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# The Gauss–Jordan and Simplex Algorithms

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## Contents



*These notes borrow extensively from the lucid expositions by David Gale [8] and Joel Franklin [6].

The simplex algorithm, a modified version of the Gauss–Jordan elimination algorithm, is used to find nonnegative solutions of linear equations. Since all linear (and quadratic) programs can be reduced to this problem, it has proven to be an extremely important tool of applied mathematics. According to George B. Dantzig [3, p. 24], the widely acknowledged originator of the algorithm,

> During the summer of 1947, Leonid Hurwicz, well-known econometrician associated with the Cowles Commission, worked with the author on techniques for solving linear programming problems. This effort and some suggestions of T. C. Koopmans resulted in the "Simplex Method."

# 1 The Gauss–Jordan method of elimination

Consider the following system of equations.

$$\begin{aligned} 3x_1 + 2x_2 &= 8 \\ 2x_1 + 3x_2 &= 7 \end{aligned}$$

The Gauss–Jordan method is a straightforward way to attack problems like this using elementary row operations.

**1 Definition** *The three **elementary row operations** on a matrix are:*

- *Interchange two rows.*
- *Multiply a row by a nonzero scalar.*
- *Add one row to another.*

*It is often useful to combine these into a fourth operation.*

- *Add a nonzero scalar multiple of one row to another row.*

*We shall also refer to this last operation as an elementary row operation.*[1]

You should convince yourself that each of these four operations is reversible using only these four operations, and that none of these operations changes the set of solutions.

The first step in using elementary row operations to solve a system of equations is to write down the so-called augmented coefficient matrix of the system, which is the $2 \times 3$ matrix of just the numbers above:

$$\left[\begin{array}{cc|c} 3 & 2 & 8 \\ 2 & 3 & 7 \end{array}\right]. \tag{1'}$$

We apply elementary row operations until we get a matrix of the form

$$\left[\begin{array}{cc|c} 1 & 0 & a \\ 0 & 1 & b \end{array}\right]$$

which is the augmented matrix of the system

$$\begin{aligned} x_1 &= a \\ x_2 &= b \end{aligned}$$

[1] The operation 'add $\alpha \times$ row $k$ to row $i$' is the following sequence of truly elementary row operations: multiply row $k$ by $\alpha$, add (new) row $k$ to row $i$, multiply row $k$ by $1/\alpha$.

and the system is solved. (If there is no solution, then the elementary row operations cannot produce an identity matrix. There is more to say about this in Section 5.) There is a simple algorithm for deciding which elementary row operations to apply, namely, the **Gauss–Jordan elimination algorithm**.

First we multiply the first row by $\frac{1}{3}$, to get a leading 1:

$$\left[\begin{array}{cc|c} 1 & \frac{2}{3} & \frac{8}{3} \\ 2 & 3 & 7 \end{array}\right]$$

We want to eliminate $x_1$ from the second equation, so we add an appropriate multiple of the first row to the second. In this case the multiple is $-2$, the result is:

$$\left[\begin{array}{cc|c} 1 & \frac{2}{3} & \frac{8}{3} \\ 2-2\cdot 1 & 3-2\cdot\frac{2}{3} & 7-2\cdot\frac{8}{3} \end{array}\right] = \left[\begin{array}{cc|c} 1 & \frac{2}{3} & \frac{8}{3} \\ 0 & \frac{5}{3} & \frac{5}{3} \end{array}\right]. \tag{2'}$$

Now multiply the second row by $\frac{3}{5}$ to get

$$\left[\begin{array}{cc|c} 1 & \frac{2}{3} & \frac{8}{3} \\ 0 & 1 & 1 \end{array}\right].$$

Finally to eliminate $x_2$ from the first row we add $-\frac{2}{3}$ times the second row to the first and get

$$\left[\begin{array}{cc|c} 1-\frac{2}{3}\cdot 0 & \frac{2}{3}-\frac{2}{3}\cdot 1 & \frac{8}{3}-\frac{2}{3}\cdot 1 \\ 0 & 1 & 1 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right], \tag{3'}$$

so the solution is $x_1 = 2$ and $x_2 = 1$.

