Neoclassical Growth Theory I: Exogenous savings rate

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The Classical View

- Thomas Malthus (1798) argued that long run per capita growth in output was impossible, because the population could and would increase faster than output.

- Karl Marx (1867) argued that capitalism provided the means to increase output through capital accumulation.

- Marx also espoused a labor theory of value, which implied subsistence wages for the working class. The value of their output exceeded their wages, and this surplus value was appropriated by the capitalists, who converted it into more capital.

- Capitalists would continue to accumulate capital until its marginal product was zero, thus rendering themselves unnecessary, and utopia would ensue.

Aggregate Production Function

$$Y = F(K, L)$$

- Y — aggregate output (flow)
- K — aggregate capital stock (stock)
- L — labor force (flow)

The Surrogate Production Function

Constant Returns to Scale

$$F(\lambda K, \lambda L) = \lambda F(K, L)$$

Euler's Theorem   If F is differentiable, then CRS is equivalent to

$$F_K(K, L)K + F_L(K, L)L = F(K, L).$$

Per Capita Analysis

Define

$$y = \frac{Y}{L} \qquad k = \frac{K}{L}$$

Then

$$y = \frac{F(K, L)}{L} = F(K/L, \underbrace{L/L}_{=1}) = f(k)$$

Savings and Population Dynamics

$$\dot{K}(t) = sY(t)$$

and

$$\frac{\dot{L}(t)}{L(t)} = n \qquad \text{or} \qquad L(t) = L_0 e^{nt}$$

where s and n are exogenous constants.

We may write K in terms of k as

$$K(t) = k(t)L(t)$$

which implies

$$\dot{K} = \dot{k}L + nkL.$$

But we may also write $\dot{K}$ in terms of Y as

$$\dot{K}(t) = sY(t) = sL(t)f\Big(k(t)\Big)$$

Combining these two expressions for $\dot{K}$ gives

$$(\dot{k} + nk)L = sLf(k)$$

or

$$\dot{k} + nk = sf(k)$$

So

$$\boxed{\dot{k}(t) = sf\Big((k(t)\Big) - nk(t) \qquad (1)}$$

and

$$\dot{y}(t) = f'\Big((k(t)\Big)\dot{k}(t).$$

Example: Leontief Production (Harrod–Domar)

$$F(K,L) = \min\{K,L\}$$

$$f(k) = \begin{cases} k & k \leqslant 1 \\ 1 & k \geqslant 1 \end{cases}$$

Then (1) becomes:

$$\dot{k} = \begin{cases} (s-n)k & k \leqslant 1, \\ s - nk & k \geqslant 1. \end{cases}$$

So assuming $k_0 < 1$ we have $\dot{k} = (s-n)k$. This dfifferential equation has the solution:

$$k(t) = k_0 e^{(s-n)t} \quad \text{as long as } k(t) \leqslant 1. \tag{2}$$

Harrod–Domar Case 1: $\boldsymbol{s > n}$.

In this case k grows over time. Define time τ to be the first time that $k(t) = 1$:

$$k_0 e^{(s-n)\tau} = 1 \qquad \Longleftrightarrow \qquad \tau = \frac{-\ln k_0}{s-n},$$

and for $t \geqslant \tau$, we have $\dot{k} = s - nk$ with $k(\tau) = 1$.

The solution to this differential equation is

$$k(t) = \left(1 - \frac{s}{n}\right) e^{-n(t-\tau)} + \frac{s}{n}, \qquad t \geqslant \tau.$$

This implies that

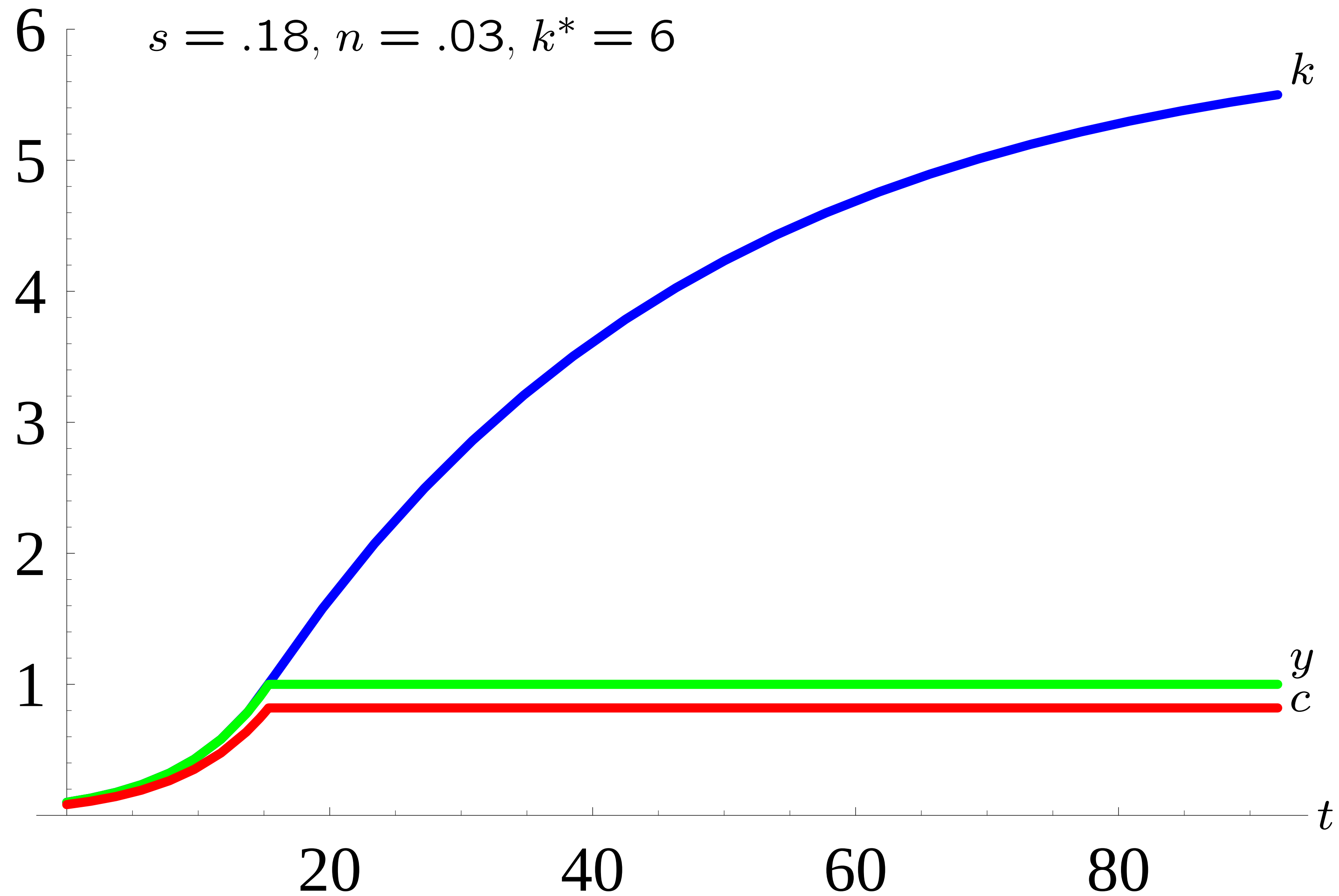
$$k(t) \to k^* = \frac{s}{n} > 1,$$

and

$$y(t) = 1, \quad t \geqslant \tau.$$

In other words, growth stops.

Output in the Harrod–Domar world with $s > n$.



