dominant strategy of truthful revelation. Groves and Ledyard (1977) are novel in that they consider the Groves mechanism for more general preferences.

[2]This was conjectured by Laffont and Maskin (1980), who provide general conditions for budget balance.

we devise a simple trick of allowing agents to announce transfers among their partners, providing just enough richness to the range of the mechanisms (without distorting incentives) to guarantee equilibrium existence in these mechanisms.

Finally, we study the stability properties of these Generalized Groves-Ledyard mechanisms, since one beneficial property of the original Groves-Ledyard mechanism is that it becomes dynamically stable under appropriate parameter values (Chen and Plott, 1996; Page and Tassier, 2004; Healy, 2006; Healy and Mathevet, 2012). It is clear that stability will become difficult to obtain in the generalized versions of the mechanism. We show that in one generalized version the best response functions' slopes depend crucially on the messages sent. With very extreme messages the best response slopes become steep enough that stability is violated. We argue that this will be a pervasive problem in any of the generalized versions of the mechanism. This suggests that the original Groves-Ledyard mechanism is not only the simplest among the generalized versions, but it is likely the only one that can be made globally stable.

II. Setup

The Economic Environment

We present here a simplified version of the Arrow-Debreu setting studied by Groves and Ledyard (1977). Specifically, there is one private good and one public good, I consumers, F firms, and a mechanism (that can be thought of as a government). The mechanism receives messages from the consumers and uses this information to procure the public good from the firms. This purchase is financed by (message-dependent) the taxes collected from the consumers.

The price of the private good is normalized to one, and the price of the public good is given by $p \in \mathbb{R}$. Quantities are given by $x \in \mathbb{R}$ for the private good and $y \in \mathbb{R}$ for the public good. Each consumer i has a consumption set $\mathscr{X}_i \subseteq \mathbb{R}^2$, a preference relation $\succeq_i$ on $\mathscr{X}_i$ that is representable by a differentiable utility function u_i, and an initial endowment of private goods $\omega_i \in \mathbb{R}$. Firms are characterized by a production set $\mathscr{Z}_f \in \mathbb{R}^2$ and a vector of profit shares $\theta_f = (\theta_f^1, \dots, \theta_f^I)$. Production vectors are given by $z_f = (z_f^x, z_f^y)$, with negative components representing inputs. Firm profits are distributed to consumers according to θ_f. An *economy* is therefore represented by $e = ((\mathscr{X}_i, \succeq_i, \omega_i)_{i=1}^I, (\mathscr{Z}_f, \theta_f)_{f=1}^F)$.

We think of there being a set of admissible economies $\mathscr{E}$, where each $e \in \mathscr{E}$ differs only in the preference profiles of the consumers. We assume that, for every $e \in \mathscr{E}$, each u_i is continuously differentiable, quasi-concave, and strictly increasing in the private good. In some cases we may discuss further restrictions on $\mathscr{E}$, such as requiring that all preferences are quasilinear ($u_i(x_i, y) = v_i(y) + x_i$ for some concave function v_i), or even quadratic-quasilinear ($v_i(y)$ is concave and quadratic).

An *allocation* is a vector $((x_i)_{i=1}^I, y, (z_f)_{f=1}^F)$ with $x_i \in \mathbb{R}$ for each consumer i, $y \in \mathbb{R}$, and $z_f \in \mathbb{R}^2$ for each firm f. An allocation is *feasible* if (i) $(x_i, y) \in \mathscr{X}_i$ for each i, (ii) $z_f \in \mathcal{Z}_f$ for each j, and (iii) $(\sum_i (x_i - \omega_i), y) \leq \sum_f z_f$. An allocation $((x_i)_{i=1}^I, y, (z_f)_{f=1}^F)$ is *Pareto optimal* if it is feasible and there is no other feasible allocation $((x_i')_{i=1}^I, y', (z_f')_{f=1}^F)$ such that $(x_i', y') \succeq_i (x_i, y)$ for all i and $(x_i', y') \succ_i (x_i, y)$ for some i.

Mechanisms and Competitive Equilibrium

A *mechanism* (or *government*) is simply tuple which specifies a message space $M = \times_i M_i$ (one for each consumer), an allocation rule $y(m, p)$ selecting a public goods level for each message profile m and (public good) price p, and a list of tax functions $t_i(m, p)$ indicating how much wealth each consumer must sacrifice for financing the public good.[3] Thus, a mechanism is given by $\Gamma = ((M_i)_{i=1}^I, y, (t_i)_{i=1}^I)$.[4]

Firm f's profit is given by $z_f^x + p z_f^y$. Its supply correspondence $\phi_f(p)$ is the set of production vectors $z_f \in \mathcal{Z}_f$ that maximize profit in $\mathcal{Z}_f$ given price p. Resulting profits are then given by $\pi_f(p) := z_f^x + p z_f^y$ for any $z_f \in \phi_f(p)$.

At price p, consumer i has wealth $w_i(\omega_i, p) := \omega_i + \sum_f \theta_f^i \pi_f(p)$. This consumer must choose a private goods consumption level x_i and a message $m_i \in M_i$ to send to the government, taking as given p and m_{-i}. She faces the constraint that (x_i, m_i) must be such that $(x_i, y(m_i, m_{-i}, p)) \in \mathscr{X}_i$ and $x_i + t_i(m_i, m_{-i}, p) \leq w_i(\omega_i, p)$. Let $B_i(m_{-i}, p)$ be the 'budget set' of choices (x_i, m_i) satisfying this constraint, and define $\beta_i(m_{-i}, p)$ to be the set of $\succeq_i$-most-preferred (or 'best response') choices in $B_i(m_{-i}, p)$.

We will define a competitive equilibrium for an economy with a given mechanism as an allocation, a price p, and a vector of messages that satisfy preference maximization and profit maximization given p. Groves and Ledyard (1977) further require exact market clearing, which implies that the government balances the budget. We allow here the possibility of governments that collect a surplus in equilibrium; equilibrium is defined formally after a few simplifications.

For simplicity, we assume a linear production technology $\mathcal{Z}_f = \{z_f \in \mathbb{R}^2 : z_f^x + \kappa z_f^y \leq 0\}$ for all j, meaning all firms have a constant marginal cost of $\kappa > 0$. Profit maximization yields $p = \kappa$ and $\pi_f(\kappa) = 0$ (so that $w_i(\omega_i, \kappa) = \omega_i$). The budget balance conditions

[3]We assume later a constant marginal cost of production, which implies that wealth is measured simply by the private good endowment.

[4]We abuse notation, letting y represent both a public good level and the function that determines this level; this should cause no confusion.

simplify to $\sum_f z_f^y = y(m,p)$ and $\sum_i x_i + \kappa y(m,p) = \sum_i \omega_i$. Since we are not studying out-of-equilibrium phenomena, we henceforth assume that p is in equilibrium and therefore equals κ.[5] We also drop p from the arguments of all functions, for brevity.