## 2 A different look at the Gauss–Jordan method

David Gale [8] gives another way to look at what we just did. The problem of finding $x$ to solve

$$\begin{aligned} 3x_1 + 2x_2 &= 8 \\ 2x_1 + 3x_2 &= 7 \end{aligned}$$

can also be thought of as finding a coefficients $x_1$ and $x_2$ to solve the vector equation

$$x_1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \end{bmatrix}.$$

That is, we want to write $b = \begin{bmatrix} 8 \\ 7 \end{bmatrix}$ as a linear combination of $a^1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $a^2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. One way to do this is to begin by writing $b$ as a linear combination of unit coordinate vectors $e^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $e^2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, which is easy:

$$8 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 7 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \end{bmatrix}.$$

We can do likewise for $a^1$ and $a^2$:

$$3\begin{bmatrix}1\\0\end{bmatrix}+2\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}3\\2\end{bmatrix},\qquad 2\begin{bmatrix}1\\0\end{bmatrix}+3\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}2\\3\end{bmatrix}.$$

We can summarize this information in the following ***tableau***,[2] which is fundamental in expositing the simplex algorithm.

| | $a^1$ | $a^2$ | $b$ |
|---|---|---|---|
| $e^1$ | 3 | 2 | 8 |
| $e^2$ | 2 | 3 | 7 |

(1)

There is a column for each of the vectors $a^1$, $a^2$, and $b$. There is a row for each element of the basis $e^1, e^2$. The columns in the *tableau* correspond to the coordinates of the vector labeling the column with respect to the ordered basis given on the left margin. Thus $a^1 = 3e^1 + 2e^2$. $b = 8e^1 + 7e^2$, etc. So far, with the exception of the margins, our *tableau* looks just like the augmented coefficient matrix (1′), as it should.

But we don't really want to express $b$ in terms of $e^1$ and $e^2$, we want to express it in terms of $a^1$ and $a^2$, so we do this in steps. Let us replace $e^1$ in our basis with either $a^1$ or $a^2$. Let's be unimaginative and use $a^1$. The new *tableau* will look something like this:

| | $a^1$ | $a^2$ | $b$ |
|---|---|---|---|
| $a^1$ | ? | ? | ? |
| $e^2$ | ? | ? | ? |

Note that the left marginal column now has $a^1$ in place of $e^1$. We now need to fill in the *tableau* with the proper coefficients. It is clear that $a^1 = 1a^1 + 0e^2$, so we have

| | $a^1$ | $a^2$ | $b$ |
|---|---|---|---|
| $a^1$ | 1 | ? | ? |
| $e^2$ | 0 | ? | ? |

I claim the rest of the coefficients are

| | $a^1$ | $a^2$ | $b$ |
|---|---|---|---|
| $a^1$ | 1 | $\frac{2}{3}$ | $\frac{8}{3}$ |
| $e^2$ | 0 | $\frac{5}{3}$ | $\frac{5}{3}$ |

(2)

That is,

$$a^1 = 1a^1 + 0e^2,\qquad a^2 = \frac{2}{3}a^1 + \frac{5}{3}e^2,\qquad b = \frac{8}{3}a^1 + \frac{5}{3}e^2.$$

or

$$\begin{bmatrix}3\\2\end{bmatrix}=1\begin{bmatrix}3\\2\end{bmatrix}+0\begin{bmatrix}0\\1\end{bmatrix},\qquad \begin{bmatrix}2\\3\end{bmatrix}=\frac{2}{3}\begin{bmatrix}3\\2\end{bmatrix}+\frac{5}{3}\begin{bmatrix}0\\1\end{bmatrix},\qquad \begin{bmatrix}8\\7\end{bmatrix}=\frac{8}{3}\begin{bmatrix}3\\2\end{bmatrix}+\frac{5}{3}\begin{bmatrix}0\\1\end{bmatrix},$$

[2]The term *tableau*, a French word best translated as "picture" or "painting," harkens back to Quesnay's *Tableau économique* [11], which inspired Leontief [10], whose work spurred the Air Force's interest in linear programming [3, p. 17].

which is correct. Now observe that the *tableau* (2) is the same as (2′).

Now we proceed to replace $e^2$ in our basis by $a^1$. The resulting *tableau* is

$$\begin{array}{c||cc|c} & a^1 & a^2 & b \\ \hline a^1 & 1 & 0 & 2 \\ a^2 & 0 & 1 & 1 \end{array} \tag{3}$$

This is the same as (3′). In other words, in terms of our original problem $x_1 = 2$ and $x_2 = 1$.

So far we have done nothing that we would not have done in the standard method of solving linear equations. The only difference is in the description of what we are doing.

> Instead of describing our steps as eliminating variables from equations one by one, we say that we are replacing one basis by another, one vector at a time.

We now formalize this notion more generally.