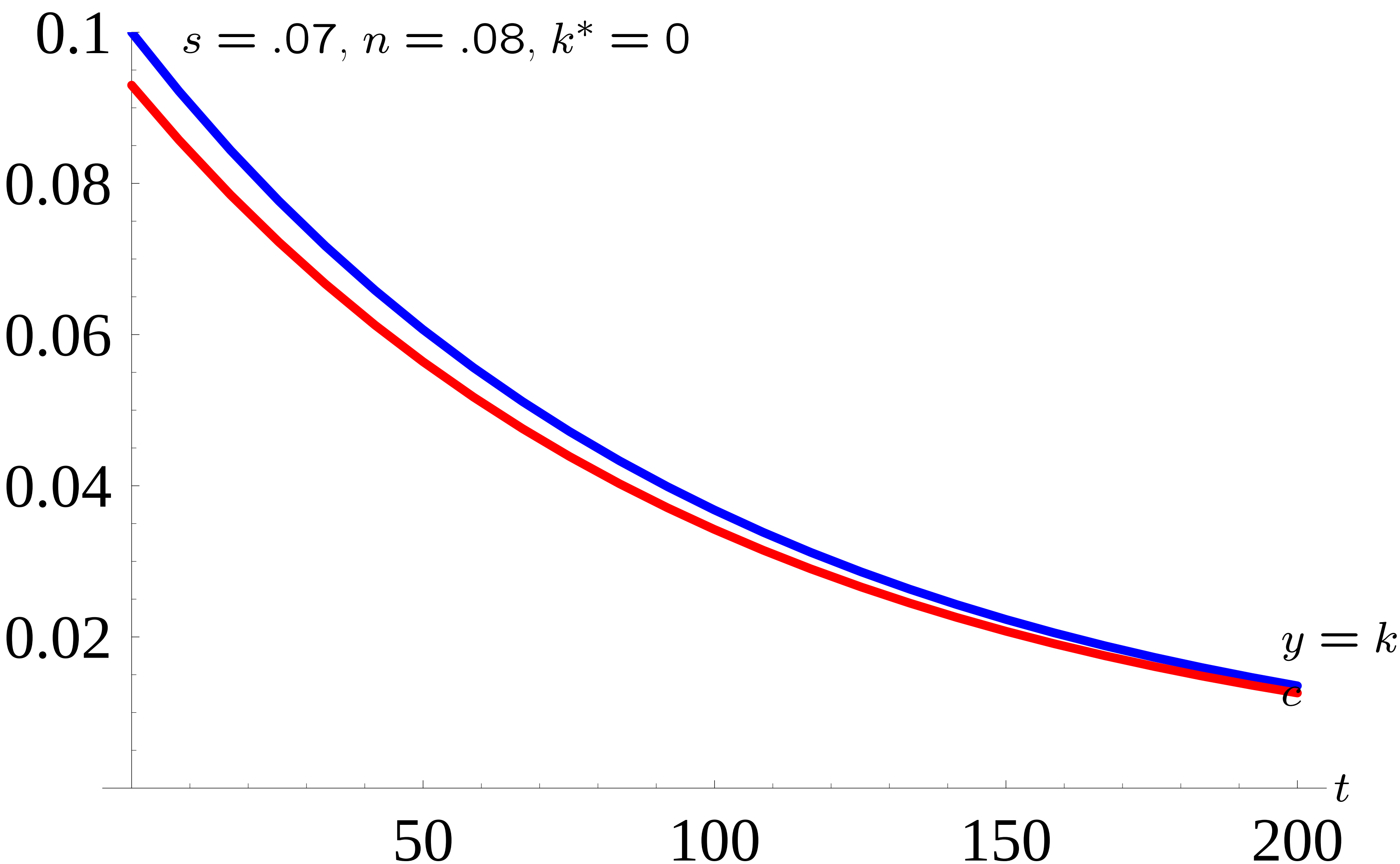
Harrod–Domar Case 2: $\boldsymbol{s < n}$

$k_0 = k(0) < 1$:

$$\dot{k} = sk - nk \qquad \Longleftrightarrow \qquad k(t) = k_0 e^{(s-n)t}$$

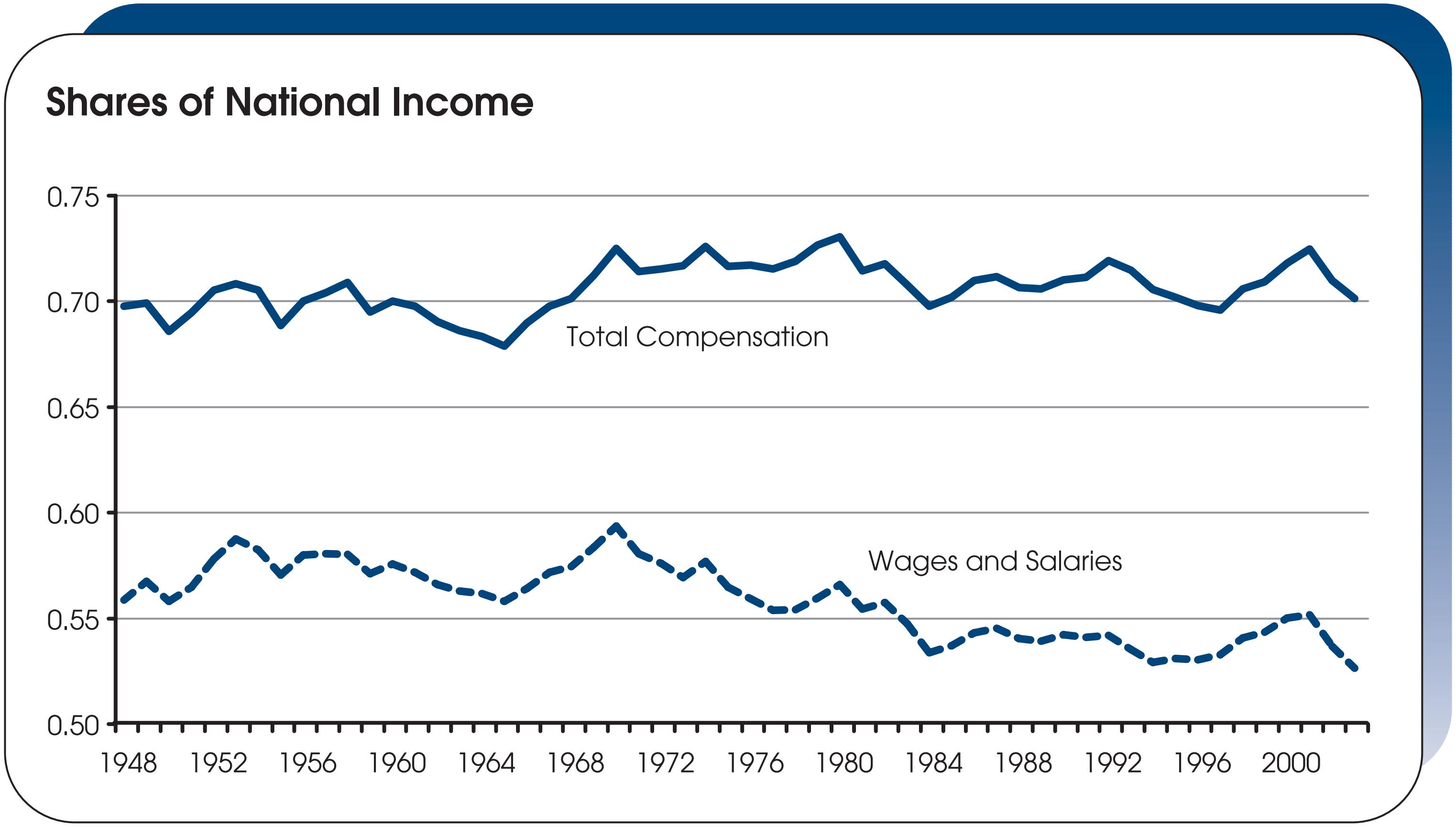
$$\boxed{y(t) = k(t) \to 0}$$

Declining output in the Harrod–Domar world with $s < n$.