We further assume that $\mathscr{X}_i = \mathbb{R}^2$ for each i, so that preference maximization implies $x_i = \omega_i - t_i(m)$. This simplifies our analysis by allowing for negative quantities of public goods. It is well known that Nash implementation of Walrasian or Lindahl equilibria is impossible when boundary equilibria are possible, so these must be ruled out either by assumptions on preferences (as in Groves and Ledyard, 1977) or by allowing an unbounded strategy space (as in Healy and Mathevet, 2012). Our goal here is to discover new mechanisms, and we want to do so under minimal assumptions on preferences, so we take the latter route. Since the choice of x_i is now trivial, we redefine $\beta_i(m_{-i})$ to be the $\succeq_i$-optimal messages in M_i, given m_{-i}.[6] This further simplifies the budget balance conditions to $\sum_f z_f^y = y(m)$ and $\sum_i t_i(m) = \kappa y(m,p)$.

Now, a *(Nash or competitive) equilibrium* is a message profile $m^* \in M$ such that (i) $m_i^* \in \beta_i(m_{-i}^*)$ for all i, and (ii) $\sum_i t_i(m^*) \geq \kappa y(m^*)$). If (ii) holds with equality, we call it a *balanced (Nash or competitive) equilibrium*. It is *unbalanced* if (ii) holds with strict inequality. Given an equilibrium m^*, the equilibrium allocation is the allocation that results from m^*. A message m_i^* is a *dominant strategy* if $m_i^* \in \beta_i(m_{-i})$ for all m_{-i}.

Our ultimate goal is to design mechanisms such that (i) at least one balanced equilibrium exists for every $e \in \mathscr{E}$, and (ii) all balanced equilibrium allocations are Pareto optimal.[7] We refer to such mechanisms as *Nash efficient* mechanisms. If a mechanism satisfies (ii) but not (i) then we say it is *conditionally Nash efficient* (conditional on existence).

Define agent i's marginal rate of substitution at any allocation (x_i, y) by

$$MRS_i(x_i, y) = \frac{\partial u_i(x_i, y)/\partial y}{\partial u_i(x_i, y)/\partial x_i}.$$

A mechanism is conditionally Nash efficient if and only if for every $e \in \mathscr{E}$ and every balanced equilibrium m^* in e,

$$\sum_i MRS_i(\omega_i - t_i(m^*), y(m^*)) = \kappa. \tag{1}$$

[5]In our section on stability, we do discuss a situation where consumers' messages m are dynamically adjusting out of equilibrium. There we assume that p does not adjust, for any deviation from $p = \kappa$ would cause firms to produce infinite (or negatively infinite) quantities. We view this as a very strong force that stabilizes p at κ even when m is out of equilibrium.

[6]Formally, $\beta_i(m_{-i}) = \{m_i \in M_i : (\omega_i - t_i(m_i, m_{-i}), y(m_i, m_{-i})) \succeq_i (\omega_i - t_i(m_i', m_{-i}), y(m_i', m_{-i}))\ \forall m_i' \in M_i\}$.

[7]Unbalanced equilibrium allocations cannot be Pareto optimal.

We refer to (1) as the Samuelson (1954) condition, since it is necessary (but not sufficient) for Pareto optimality. If, in addition, the allocation is budget-balanced then efficiency obtains. Thus, if existence of a balanced equilibrium is guaranteed for every $e \in \mathscr{E}$, then conditional Nash efficiency implies Nash efficiency.

Direct, Quasi-Direct, and Indirect Mechanisms

A mechanism is called *direct* if each M_i is the space of possible preferences (or utility functions) for consumer i. For example, the Vickrey-Clarke-Groves mechanism introduced below is a direct mechanism when preferences are quasi-linear.

We introduce here the notion of a *quasi-direct* mechanism, in which agents can be thought of as announcing their preferred allocation, rather than an abstract message.

Definition 1. Mechanism $\Gamma = ((M_i)_{i=1}^I, y, (t_i)_{i=1}^I)$ is *quasi-direct* if

(1.1) $y(\cdot, m_{-i})$ is surjective for each i and m_{-i} (meaning, for each m_{-i} and y_i there exists some m_i such that $y(m_i, m_{-i}) = y_i$), and

(1.2) each $t_i(\cdot)$ is measurable with respect to $\{(y(m), m_{-i})\}_{m \in M}$ (meaning, $y(m_i, m_{-i}) = y(m_i', m_{-i})$ implies $t_i(m_i, m_{-i}) = t_i(m_i', m_{-i})$).

We will see a direct mechanism that is also quasi-direct; these definitions are not mutually exclusive. A mechanism is said to be *indirect* if it is neither direct nor quasi-direct.

In a quasi-direct mechanism, choosing m_i (given m_{-i}) is equivalent to choosing a public goods level $y_i(m_i, m_{-i})$, and then paying a tax that depends only on y_i and m_{-i}. With quasi-direct mechanisms we sometimes view the message as y_i rather than m_i, and re-write the tax function as $t_i(y_i, m_{-i})$.

Consider an agent in a quasi-direct mechanism choosing $y_i \in \mathbb{R}$ to maximize $u_i(\omega_i - t_i(y_i, m_{-i}), y_i)$ given m_{-i}. Suppose $t_i(y_i, m_{-i})$ is differentiable in y_i. The individually-optimal choice y_i^* will be characterized by the first-order condition for utility maximization that

$$\frac{\partial t_i(y_i^*, m_{-i})}{\partial y_i} = MRS_i(\omega_i - t_i(y_i^*, m_{-i}), y_i^*). \tag{2}$$

Combining this with the Samuelson condition (1) (and our assumption of quasiconcave utilities) gives the following lemma.

Lemma 1. A quasi-direct mechanism is conditionally Nash efficient if and only if for every balanced equilibrium m^*

$$\sum_i \frac{\partial t_i(y(m^*), m^*_{-i})}{\partial y_i} = \kappa.$$

It is not hard to construct quasi-direct mechanisms that are conditionally Nash efficient. For example, consider the proportional tax mechanism described by Groves and Ledyard (1977).

Definition 2. The **proportional tax mechanism** Γ^P is given by

(2.1) $M_i^P = \mathbb{R}^1$ for each i,
(2.2) $y^P(m) = \sum_i m_i$,
(2.3) $t_i^P(m) = \alpha_i \kappa y^P(m)$,

where $(\alpha_i)_{i=1}^I$ is any vector of parameters satisfying $\sum_i \alpha_i = 1$.

The proportional tax mechanism is quasi-direct; we can view the agent as choosing y_i (by setting $m_i = y_i - \sum_{j\neq i} m_j$) and then paying $t_i^P(y_i, m_{-i}) = \alpha_i \kappa y_i$. By Lemma 1, this mechanism is conditionally Nash efficient. The difficulty, however, is that equilibrium rarely exists. At any Nash equilibrium m^*, all agents must select the same value of y_i. Call this value y^*. By equation (2) we have $MRS_i(\omega_i - \alpha_i \kappa y^*, y^*) = \alpha_i \kappa$ for each i. Thus, we have equilibrium existence only in the knife-edge economies in which there exists a y satisfying this requirement for all agents. In those economies the equilibrium will be balanced and will generate Pareto optimal allocations. In all other economies no equilibrium will exist.

The Original Groves-Ledyard Mechanism

The Groves-Ledyard mechanism can be viewed as an extension of the proportional tax mechanism that guarantees existence of equilibrium by adding a quadratic penalty term.