Cobb–Douglas Production Function

Constancy of the Share of Labor



Example: Cobb–Douglas Production (Solow)

$$F(K, L) = K^{\alpha} L^{1-\alpha}$$

$$f(k) = k^{\alpha}$$

$$\dot{k} = sk^{\alpha} - nk$$

$$k(t) = \left[\left(k_0^{1-\alpha} - \frac{s}{n} \right) e^{-n(1-\alpha)t} + \frac{s}{n} \right]^{\frac{1}{1-\alpha}}$$

So

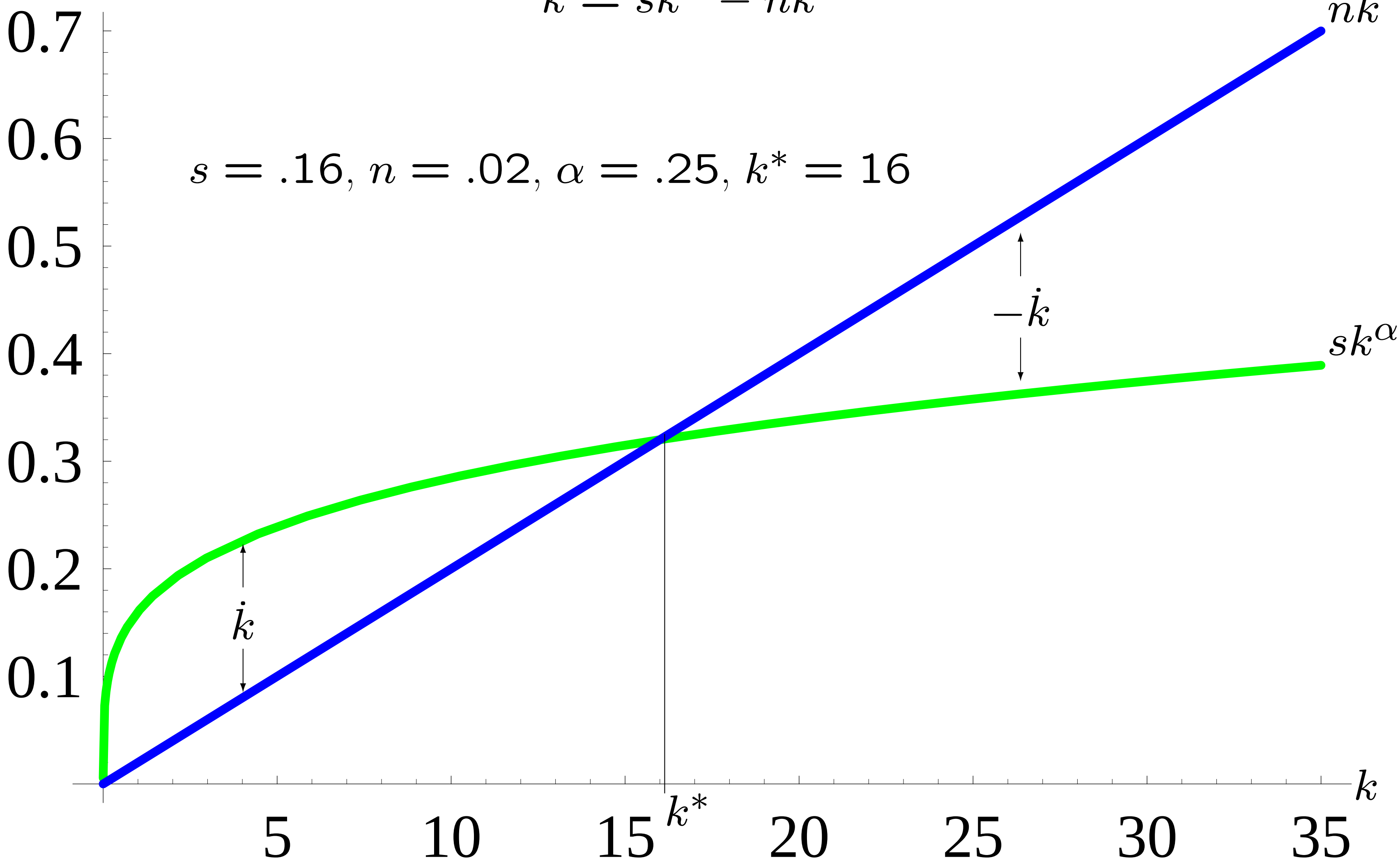
$$
\boxed{
\begin{aligned}
k(t) \rightarrow k^* &= \left(\frac{s}{n}\right)^{\frac{1}{1-\alpha}} \\
y(t) \rightarrow y^* &= \left(\frac{s}{n}\right)^{\frac{\alpha}{1-\alpha}}
\end{aligned}
}
$$

Again growth stops.

This differential equation is more difficult, and we shall come back to it in a moment. But the convergence result is intuitive.

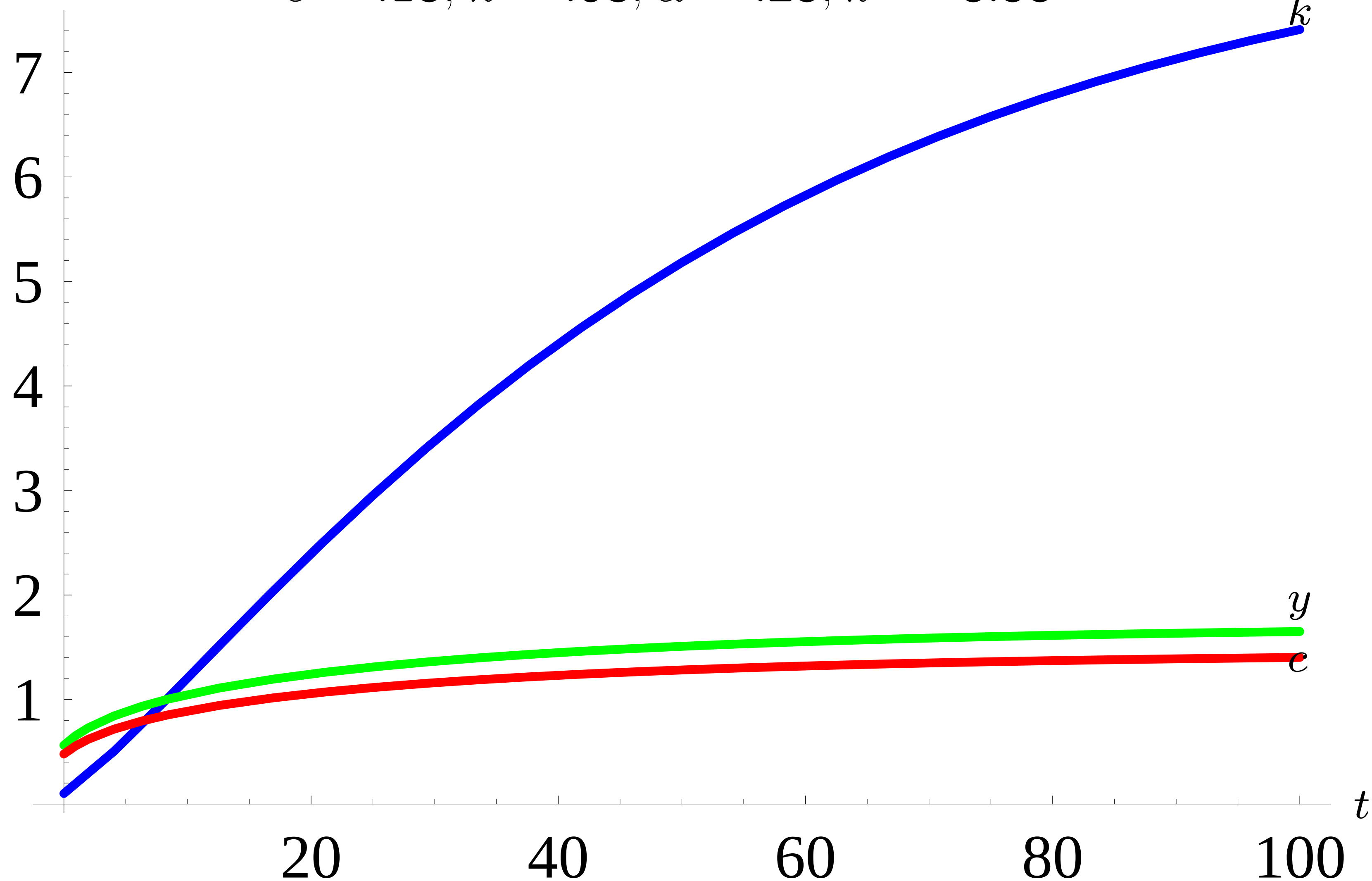
Global stability of k^*.

$$\dot{k} = sk^{\alpha} - nk$$



Long-run per capita variables for a Cobb-Douglas case.

$s = .15$, $n = .03$, $\alpha = .25$, $k^* = 8.55$



Cobb–Douglas Steady states

$$
\begin{aligned}
k^* &= \left(\frac{s}{n}\right)^{\frac{1}{1-\alpha}} \\
y^* &= \left(\frac{s}{n}\right)^{\frac{\alpha}{1-\alpha}} \\
c^* &= (1-s)\left(\frac{s}{n}\right)^{\frac{\alpha}{1-\alpha}}
\end{aligned}
$$

k^*, y^* decrease with n and increase with s.
c^* decreases with n, but is maximized when $s = \alpha$.

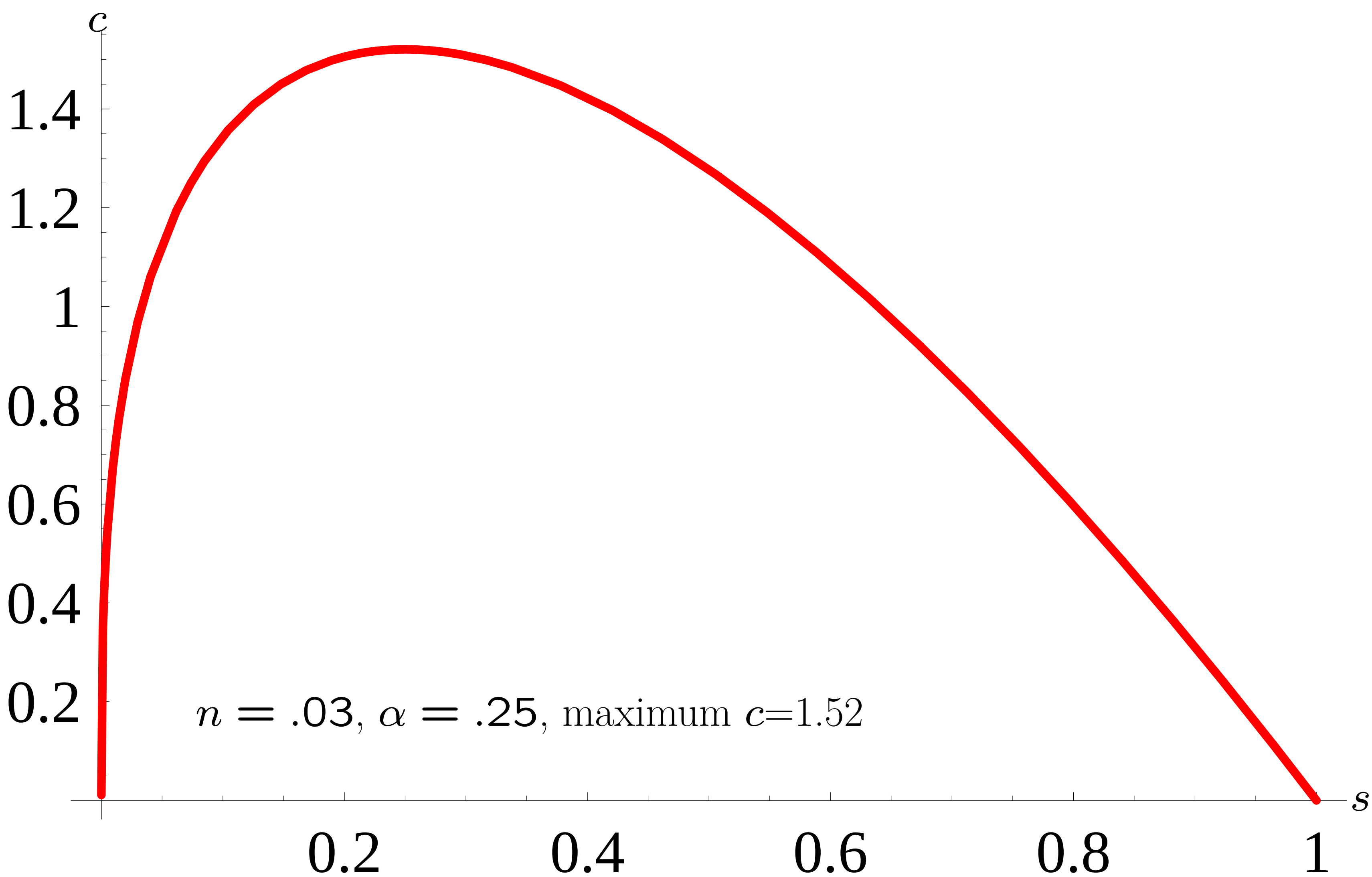
Which savings rate s leads to the highest steady state per capita consumption? First order condition for a maximum of c^* with respect to s:

$$\frac{dc^*}{ds} = -\left(\frac{s}{n}\right)^{\frac{\alpha}{1-\alpha}} + (1-s)\frac{\alpha}{1-\alpha}\frac{1}{n}\left(\frac{s}{n}\right)^{\frac{\alpha}{1-\alpha}-1} = 0.$$