Definition 3. The **Groves-Ledyard mechanism** Γ^L is given by

(3.1) $M_i^L = \mathbb{R}^1$ for each i,
(3.2) $y^L(m) = \sum_i m_i$,
(3.3) $t_i^L(m) = \alpha_i \kappa y(m) + \frac{\gamma}{2}\left[\frac{I-1}{I}(m_i - \bar{m}_{-i})^2 - \sigma^2(m_{-i})\right]$,

where

- $((\alpha_i)_{i=1}^I, \gamma)$ is a vector of parameters satisfying $\sum_i \alpha_i = 1$ and $\gamma > 0$,
- $\bar{m}_{-i} = \frac{1}{I-1}\sum_{j\neq i} m_j$, and

- $\sigma^2(m_{-i}) = \frac{1}{I-2}\sum_{j\neq i}(m_j - \bar{m}_{-i})^2$.

Groves and Ledyard (1977, 1980) show that this mechanism is (unconditionally) Nash efficient. One deficiency is that it may violate individual rationality, meaning there are economies in which some agents prefer the initial endowment over the equilibrium allocation of the mechanism. Hurwicz (1979a) shows that if one wants both Nash efficiency and individual rationality for every economy (subject to a continuity condition), then one must select a mechanism whose equilibrium outcomes are Lindahl allocations.[8] We refer to such mechanisms as *Nash-Lindahl* mechanisms. One of the first Nash-Lindahl mechanisms was identified by Walker (1981). Healy and Mathevet (2012) provide a full characterization of continuously differentiable Nash-Lindahl mechanisms when the set of admissible economies is sufficiently rich. They prove that, for any Nash-Lindahl mechanism, $t_i(m)$ must be of the form $t_i(m) = q_i(m_{-i})y(m)$. The proportional tax mechanism and the Walker mechanism satisfy this property, while the Groves-Ledyard mechanism does not. Other Nash-Lindahl mechanisms have been proposed by Hurwicz (1979b); Tian (1990); Kim (1993); Chen (2002); Healy and Mathevet (2012), and Van Essen (2013), among others.

III. Using Groves Mechanisms to Construct Nash Efficient Mechanisms

Groves Mechanisms

We begin with the definition of the Groves mechanisms.

Definition 4. A **Groves mechanism** Γ^G is given by

(4.1) M_i^G is a set of strictly concave, differentiable functions m_i such that, for all m, $\arg\max_y \left\{\sum_i m_i(y) - \kappa y\right\}$ is not empty,

(4.2) $y^G(m) = \arg\max_y \left\{\sum_i m_i(y) - \kappa y\right\}$

(4.3) $t_i^G(m) = \alpha_i \kappa y^G(m) - \sum_{j\neq i}\left(m_j(y^G(m)) - \alpha_j \kappa y^G(m)\right) + h_i(m_{-i})$,

where $\sum_i \alpha_i = 1$ and $h_i(m_{-i})$ is any function that does not depend on m_i.

The tax function t_i^G is measurable in $\{(y^G(m), m_{-i})\}_m$. Furthermore, for any y_i and m_{-i}, player i can announce any $m_i \in M_i^G$ with $m_i'(y_i) = \kappa - \sum_{j\neq i} m_j'(y_i)$ to get $y^G(m_i, m_{-i}) = y_i$. Thus, Groves mechanisms are quasi-direct. If we restrict ourselves to economies where the actual preferences are quasi-linear and the v_i functions are in M_i^G, then the Groves mechanisms are also direct mechanisms, and truth-telling is a dominant strategy (Groves, 1973). In the literature, Groves mechanisms are almost always applied to

[8] A Lindahl allocation is a vector $((x_i^*)_{i=1}^I, y^*)$ such that there exists some $(p_1, \ldots, p_I)$ with $\sum_i p_i = \kappa$ for which the maximizers $(\hat{x}_i, \hat{y}_i) \in \arg\max\{u_i(x_i, y_i) : x_i + p_i y_i \leq \omega_i\}$ satisfy $\hat{x}_i = x_i^*$ and $\hat{y}_i = y^*$ for each i.

quasi-linear economies and viewed as direct mechanisms. The key innovation of Groves and Ledyard (1977) is to view them instead as quasi-direct mechanisms for more general economies.

Groves and Ledyard (1977) prove the following properties of Groves mechanisms in general economies.

Lemma 2 (Groves and Ledyard (1977), Theorem 3.1)**.** Consider any $e \in \mathscr{E}$ and any equilibrium m^* of a Groves mechanism Γ^G in e.

(2.1) Each agent's message reveals their marginal rate of substitution at the realized allocation:[9]

$$\frac{\partial m_i^*(y^G(m^*))}{\partial y} = MRS_i(\omega_i - t_i^G(m^*), y^G(m^*)).$$

(2.2) The Samuelson condition is satisfied: $\sum_i MRS_i(\omega_i - t_i^G(m^*), y^G(m^*)) = \kappa$.

A proof appears in the appendix.

Remark 1. There are many possible best responses to a given m_{-i}, even with quasi-linear preferences. For example, consumer i can directly affect the taxes of others by adding a scalar to their message. The resulting message will still be a best response, and the added scalar will not alter the public good level or their own tax. Other similar manipulations are possible.

Remark 2. In quasi-linear economies, any m_i with $m_i'(y^G(m_i, m_{-i})) = v_i'(y^G(m_i, m_{-i}))$ is a best response to m_{-i}. If other agents change their message to some $\hat{m}_{-i}$ at which $m_i'(y^G(m_i, \hat{m}_{-i})) \neq v_i'(y^G(m_i, \hat{m}_{-i}))$, then m_i is no longer a best response to $\hat{m}_{-i}$. Thus, m_i is not a dominant strategy. If, however, $m_i \equiv v_i$ (up to an additive scalar), then m_i is a best response regardless of the value of y^G. Thus, truth-telling is a dominant strategy in Groves mechanisms with quasi-linear preferences. With general preferences, however, consumer i's MRS_i depends on the level of taxes; as others change their message (for example, by adding scalars), consumer i must alter her message in response to match the resulting change in MRS_i.

That the Samuelson condition is satisfied is only one step toward establishing Nash efficiency. What remains is to ensure that an equilibrium exists, and that it is balanced. For General Groves mechanisms, equilibrium existence is impossible if the mechanism fails to collect sufficient funds to finance $y^G(m)$. The VCG mechanism cures this problem by ensuring that $h_i(m) \geq \alpha_i \kappa y(m)$ for all m.

[9] Groves and Ledyard (1977) actually show that agents reveal their 'marginal willingness to pay', but it is equivalent here to the MRS_i function.

Definition 5. A mechanism $\Gamma^V = ((M_i^V)_i, y^V, (t_i^V)_i)$ is the **VCG mechanism** if it is a Groves mechanism in which

(5.1) $h_i(m_{-i}) = \max_{y_{-i}} \sum_{j \neq i} \left(m_j(y_{-i}) - \alpha_j \kappa y_{-i}\right)$.

Though the VCG mechanism will never run a shortage, it collects excess funds in many economies. The excess cannot be returned to agents in the economy without altering their preferences, and therefore represents social inefficiency. To achieve Pareto optimality, we must use a Groves mechanism that is exactly balanced.

Unfortunately, it is well-known that balanced Groves mechanisms are impossible when M_i^G is sufficiently large.[10] Thus, we must look to smaller domains of quasilinear preferences to achieve balance.

Quadratic Preferences and the Groves-Ledyard Mechanism

Groves and Loeb (1975) show that the Groves mechanism can be exactly balanced when preferences are quadratic-quasilinear and only vary in the intercept of the marginal utility for y. Groves and Ledyard (1977) use this space of preferences to construct their mechanism in general economies. Specifically, they apply the Groves mechanism with message space

$$M_i^Q := \left\{ v_i(\cdot|\theta_i) : v_i(y|\theta_i) = (\gamma\theta_i + \alpha_i\kappa)y - \frac{\gamma}{2I}y^2,\ \theta_i \in \mathbb{R} \right\}, \tag{3}$$

where $((\alpha_i)_{i=1}^I, \gamma)$ is a fixed vector of parameters satisfying $\sum_i \alpha_i = 1$ and $\gamma > 0$. Each $\theta_i \in \mathbb{R}$ is a free parameter that indexes the set of functions in M_i^Q. Thus, we can view consumers as selecting a $\theta_i \in \mathbb{R}$ and then submitting $m_i = v_i(\cdot|\theta_i)$ to the VCG mechanism. Letting θ be the vector of chosen parameter values, a few calculations reveals that

$$\begin{aligned} y^G(\theta) &= \sum_i \theta_i, \\ t_i^G(\theta) &= \alpha_i\kappa + \frac{\gamma}{2}\left[\frac{I-1}{I}\left(\theta_i - \bar{\theta}_{-i}\right)^2 - \sigma^2(\theta_{-i})\right]. \end{aligned}$$

This is exactly the Groves-Ledyard mechanism Γ^L. Because it is balanced, we have from Lemma 2 that this mechanism is conditionally Nash efficient. Furthermore, Groves and Ledyard (1980) show that equilibrium exists for a wide class of economies, giving the following result.

Proposition 1 (Groves and Ledyard (1977,1980))**.** Let Γ be a Groves mechanism in which $M_i^G = M_i^Q$ (the space of parameterized quadratic-quasilinear functions given in

[10]See footnote 22 of Groves and Ledyard (1977).

(3)). Then Γ is equivalent to the Groves-Ledyard mechanism Γ^L, has a balanced equilibrium for every e, and is Nash efficient.

Tian's Preferences and the Generalized Groves-Ledyard Mechanism

Originally, it was conjectured that quadratic-quasilinear preferences are the only preferences for which Groves mechanisms can be balanced (see Laffont and Maskin, 1980). Then Tian (1996) showed that budget balance could be obtained on a wider class of polynomial functions that may have an order higher than two. Liu and Tian (1999) extended this line of inquiry by providing a characterization of domains of functions on which there exists a balanced Groves mechanism.

We can now replicate Groves and Ledyard's construction of their mechanism, but with the more general budget-balancing preferences of Tian (1996). The resulting mechanism will generalize the original Groves-Ledyard mechanism.

Tian (1996)'s preferences were derived for the case of $\kappa = 0$. And his proof that these preferences result in a balanced budget rely on the arguments of Laffont and Maskin (1980), who also assume $\kappa = 0$. For completeness we now verify that the Laffont-Maskin result holds for $\kappa > 0$ and then re-derive Tian's preferences, accounting for κ.[11]

Lemma 3 (Based on Laffont and Maskin (1980), Theorem 4.1)**.** Assuming $y^G(m)$ is differentiable and $\kappa \geq 0$, there exists a balanced Groves mechanism if and only if there exist $(\alpha_i)_{i=1}^I$ such that $\sum_i \alpha_i = 1$ and

$$\sum_{i=1}^{I} \frac{\partial^{I-1}}{\partial m_{-i}} \left[\left(v_i'(y(m)|\theta_i) - \alpha_i \kappa \right) \frac{\partial y}{\partial \theta_i} \right] \equiv 0.$$

Lemma 4 (Based on Tian (1996), Theorem 1)**.** Assume $y^G(m)$ is differentiable and $\kappa \geq 0$. Fix any $(\alpha_i)_{i=1}^I$ such that $\sum_i \alpha_i = 1$, any $c > 0$ and $d \geq 0$, and any continuously differentiable functions $f(y)$ and $(\psi_i(\theta_i))_i$ such that $f(y) \geq 0$ for all y, $f'(y) \neq 0$ for all y, f is invertible, and $\psi_i'(\theta_i) \neq 0$ for all i and θ_i. For each $r \in \{2, 3, \ldots, I-1\}$, there exists a balanced Groves mechanism if $M_i^G = M_i^{Tr}$, where

$$M_i^{Tr} = \left\{ v_i(\cdot|\theta_i) : v_i(y|\theta_i) = \alpha_i \kappa y + f(y)\psi_i(\theta_i) - \frac{r-1}{rc}(cf(y)+d)^{r/(r-1)},\ \theta_i \in \mathbb{R}^1 \right\}.$$

We now describe how Tian (1996)'s preferences are a generalization of the quadratic preferences M_i^Q above. First, the public good level y can be transformed by an invertible, non-negative function $f(y)$ which is identical across agents. Second, the type of the agent θ_i can be transformed an increasing or decreasing function $\psi_i(\theta_i)$ which can differ

[11] Proofs for both lemmata follow closely the original proofs and are available upon request.

across agents. Third, the quadratic term can be raised to any fractional power $r/(r-1)$, where $r \in \{2,3,\ldots,I-1\}$. Finally, the quadratic term can be linearly transformed by multiplicative scalar $c>0$ and additive scalar $d \geq 0$. Tian's preferences are identical to quadratic preferences when $r=2$, $f(y)=y$, $\psi_i(\theta_i)=\gamma\theta_i$ for any $\gamma>0$, $c=\gamma/I$, and $d=0$.

Now we apply the Tian preferences to generate a family of Generalized Groves-Ledyard mechanisms, in exactly the same way that Groves and Ledyard (1977) applied quadratic preferences to develop the original Groves-Ledyard mechanism. Specifically, we note that announcing one's preferences (in either the Tian family or the quadratic family) requires announcing only a single real number θ_i, so we take the balanced Groves mechanism and perform simple algebraic manipulation to derive a simpler form for the expression of the Groves mechanism in terms of those single-number announcements.

Proposition 2. (Generalized Groves-Ledyard Mechanisms.) Fix $r \in \{2,3,\ldots,I-1\}$ and parameters $(\alpha_i)_{i=1}^I$ such that $\sum_i \alpha_i = 1$, $c>0$, $d \geq 0$, $f(\cdot)$, and $(\psi_i(\cdot))_i$. Using message space M_i^{Tr} in the Groves mechanism gives the following Generalized Groves-Ledyard mechanism.