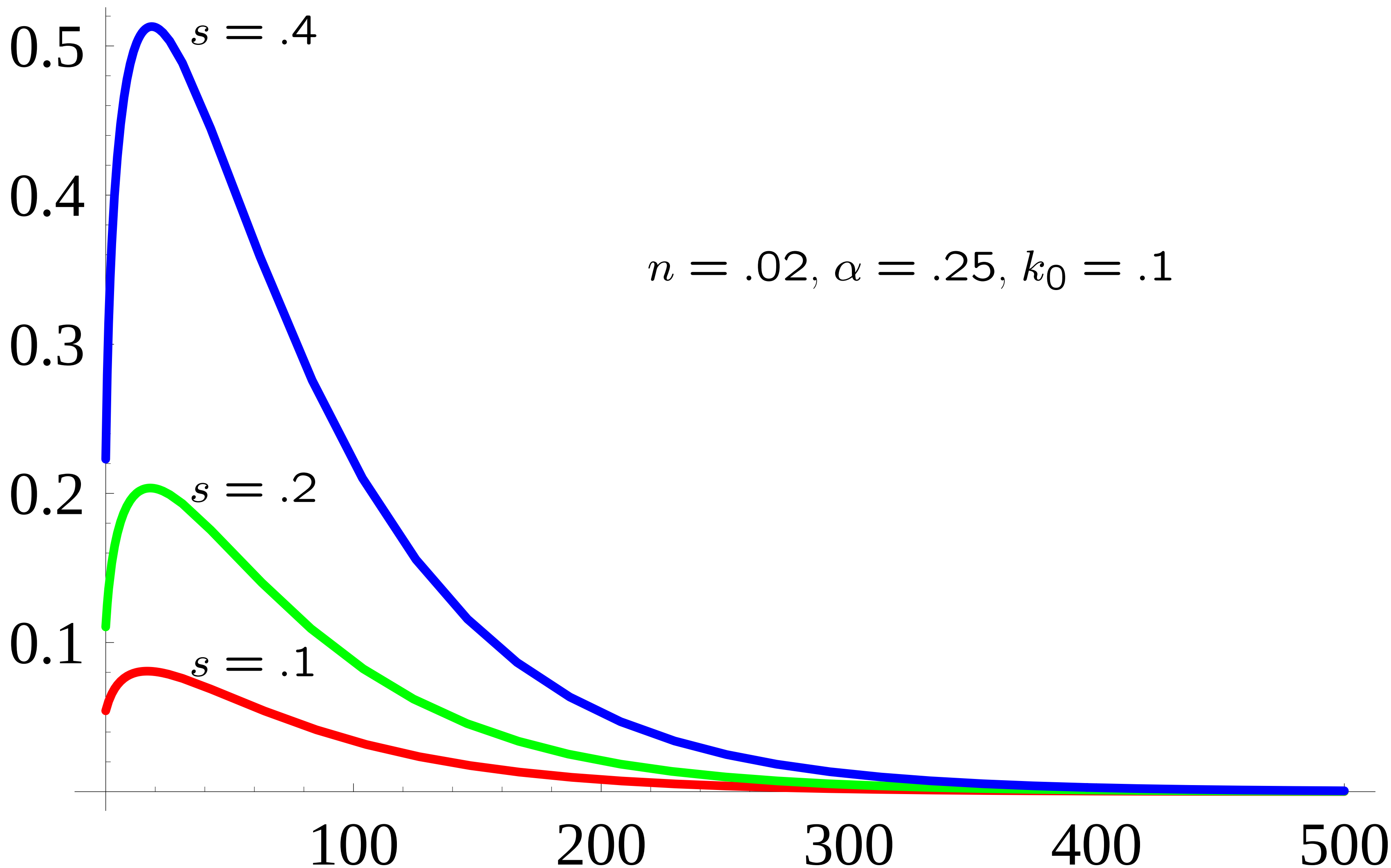
$$\begin{aligned}
\left(\frac{s}{n}\right)^{\frac{\alpha}{1-\alpha}} &= (1-s)\frac{\alpha}{1-\alpha}\frac{1}{n}\left(\frac{s}{n}\right)^{\frac{\alpha}{1-\alpha}-1} \\
1 &= (1-s)\frac{\alpha}{1-\alpha}\frac{1}{s} \\
\frac{s}{1-s} &= \frac{\alpha}{1-\alpha} \\
s &= \alpha.
\end{aligned}$$

This says that in order to maximize the long-run consumption per worker, all the return on capital should be saved and reinvested. This is more or less the way Marx saw capitalists.

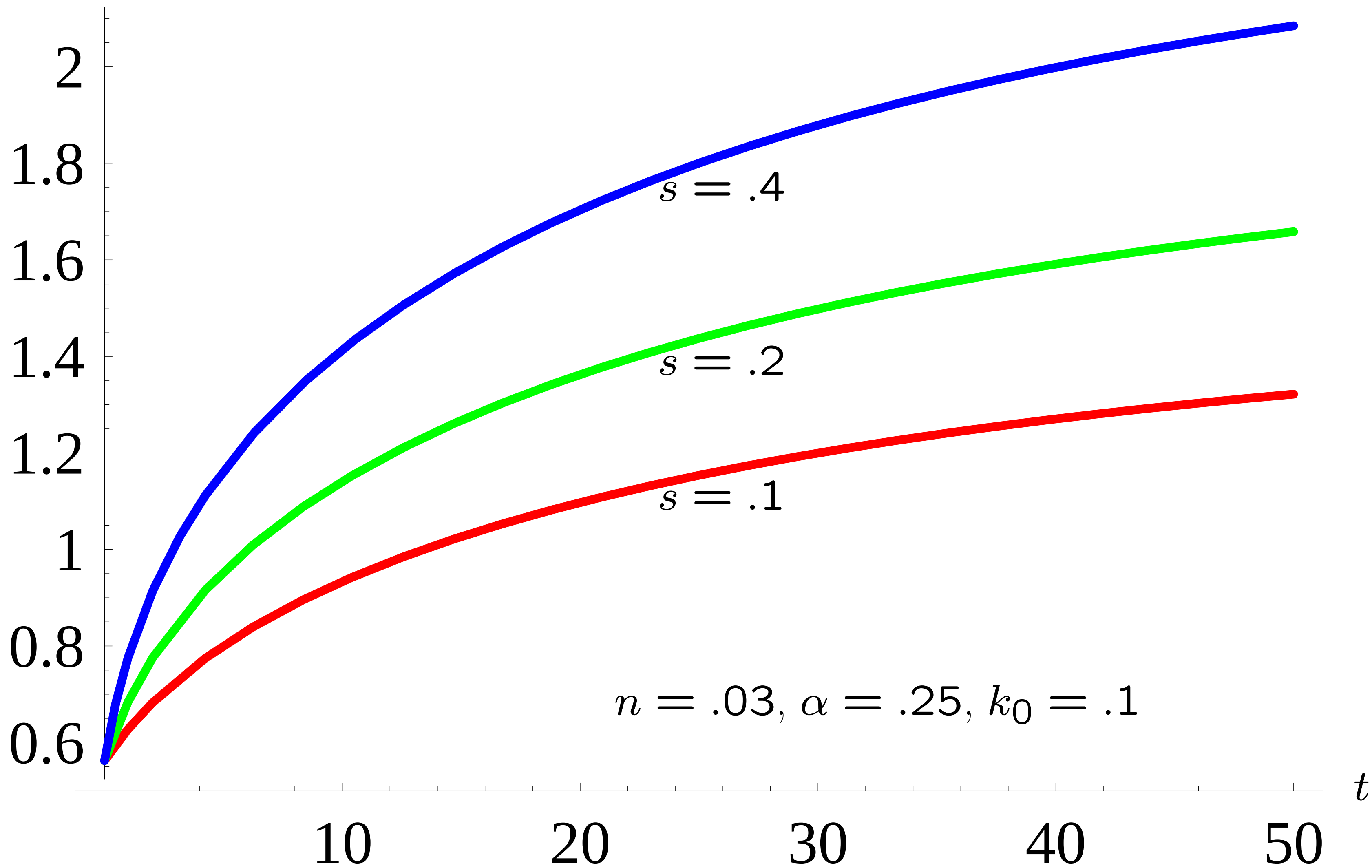
Steady state consumption as a function of the saving rate s.