(2.1) $M_i^* = \mathbb{R}$,
(2.2) $y^*(\theta) = f^{-1}\left(\frac{1}{c}\left(\bar{\psi}(\theta)^{r-1}-d\right)\right)$,
(2.3) $t_i^*(\theta) = \alpha_i \kappa y^*(\theta) - (I-1)\left[\frac{1}{c}\left(\bar{\psi}(\theta)^{r-1}-d\right)\bar{\psi}_{-i}(\theta_{-i}) + \frac{r-1}{rc}\bar{\psi}(\theta)^r\right] + h_i(\theta_{-i})$,

where $\bar{\psi}(\theta) = \sum_i \psi_i(\theta_i)/I$ and $\bar{\psi}_{-i}(\theta_{-i}) = \sum_{j\neq i}\psi_j(\theta_j)/(I-1)$. The mechanism is balanced if, in addition,

$$(2.4) \qquad h_i(\theta_{-i}) = \sum_{q=1}^{r}\sum_{\mathcal{T}_{q,r}}\left[\frac{I-1}{I^{r-1}}\frac{2r-1}{cr}\frac{\binom{r}{t_1,t_2,\ldots,t_q}}{I-q}\left(\sum_{i_1\neq i}\sum_{i_2\notin\{i,i_k\}_{k<2}}\cdots\sum_{i_q\notin\{i,i_k\}_{k<q}}\prod_{k=1}^{q}\psi_{i_k}(\theta_{i_k})^{t_k}\right)\right] - \frac{d}{c}\sum_{j\neq i}\psi_j(\theta_j),$$

where

$$(2.5) \qquad \mathcal{T}_{q,r} = \left\{(t_1,\ldots,t_q)\in\mathbb{N}^q : \sum_{k=1}^{q} t_k = r \text{ and } (\forall k>1)\, t_{k-1} > t_k\right\}$$

is the set of all strictly decreasing length-q sequences of positive integers that sum to r, and

$$(2.6) \qquad \binom{r}{t_1,t_2,\ldots,t_q} = \frac{r!}{t_1!t_2!\cdots t_q!}$$

is the multinomial coefficient.

A detailed derivation appears in the appendix. We have also verified that the original Groves-Ledyard mechanism is obtained in the special case where Tian's preferences reduce to the quadratic preferences ($r = 2$, $f(y) = y$, $\psi_i(\theta_i) = \gamma\theta_i$ for any $\gamma > 0$, $c = \gamma/I$, and $d = 0$).

Example Mechanisms

In this section we adapt the parametric restrictions that imply the Groves-Ledyard mechanism, but consider higher-order preferences represented by $r > 2$.[12] For any $r \in \{2, 3, \dots, I-1\}$, define

$$M_i^* = \left\{ v_i(\cdot|\theta_i) : v_i(y|\theta_i) = \alpha_i \kappa y + \gamma\theta_i f(y) - \frac{\gamma(r-1)}{rI} f(y)^{\frac{r}{r-1}}, \theta_i \in \mathbb{R} \right\}.$$

The resulting public good function is characterized by

$$f(y^*(\theta)) = (\sum_i \theta_i)^{r-1},$$

and the tax function is given by

$$t_i^*(\theta) = \alpha_i \kappa y^*(\theta) - \sum_{j \neq i} \left[\gamma\theta_j (\sum_k \theta_k)^{r-1} - \frac{\gamma(r-1)}{rI} (\sum_k \theta_k)^r \right] + h_i(\theta_{-i}). \tag{4}$$

Algebraic manipulation of equation 4 yields a version of the tax function somewhat similar to the original Groves-Ledyard mechanism. Specifically, we find

$$\begin{aligned} t_i^*(\theta) =& \alpha_i \kappa (\sum_{i=1}^{I} \theta_i)^{r-1} + \frac{\gamma(r-1)}{r} \left(\frac{I-1}{I} \left(\theta_i - \bar{\theta}_{-i} \right)^r + \sum_{j=1}^{r-1} \theta_i^{r-j} b_j (I-1)^j \bar{\theta}_{-i}^j \right) \\ &+ \gamma \frac{I-1}{I} \frac{(r-1)}{r} b_r (I-1)^r \bar{\theta}_{-i}^r + h_i(\theta_{-i}), \end{aligned}$$

where

$$b_i = -\Big(\frac{(-1)^i \binom{r}{i}}{(I-1)^i} + \frac{\frac{r}{r-1} I \binom{r-1}{i-1} - (I-1)\binom{r}{i}}{I-1} \Big).$$

For the case of $r = 2$ and $f(y) = y$, we have $y^*(\theta) = \sum_i \theta_i$, as in Groves-Ledyard, and

$$h_i(\theta_{-i}) = \frac{\gamma}{2}(I-1) \left[\frac{1}{I-1} \sum_{j \neq i} \theta_j^2 + \frac{2}{I-2} \sum_{j \neq i} \sum_{k \neq i,j} \theta_j \theta_k \right].$$

Further manipulation gives the exact form for t_i^L above.

Finally, if $r = 3$, and $f(y) = y$, we have

$$y^*(\theta) = (\sum_i \theta_i)^2$$

[12] Technically, these preferences deviate from from M^{Tr} defined above by adding a scalar multiple to the $f(y)^{r/(r-1)}$ term, but it can be shown easily that this does not impact budget balance.

and

$$t_i^*(\theta) = \alpha_i \kappa \Big(\sum_{i=1}^{I} \theta_i\Big)^2 + \frac{\gamma(r-1)}{r}\Big(\frac{I-1}{I}\Big(\theta_i - \bar\theta_{i-1}\Big)^3 + \sum_{j=1}^{2} \theta_i^{3-j} b_j (I-1)^j \bar\theta_{-i}^j\Big) + \frac{2}{3}\gamma \frac{I-1}{I} b_3 (I-1)^3 \bar\theta_{-i}^3 + h_i(\theta_{-i}),$$

where

$$h_i(\theta_{-i}) = \frac{2\gamma}{3}(I-1)[\frac{1}{I-1}\sum_{j\neq i}\theta_j^3 + \frac{3}{I-2}\sum_{j\neq i}\sum_{k\neq i,j}\theta_j^2\theta_k + \frac{1}{I-3}\sum_{j\neq i}\sum_{k\neq i,j}\sum_{l\neq i,j,k}\theta_j\theta_k\theta_l].$$

IV. Equilibrium Existence and Two-Dimensional Mechanisms

Groves and Ledyard (1980) prove that a competitive equilibrium exists for their mechanism in a wide range of economies. Here we find conditions on the strategy space under which equilibrium existence is assured for Groves mechanisms, and then ask whether or not those conditions can translate to the Generalized Groves-Ledyard mechanisms with their more restrictive strategy spaces.

Proposition 3. Suppose Γ^G is a Groves mechanism such that, for every i, $m_i \in M_i^G$, and scalar a_i, the function $m_i(\cdot) + a_i$ is also in M_i^G. If an economy e has an allocation $((x_i^o)_i, y^o)$ satisfying the Samuelson condition (eq. 1) and there is a message $\hat m \in M^G$ for which $\hat m_i'(y^o) = MRS_i(x_i^o, y^o)$ for all i, then there exists a Nash equilibrium of Γ^G in e whose allocation is $((x_i^o)_i, y^o)$.

Unfortunately, this existence result does not apply to the Generalized Groves-Ledyard mechanisms proposed above because their message spaces do not contain functions that differ only by a scalar. But if we add a scalar to each function as a second parameter—which the agent must also announce—then the message space becomes rich enough to permit equilibrium existence via Proposition 3. We now briefly formalize the development of the resulting mechanism.