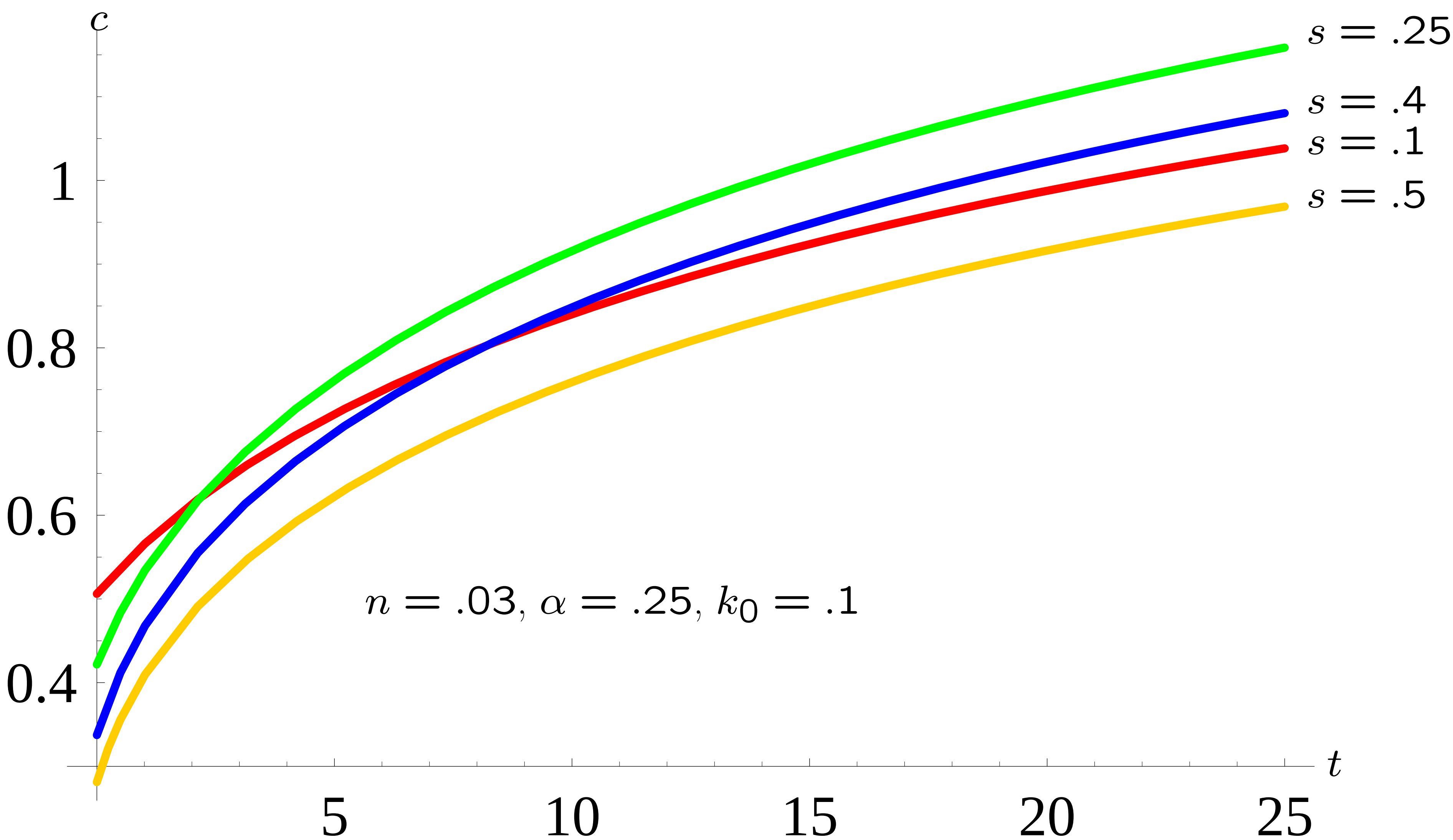
Behavior of $\dot{k}(t) = sk^{\alpha} - nk$.



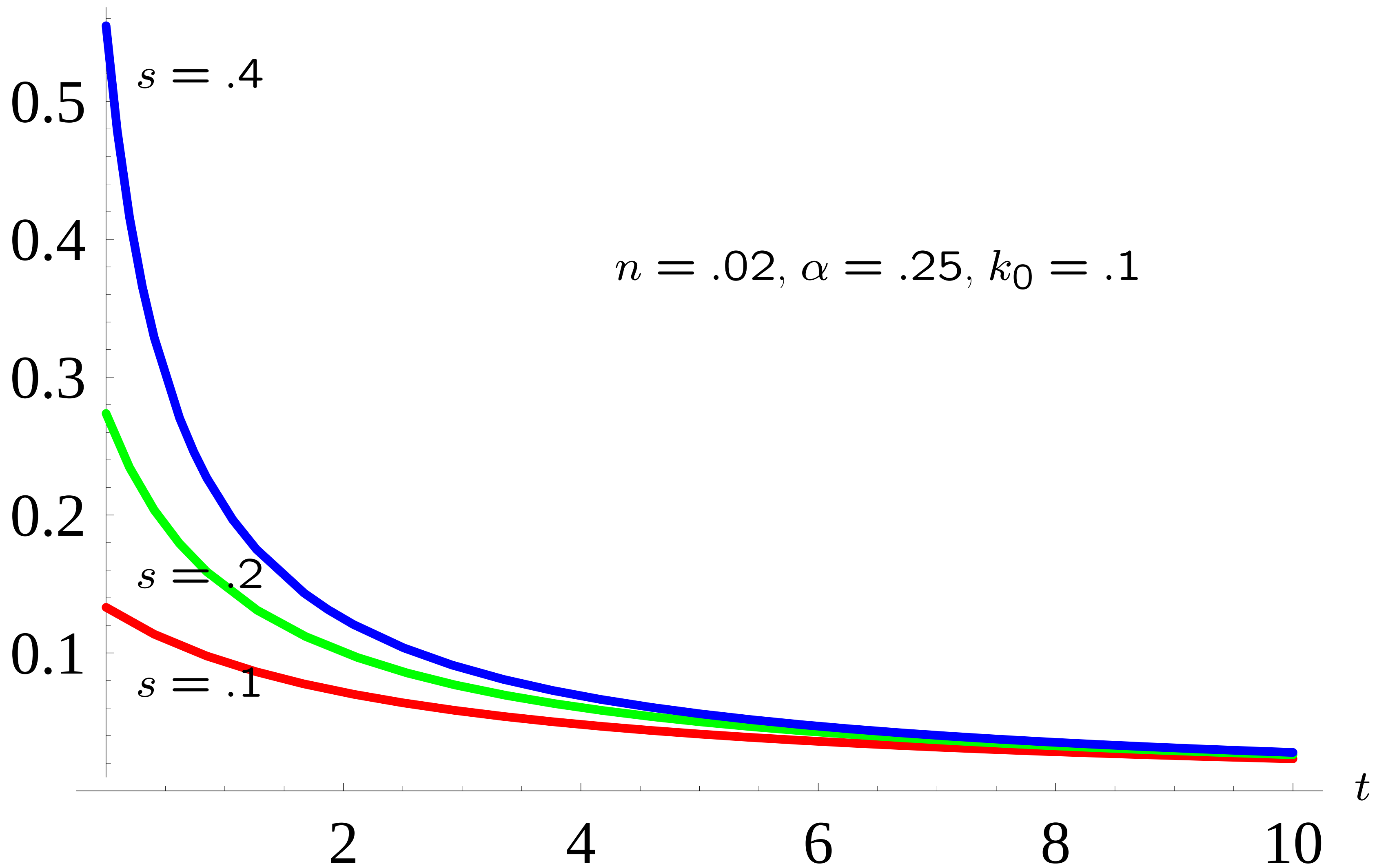
The effect of savings rates on income levels $y(t)$.