Fix any Generalized Groves-Ledyard mechanism, which is a Groves mechanism $\Gamma^G = ((M_i^{Tr})_i, y^G, (t_i^G)_i)$ with Tian preferences M^{Tr} for some $r \in \{2, \ldots, I-1\}$. Now, construct a new 'shifting' mechanism $\Gamma^S = ((M_i^S)_i, y^S, (t_i^S)_i)$ as follows. First, set

$$M_i^S = \Big\{v_i(\cdot|\theta_i, \beta_i) = v_i(\cdot|\theta_i) + \beta_i : v_i(\cdot|\theta_i) \in M_i^{Tr},\ \beta_i \in \mathbb{R}\Big\}.$$

We can equivalently view agents as submitting (θ_i, β_i) instead of m_i. Next, set $y^S(\theta, \beta) = y^G(\theta)$. Now, if we simply apply the Groves tax function with these new preferences, we would have that $t_i^S(\theta, \beta) = t_i^G(\theta) - \sum_{j\neq i}\beta_j$ for each i. This is undesirable; if the original mechanism were balanced, this new mechanism would be unbalanced by a factor of $\sum_i \sum_{j\neq i}\beta_j$. To compensate, we need to add this amount into $\sum_i h_i(\theta_{-i}, \beta_{-i})$. This can be done by adding $(I-1)\beta_{i+1}$ to each $h_i(\theta_{-i}, \beta_{-i})$, where $i+1$ is taken modulo I. In total, we now have

$$t_i^S(\theta,\beta) = t_i^G(\theta) - \sum_{j\neq i} \beta_j + (I-1)\beta_{i+1}.$$

The added terms clearly balance out in aggregate, preserving the balance properties of the original mechanism.

Since β_i does not affect consumer i's allocation, this added dimension does not alter the strategic properties of the mechanism. It does, however, give us enough flexibility to use the above existence result. Thus, we have the following proposition.

Proposition 4. For any Generalized Groves-Ledyard mechanism Γ^*, the two-dimensional 'shifting' mechanism Γ^S defined above

(4.1) satisfies the Samuelson condition (Lemma 2) at any equilibrium,
(4.2) is budget balanced ($\sum_i t_i^S(\theta,\beta) = \kappa y^S(\theta,\beta)$) for every θ and β, and
(4.3) has an equilibrium in every economy $e \in \mathscr{E}$.

Consequently, Γ^S is a Nash efficient mechanism.

A simple interpretation of this mechanism is that it is identical to Γ^*, except each agent i also gets to take any amount of private good from their neighbor and redistribute it equally among $j \neq i$. This makes any vector of private goods consumption obtainable at any public goods level, subject to the aggregate resource constraint. Any Generalized Groves-Ledyard mechanism can be augmented in this way to guarantee existence, including the original Groves-Ledyard mechanism.

V. Issues with stability

The original Groves-Ledyard mechanism has nice stability properties. With quasi-linear preferences and a suitable choice of γ, the mechanism always induces a game with contractive best response functions.[13] Formally, if $\beta_i(m_{-i})$ is player i's best response function, then for any $j \neq i$,

$$\frac{\partial \beta_i(m_{-i})}{\partial m_j} = \frac{\gamma + I v_i''(y(m))}{\gamma(I-1) - I v_i''(y(m))}. \tag{5}$$

Assuming v_i'' is bounded and letting γ grow large, this slope converges to $1/(I-1)$ from below. And this is true at every message profile $m \in \mathbb{R}^I$, since messages only affect the

[13]See Healy and Mathevet (2012) for a discussion of contraction as a notion of stability.

slope through v_i''. Thus, $\sum_j \partial\beta_i/\partial m_j \in (0,1)$ for large γ, establishing that the mechanism is both contractive and supermodular (hence, stable).[14]

A natural question is whether the Generalized Groves-Ledyard mechanisms inherit the same stability properties. We conjecture that, generically, the answer is negative. Consider the example mechanism with $r = 3$ and $f(y) = y$ from the end of Section III. If $\beta_i(\theta_{-i})$ is player i's best response function, then for any $j \neq i$,

$$\frac{\partial\beta_i(\theta_{-i})}{\partial\theta_j} = \frac{A_i(\theta) - 2\frac{(I-1)^2}{I-1}b_2\bar{\theta}_{-i} + \frac{1}{I-1}[4\frac{I-1}{I}\gamma(\theta_i - \bar{\theta}_{-i}) - 2(I-1)b_1\theta_i]}{4\frac{I-1}{I}\gamma(\theta_i - \bar{\theta}_{-i}) + 2(I-1)b_1\bar{\theta}_{-i} - A_i(\theta)}, \tag{6}$$

where $\bar{\theta}_{-i} = \sum_{j\neq i}\theta_j/(I-1)$ and

$$A_i(\theta) = 2v_i'(y(\theta)) + 4v_i''(y(\theta))(\sum_j \theta_j)^2 - 2\alpha_i\kappa.$$

The difficulty with establishing stability is that the slope of the best response function depends directly on the message profile $\theta \in \mathbb{R}^I$. In the original mechanism the slope depended on θ only through v_i''. Messages can be arbitrarily large or small, which means it may be difficult to bound the slope when messages affect the slope directly. For example, the slope given in (6) converges to -1 as θ_i grows large (holding θ_{-i} fixed). Since $\sum_{j\neq i}\partial\beta_i/\partial\theta_j$ converges to $-(I-1)$, the mechanism is not stable for extreme messages.

It is clear from inspection that the best response slopes will depend on messages directly (rather than through v_i'') for any $r > 2$. Even with $r = 2$ this appears to be a problem unless f and each ψ_i are all linear. Thus, we conjecture that the only Generalized Groves-Ledyard mechanisms that achieve stability are those for which $r = 2$ and f and each ψ_i are linear. In our view, this represents a trivial generalization, so we conjecture that the original Groves-Ledyard mechanism is essentially the only one for which stability can be assured.

VI. Discussion

We do not claim that the Generalized Groves-Ledyard mechanisms are in any way more practical or useful than the original mechanism. Instead, our goal here is to highlight how the method of creating mechanisms used by Groves and Ledyard (1977) could be extended to find other mechanisms. In some sense, we view our results as negative; the Tian preferences are the broadest class of preferences for which it is known that

[14]Quasi-linearity appears important to establish this result. For example, Kim (1987)—building off of Jordan (1986)—shows that stability of any Nash-efficient mechanism is impossible to achieve if a broad enough family of (non-quasilinear) preferences is considered.

the Groves mechanism can be balanced, and we fully characterize the mechanisms that can be created using these preferences. For all but the simplest quadratic preferences, the resulting mechanisms are quite unruly. They are both mathematically complex and possibly cannot be made stable. But, if there are no other preferences that balance the Groves mechanism, then there are no other mechanisms that can be constructed in this way.

Of course, this is also not the only way to construct Nash efficient mechanisms. Brock (1980) used a differential approach to show how such mechanisms might be constructed.[15] Healy and Mathevet (2012) used this approach to characterize Nash-Lindahl mechanisms. In their setting it is easy to see that conditional Nash efficiency (ignoring the Lindahl requirement) is characterized by $\sum_i (\partial t_i/\partial m_i)/(\partial y/\partial m_i) = \kappa$ for all m. And budget balance is given by $\sum_i t_i(m) = \kappa y(m)$ for all m. These are fairly easy to satisfy. Thus, it is the equilibrium existence requirement that is the real challenge, not conditional Nash efficiency.

We study a two-good economy for simplicity. Groves and Ledyard (1977) study economies with arbitrary finite numbers of goods. They demonstrate that the procedure of using the Groves mechanisms to construct Nash efficient mechanisms faces no difficulty with the larger number of goods. Messages simply become vectors instead of scalars. Thus, it is immediately clear that our more generalized construction will similarly apply to the larger setting.

Though our 'shifting' mechanism has the benefit of equilibrium existence, it has a few undesirable properties. It is not symmetric. Agents are forced to treat their neighbors (and to be treated by their neighbors) differently than everyone else. This creates obvious political difficulties in practice. But it appears there is no simple way to make the mechanism symmetric without sacrificing either budget balance or equilibrium existence. Behavior would also be distorted by the slightest degree of social preferences. Each agent essentially engages in a dictator game in which they unilaterally redistribute money among the other parties. The required vector of redistribution that guarantees equilibrium for some economy may be seen by some agents as undesirable, leading again to a non-existence result. Agents in a larger context may also fear reciprocity for their redistribution decisions. Finally, equilibrium existence may require a very specific vector of transfers, to be executed simultaneously by all parties, all of whom

[15] Groves and Ledyard (1987) describe the procedure in detail. Like us, Brock (1980) finds that the Groves-Ledyard mechanism is the 'simplest' mechanism one might hope to construct that guarantees existence. Another notably simple mechanism was constructed by Walker (1981).

are indifferent over their own choices. It seems unlikely that equilibrium would obtain in practice.

APPENDIX A. PROOFS

Proof of Lemma 2

In any quasi-direct mechanism, agent i chooses y_i in response to m_{-i} to maximize

$$u_i(w_i - t_i(y_i, m_{-i}), y_i).$$

At any best response y_i^* we have the first-order condition

$$\frac{\partial t_i\left(y_i^*, m_{-i}\right)}{\partial y} = \frac{\partial u_i\left(w_i - t_i\left(y_i^*, m_{-i}\right), y_i^*\right)/\partial y}{\partial u_i\left(w_i - t_i\left(y_i^*, m_{-i}\right), y_i^*\right)/\partial x_i}.$$

The best response message m_i^* is that which generates $y(m_i^*, m_{-i}) = y_i^*$.

The Groves mechanism is quasi-direct. By the definition of t_i^G, we have that

$$\frac{\partial t_i^G(y_i, m_{-i})}{\partial y} = q - \sum_{j \neq i} m'_j(y_i).$$

Recall $y^G(m)$ sets $\sum_i m'_i\left(y^G(m)\right) = \kappa$. Thus, at any y_i and m_{-i}, the message m_i that will lead to $y_i = y^G(m)$ must satisfy

$$m'_i(y_i) = \frac{\partial t_i^G(y_i, m_{-i})}{\partial y}.$$

If the message is in equilibrium then it is a best response, and so $m'_i\left(y^G(m)\right)$ is agent i's marginal rate of substitution at $\left(\left(\omega_i - t_i^G(m)\right)_{i=}^{I}, y^G(m)\right)$. The Samuelson condition then follows from $\sum_i m'_i\left(y^G(m)\right) = \kappa$.

Proof of Proposition 2

Consider an agent with Tian preferences that submits them truthfully to the Groves mechanism Γ^G. Recall that, with Tian preferences there is only one free parameter θ_i, and the v_i function is given by

$$v_i(y|\theta_i) = \alpha_i \kappa y + f(y)\psi_i(\theta_i) - \frac{r-1}{rc}(cf(y)+d)^{r/(r-1)}.$$

Thus,

$$v'_i(y|\theta_i) = \alpha_i \kappa + f'(y)\psi_i(\theta_i) - f'(y)(cf(y)+d)^{1/r-1}$$

Because $y^G(\theta)$ equates $\sum_i v'_i$ with κ, we have that $y^G(\theta)$ solves

$$\sum_i \psi_i(\theta_i) - I\left(cf\left(y^G(\theta)\right) + d\right)^{\frac{1}{r-1}} = 0$$

at every θ. Or, equivalently,

$$\left(cf\left(y^G(\theta)\right)+d\right)^{\frac{1}{r-1}}=\bar{\psi}(\theta). \tag{7}$$

The Generalized Groves-Ledyard mechanism uses the same public good function as the Groves mechanism, so we have

$$y^*(\theta)\equiv f^{-1}\left(\frac{1}{c}\bar{\psi}(\theta)^{r-1}-\frac{d}{c}\right).$$

The Groves tax function is

$$t_i(\theta)=\alpha_i\kappa y(\theta)-\sum_{j\neq i}\left[v_j\left(y(\theta)|\theta_j\right)-\alpha_j\kappa y(\theta)\right]+h_i(\theta_{-i}),$$

where h_i is arbitrary. Substituting in the parameterized preferences and using the fact that $(cf(y(\theta))+d)^{1/(r-1)}=\bar{\psi}(\theta)$, this reduces to

$$t_i^*(\theta)=\alpha_i\kappa y^*(\theta)-(I-1)\left[\bar{\psi}_{-i}(\theta_{-i})f\left(y^*(\theta)\right)+\frac{r-1}{rc}\bar{\psi}(\theta)^r\right]+h_i(\theta_{-i}).$$

Plugging in $f(y(\theta))$ gives

$$t_i^*(\theta)=\alpha_i\kappa y^*(\theta)-\frac{I-1}{c}\left[\bar{\psi}_{-i}(\theta_{-i})\left(\bar{\psi}(\theta)^{r-1}-d\right)+\frac{r-1}{r}\bar{\psi}(\theta)^r\right]+h_i(\theta_{-i}).$$

To get balance we need

$$\sum_i h_i(\theta_{-i})=\sum_i(I-1)\left[\bar{\psi}_{-i}(\theta_{-i})f\left(y^*(\theta)\right)+\frac{r-1}{rc}\bar{\psi}(\theta)^r\right].$$

After manipulation, this yields

$$\sum_i h_i(\theta_{-i})=\frac{1}{c}\left[\frac{I-1}{I^{r-1}}\frac{2r-1}{r}\left(\sum_i\psi_i(\theta_i)\right)^r-d\sum_i\sum_{j\neq i}\psi_j\left(\theta_j\right)\right].$$

The goal is to express the right-hand side as sums over i of terms that do not depend on θ_i. To accomplish this, we need to break apart the $\left(\sum_i\psi_i(\theta_i)\right)^r$ term. The following describes the procedure.