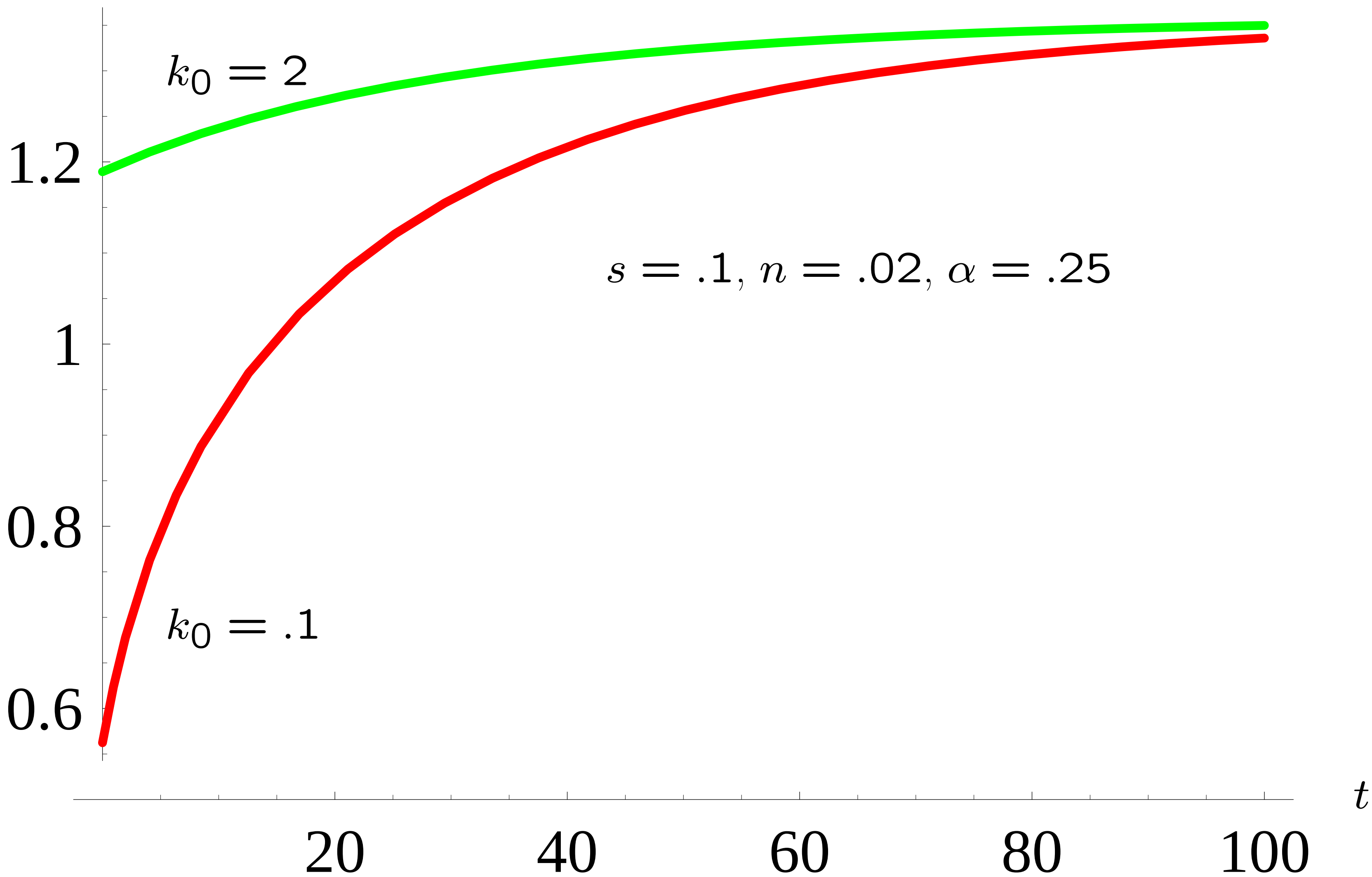
The effect of savings rates on consumption levels $c(t)$.



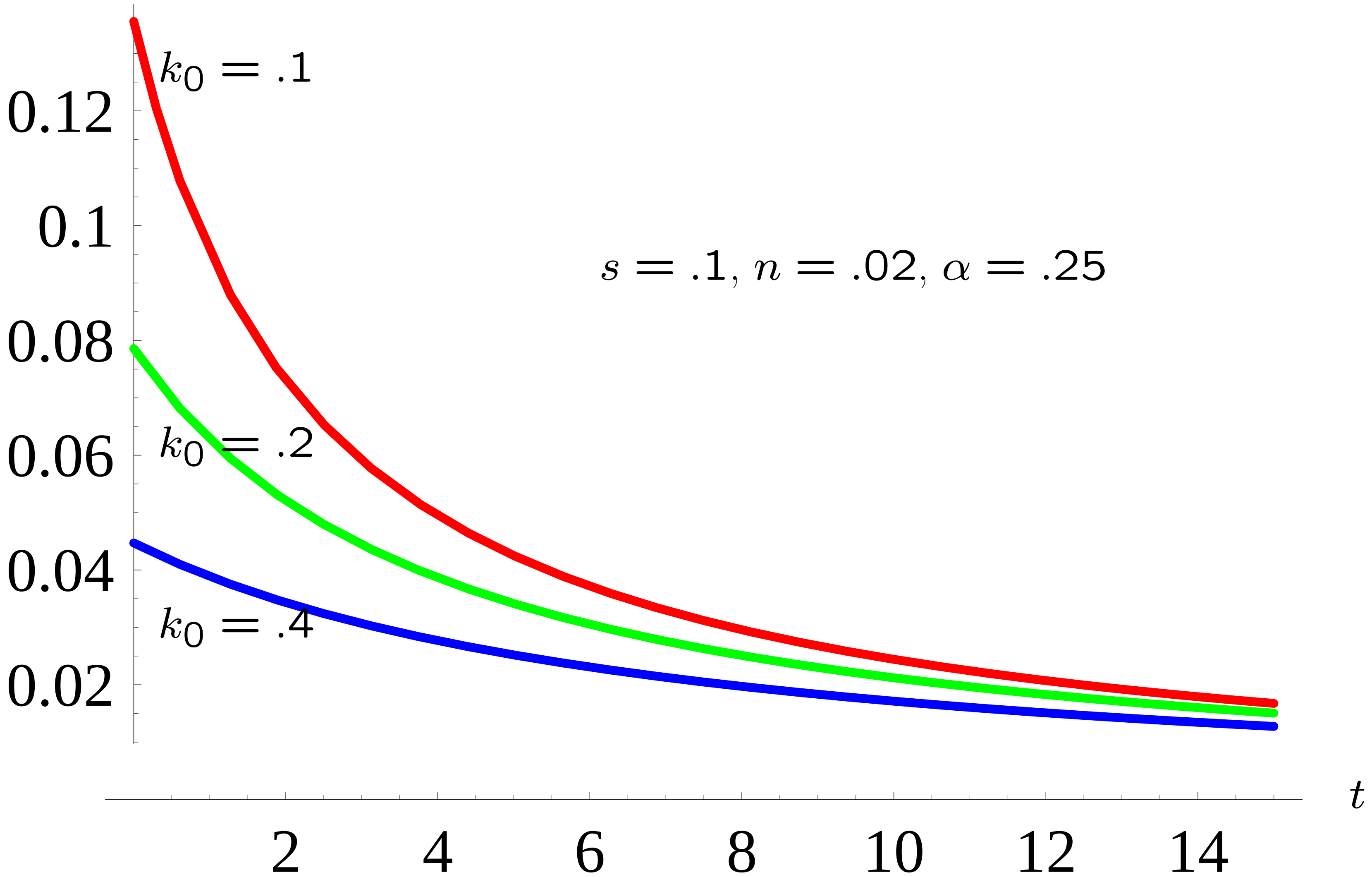
The effect of savings rates on income growth rates $\dot{y}(t)/y(t)$.



The effect of initial capital on income levels $y(t)$.



The effect of initial capital on income growth rates $\dot{y}(t)/y(t)$.



Room to grow

One way to put growth into the model for now is to assume a growing technological progress. Let $h(t)$ be the level of labor-augmenting technology at time t. Assume that

$$Y(t) = F\Big(K(t), h(t)L(t)\Big),$$

and

$$h(t) = e^{ht}.$$

If we define $H(t)$ to be the augmented labor

$$H(t) = h(t)L(t),$$

We can analyze the model as before, replacing L by H, and noting that H grows at the rate $(n+h)$. Defining

$$\kappa(t) = \frac{K(t)}{H(t)} = \frac{k(t)}{h(t)}.$$

Then we see that κ obeys the differential equation

$$\dot{\kappa}(t) = sf\Big(\kappa(t)\Big) - (n + h)\kappa(t).$$

The Cobb–Douglas Case with Augmented Labor

From our previous analysis we know that if

$$Y = F(K, H) = K^{\alpha} L^{1-\alpha},$$

then

$$\kappa(t) = \left[\left(k_0^{1-\alpha} - \frac{s}{n+h}\right) e^{-(n+h)(1-\alpha)t} + \frac{s}{(n+h)}\right]^{\frac{1}{1-\alpha}}$$

Thus $\kappa(t) \to (s/(n+h))^{1/(1-\alpha)}$, and $Y(t)/H(t) \to (s/(n+h))^{\alpha/(1-\alpha)}$. But per capita income

$$y(t) = \frac{Y(t)}{L(t)} = \frac{Y(t)}{H(t)} h(t),$$

grows at the asymptotic rate h. This does not depend on the saving rate r or capital's share α, although these affect the level of $y(t)$, just not its growth rate.

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Appendix on Solving the Differential Equations

Summary

$$\dot{k} = s - nk \qquad (t > \tau,\ k(\tau) = 1)$$

$$k(t) = \left(1 - \frac{s}{n}\right) e^{-n(t-\tau)} + \frac{s}{n}$$

$$\dot{k} = sk^{\alpha} - nk \qquad (k(0) = k_0)$$

$$k(t) = \left[\left(k_0^{1-\alpha} - \frac{s}{n}\right) e^{-n(1-\alpha)t} + \frac{s}{n}\right]^{\frac{1}{1-\alpha}}$$

Theorem 1 (First order linear differential equations) Assume P, Q are continuous on the open interval I. Let $a \in I$, $b \in \boldsymbol{R}$.

Then there is one and only one function $y = f(t)$ that satisfies the initial value problem

$$y' + P(t)y = Q(t)$$

with $f(a) = b$. It is given by

$$f(t) = be^{-A(t)} + e^{-A(t)} \int_a^t Q(x)e^{A(x)}\, dx$$

where

$$A(t) = \int_a^t P(x)\, dx.$$

For a proof see [1, Theorems 8.2 and 8.3, pp. 309–310].

Solution to the linear differential equation $\dot{\boldsymbol{k}} = \boldsymbol{s} - \boldsymbol{n}\boldsymbol{k}$ $(\boldsymbol{t} > \boldsymbol{\tau})$

Apply Theorem 1 with $a = \tau$, $b = 1$, $P(t) = n$, $Q(t) = s$. Then

$$A(t) = \int_\tau^t P(x)\,dx = \int_\tau^t n\,dx = n(t-\tau),$$

$$\int_\tau^t Q(x)e^{A(x)}\,dx = \int_\tau^t se^{n(x-\tau)}\,dx = \frac{s}{n}e^{n(x-\tau)}\Big|_\tau^t = \frac{s}{n}\left(e^{n(t-\tau)} - 1\right)$$

$$\begin{aligned} k(t) &= e^{-n(t-\tau)} + e^{-n(t-\tau)}\frac{s}{n}\left(e^{n(t-\tau)} - 1\right) \\ &= e^{-n(t-\tau)} + \frac{s}{n}\left(1 - e^{-n(t-\tau)}\right) \\ &= \left(1 - \frac{s}{n}\right)e^{-n(t-\tau)} + \frac{s}{n} \end{aligned}$$

Theorem 2 (Separable differential equations) Let y=Y(t) be any solution of the separable differential equation

$$A(y)y' = Q(t), \tag{3}$$

such that Y' is continuous on an open interval I. Assume that both Q and $A \circ Y$ are continuous on I. Let G be any primitive of A, that is, $G' = A$. Then the solution Y satisfies the implicit formula

$$G(y) = \int Q(t)\, dt + C \tag{4}$$

for some constant C. Conversely if y satisfies (4), then y is a solution of (3).

For a proof see [1, Theorem 8.10, pp. 345–346].

Solution to the nonlinear differential equation $\dot{\boldsymbol{k}} = \boldsymbol{sk^{\alpha} - nk}$

Rewrite this as:

$$\frac{\dot{k}}{sk^{\alpha} - nk} = 1.$$

To apply Theorem 2, we need to find a function G satisfying

$$G'(k) = \frac{1}{sk^{\alpha} - nk}.$$

This is not easy. The answer just happens to be

$$G(k) = \frac{\alpha \ln k - \ln(nk - sk^{\alpha})}{n(1 - \alpha)}.$$

I used Mathematica to find the primitive, since I couldn't find it in my table of integrals. To verify:

$$
\begin{aligned}
\frac{d}{dk}\frac{1}{(1-\alpha)n}(\alpha\ln k - \ln(nk - sk^\alpha)) &= \frac{1}{(1-\alpha)n}\left(\frac{\alpha}{k} + \frac{n-\alpha sk^{\alpha-1}}{sk^\alpha - nk}\right) \\
&= \frac{1}{(1-\alpha)n}\,\frac{\alpha(sk^\alpha - nk) + k(n-\alpha sk^{\alpha-1})}{k(sk^\alpha - nk)} \\
&= \frac{1}{(1-\alpha)n}\,\frac{-\alpha nk + kn}{k(sk^\alpha - nk)} \\
&= \frac{1}{(1-\alpha)n}\,\frac{(1-\alpha)nk}{k(sk^\alpha - nk)} \\
&= \frac{1}{sk^\alpha - nk}.
\end{aligned}
$$

Use Theorem 2 to observe that $k(t)$ must satisfy

$$\frac{\alpha \ln k - \ln(nk - sk^{\alpha})}{n(1-\alpha)} = \int 1\, dt + C = t + C.$$

Rearrange to get

$$\alpha \ln k - \ln(nk - sk^{\alpha}) = n(1-\alpha)t + C.$$

Exponentiate to get

$$\frac{k^{\alpha}}{nk - sk^{\alpha}} = Ce^{n(1-\alpha)t}$$

$$k^{\alpha} = C(nk - sk^{\alpha})e^{n(1-\alpha)t}$$

$$(1 + Cse^{n(1-\alpha)t})k^{\alpha} = Cnke^{n(1-\alpha)t}$$

$$\frac{1 + Cse^{n(1-\alpha)t}}{Cne^{n(1-\alpha)t}} = k^{1-\alpha}$$

So

$$Ae^{-n(1-\alpha)t} + \frac{s}{n} = k^{1-\alpha}$$

where $A = \frac{1}{Cn}$. Solve for $k(t)$:

$$\boxed{k(t) = \left(Ae^{-n(1-\alpha)t} + \frac{s}{n}\right)^{\frac{1}{1-\alpha}}}$$

To determine A, evaluate at $t = 0$:

$$k_0 = \left(A + \frac{s}{n}\right)^{\frac{1}{1-\alpha}}$$

so

$$A = k_0^{1-\alpha} - \frac{s}{n}.$$

That is,

$$\boxed{k(t) = \left[\left(k_0^{1-\alpha} - \frac{s}{n}\right) e^{-n(1-\alpha)t} + \frac{s}{n}\right]^{\frac{1}{1-\alpha}}}$$

Q.E.D.