For this notational simplicity, assume temporarily that $\psi_i(\theta_i)=\theta_i$ for all i. It is easy to see that

$$\left(\sum_i\theta_i\right)^r=\sum_{i_1}\sum_{i_2}\cdots\sum_{i_r}\theta_{i_1}\theta_{i_2}\ldots\theta_{i_r}. \tag{8}$$

Now we can break that sum into partial sums based on exponent groupings, to get

$$\left(\sum_i\theta_i\right)^r=\sum_{i_1}\theta_{i_1}^r+\sum_{i_1}\sum_{i_2\neq i_1}\theta_{i_1}^{r-1}\theta_{i_2}+\sum_{i_1}\sum_{i_2\neq i_1}\theta_{i_1}^{r-2}\theta_{i_2}^2+\sum_{i_1}\sum_{i_2\neq i_1}\sum_{i_3\notin\{i_k\}_{k<3}}\theta_{i_1}^{r-2}\theta_{i_2}\theta_{i_3}+\cdots.$$

Take any one grouping in this sum, written generically as

$$\sum_{i_1}\sum_{i_2\neq i_1}\cdots\sum_{i_q\notin\{i_k\}_{k<q}}\theta_{i_1}^{t_1}\theta_{i_2}^{t_2}\cdots\theta_{i_q}^{t_q},$$

where $\sum_{k=1}^{q} t_k = r$ and $t_k > t_{k+1}$ for all $k < q$. Now, suppose for each i, we remove all terms containing the i^{th} index, but then sum the entire expression over i. That would be

$$\sum_i\left(\sum_{i_1\neq i}\sum_{i_2\notin\{i,i_k\}_{k<2}}\cdots\sum_{i_q\notin\{i,i_k\}_{k<q}}\theta_{i_1}^{t_1}\theta_{i_2}^{t_2}\cdots\theta_{i_q}^{t_q}\right).\tag{9}$$

There are $(I-1)(I-2)\cdots(I-q)$ terms inside the bracket. But we have I total brackets, so in total this expression counts $I(I-1)\cdots(I-q)$ terms.

If we return to the original sum (8), how many of those terms can be written as $\theta_{i_1}^{t_1}\theta_{i_2}^{t_2}\cdots\theta_{i_q}^{t_q}$? In total, there are

$$I\binom{r}{t_1}(I-1)\binom{r-t_1}{t_2}(I-2)\binom{r-t_1-t_2}{t_3}\cdots(I-q+1)\binom{r-\sum_{k=1}^{q-2}t_k}{t_{q-1}}$$

such terms in the original sum. This reduces to

$$\binom{r}{t_1,t_2,\ldots,t_q}I(I-1)\cdots(I-q+1),$$

where the bracketed term is the multinomial coefficient $r!/(t_1!t_2!\cdots t_q!)$.

Thus, in expression (9) we 'overcounted' terms by a factor of

$$\frac{I(I-1)\cdots(I-q)}{\binom{r}{t_1,t_2,\ldots,t_q}I(I-1)\cdots(I-q+1)}=\frac{(I-q)}{\binom{r}{t_1,t_2,\ldots,t_q}}.$$

To adjust for this, we alter (9) to get

$$\sum_i\frac{\binom{r}{t_1,t_2,\ldots,t_q}}{I-q}\left(\sum_{i_1\neq i}\sum_{i_2\notin\{i,i_k\}_{k<2}}\cdots\sum_{i_q\notin\{i,i_k\}_{k<q}}\theta_{i_1}^{t_1}\theta_{i_2}^{t_2}\cdots\theta_{i_q}^{t_q}\right).\tag{10}$$

Return now to the expression

$$\left(\sum_i\theta_i\right)^r=\sum_{i_1}\theta_{i_1}^r+\sum_{i_1}\sum_{i_2\neq i_1}\theta_{i_1}^{r-1}\theta_{i_2}+\sum_{i_1}\sum_{i_2\neq i_1}\theta_{i_1}^{r-2}\theta_{i_2}^2+\sum_{i_1}\sum_{i_2\neq i_1}\sum_{i_3\notin\{i_k\}_{k<3}}\theta_{i_1}^{r-2}\theta_{i_2}\theta_{i_3}+\cdots.$$

For any r and q, let the set of decreasing sequences of positive integers that sum to r be given by

$$\mathscr{T}_{q,r}=\left\{(t_1,\ldots,t_q)\in\mathbb{N}^q:\sum_{k=1}^{q}t_k=r\ \&\ (\forall k>1)\ t_{k-1}>t_k\right\}.$$

Using (10), we now have

$$\left(\sum_i \theta_i\right)^r = \sum_i \sum_{q=1}^{r} \sum_{\mathcal{T}_{q,r}} \left[\frac{\binom{r}{t_1,t_2,\ldots,t_q}}{I-q} \left(\sum_{i_1 \neq i} \sum_{i_2 \notin \{i,i_k\}_{k<2}} \cdots \sum_{i_q \notin \{i,i_k\}_{k<q}} \theta_{i_1}^{t_1} \theta_{i_2}^{t_2} \cdots \theta_{i_q}^{t_q} \right) \right].$$

Returning to the question of budget balance, we derive that

$$\sum_i h_i(\theta_{-i}) = \sum_i \frac{1}{c} \left\{ \sum_{q=1}^{r} \sum_{\mathcal{T}_{q,r}} \left[\frac{I-1}{I^{r-1}} \frac{2r-1}{r} \frac{\binom{r}{t_1,t_2,\ldots,t_q}}{I-q} \left(\sum_{i_1 \neq i} \cdots \sum_{i_q \notin \{i,i_k\}_{k<q}} \prod_{k=1}^{q} \psi_{i_k}\left(\theta_{i_k}\right)^{t_k} \right) \right] - d \sum_{j \neq i} \psi_j\left(\theta_j\right) \right\},$$

giving the desired expression for each $h_i(\theta_{-i})$.

Proof of Proposition 3

Fix any allocation $((x_i^o)_i, y^o)$ in the range of Γ^G satisfying the Samuelson condition

$$\sum_i MRS_i(x_i^o, y^o) = \kappa,$$

and some message profile $\hat{m}$ such that $\hat{m}_i'(y^o) = MRS_i(x_i^o, y^o)$ for all i. Because $\sum_i \hat{m}_i'(y^o) = \kappa$, the mechanism selects $y^G(\hat{m}) = y^o$. Note that if we add a scalar a_j to each $\hat{m}_j$ then $y^G(\cdot)$ is unaffected, but each $t_i^G(\cdot)$ decreases by $\sum_{j \neq i} a_j$. Define $m_i^a(\cdot) = \hat{m}_i(\cdot) + a_i$ for each i and let $(a_i)_{i=1}^I$ be the unique vector of scalars such that $\omega_i - t_i^G(m^a) = x_i^o$ for all i.[16] Suppose all agents $j \neq i$ announce m_j^a. By the arguments in Lemma 1, agent i's best response will be a message m_i^* satisfying

$$MRS_i(\omega_i - t_i^G(m_i^*, m_{-i}^a), y^G(m_i^*, m_{-i}^a)) = \kappa - \sum_{j \neq i} \frac{dm_j^a(y^G(m_i^*, m_{-i}^a))}{dy}.$$

The message m_i^a satisfies this condition, because $MRS_i(\omega_i - t_i^G(m^a), y^G(m^a)) = MRS_i(x_i^o, y^o)$, $dm_j^a(y^G(m^a))/dy = MRS_j(x_j^o, y^o)$ for all j, and $\sum_i MRS_i(x_i^o, y^o) = \kappa$. This is true for all i, so m^a is a Nash equilibrium.

[16]To find a, let A be the $I \times I$ matrix with zeros on the main diagonal and ones in every other entry. Then $a = x^o \cdot A^{-1}$.

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