## 3 The replacement operation

Let $\mathscr{A} = \{a^1, \dots, a^n\}$ be a set of vectors in some vector space, and let $\{b^1, \dots, b^m\}$ span $\mathscr{A}$. That is, each $a^j$ can be written as a linear combination of $b^i$s. Let $T = [t_{i,j}]$ be the $m \times n$ matrix of coordinates of the $a^j$s with respect to the $b^i$s.[3] That is,

$$a^j = \sum_{k=1}^{m} t_{k,j} b^k, \qquad j = 1, \dots, n. \tag{4}$$

We express this as the following *tableau*:

$$\begin{array}{c||ccccc} & a^1 & \dots & a^j & \dots & a^n \\ \hline b^1 & t_{1,1} & \dots & t_{1,j} & \dots & t_{1,n} \\ \vdots & \vdots & & \vdots & & \vdots \\ b^i & t_{i,1} & \dots & t_{i,j} & \dots & t_{i,n} \\ \vdots & \vdots & & \vdots & & \vdots \\ b^m & t_{m,1} & \dots & t_{m,j} & \dots & t_{m,n} \end{array}$$

- It is obvious that interchanging any two rows or interchanging any two columns represents the same information, namely that each vector listed in the top margin is a linear combination of the vectors in the left margin, with the coefficients being displayed in the *tableau*.

- We can rewrite (4) in terms of the coordinates of the vectors as

$$a_i^j = \sum_{k=1}^{m} t_{k,j} b_i^k$$

[3] If the $b^i$s are linearly dependent, $T$ may not be unique.

or perhaps more familiarly as the matrix equation

$$BT = A,$$

where $A$ is the matrix $m \times n$ matrix whose columns are $a^1, \dots, a^n$, $B$ is the matrix $m \times m$ matrix whose columns are $b^1, \dots, b^m$, and $T$ is the $m \times n$ matrix $[t_{i,j}]$.

The usefulness of the *tableau* is the ease with which we can change the basis of a subspace. The next lemma is the key.

**2 Replacement Lemma** *If $\{b^1, \dots, b^m\}$ is a linearly independent set that spans $\mathscr{A}$, then*

$$t_{k,\ell} \neq 0 \quad \text{if and only if} \quad \{b^1, \dots, b^{k-1}, a^\ell, b^{k+1}, \dots, b^m\} \text{ is independent and spans } \mathscr{A}.$$

*Moreover, in the latter case* ***the new* tableau *is derived from the old one by applying elementary row operations that transform the $\ell^{\text{th}}$ column into the $k^{\text{th}}$ unit coordinate vector****. That is, the* tableau

| | $a^1$ | $\dots$ | $a^{\ell-1}$ | $a^\ell$ | $a^{\ell+1}$ | $\dots$ | $a^n$ |
|---|---|---|---|---|---|---|---|
| $b^1$ | $t'_{1,1}$ | $\dots$ | $t'_{1,\ell-1}$ | $0$ | $t'_{1,\ell+1}$ | $\dots$ | $t'_{1,n}$ |
| $\vdots$ | $\vdots$ | | $\vdots$ | $\vdots$ | $\vdots$ | | $\vdots$ |
| $b^{k-1}$ | $t'_{k-1,1}$ | $\dots$ | $t'_{k-1,\ell-1}$ | $0$ | $t'_{k-1,\ell+1}$ | $\dots$ | $t'_{k-1,n}$ |
| $a^\ell$ | $t'_{k,1}$ | $\dots$ | $t'_{k,\ell-1}$ | $1$ | $t'_{k,\ell+1}$ | $\dots$ | $t'_{k,n}$ |
| $b^{k+1}$ | $t'_{k+1,1}$ | $\dots$ | $t'_{k+1,\ell-1}$ | $0$ | $t'_{k+1,\ell+1}$ | $\dots$ | $t'_{k+1,n}$ |
| $\vdots$ | $\vdots$ | | $\vdots$ | $\vdots$ | $\vdots$ | | $\vdots$ |
| $b^m$ | $t'_{m,1}$ | $\dots$ | $t'_{m,\ell-1}$ | $0$ | $t'_{m,\ell+1}$ | $\dots$ | $t'_{m,n}$ |

*is obtained by dividing the $k^{\text{th}}$ row by $t_{k,\ell}$,*

$$t'_{k,j} = \frac{t_{k,j}}{t_{k,\ell}}, \qquad j = 1, \dots, n,$$

*and adding $-\dfrac{t_{i,\ell}}{t_{k,\ell}}$ times row $k$ to row $i$ for $i \neq k$,*

$$t'_{i,j} = t_{i,j} - \frac{t_{i,\ell}}{t_{k,\ell}} t_{k,j} \quad \left(= t_{i,j} - t_{i,\ell} t'_{k,j}\right), \qquad \begin{array}{c} i = 1, \dots, m,\ i \neq k \\ j = 1, \dots, n \end{array}.$$

*Proof*: If $t_{k,\ell} = 0$, then

$$a^\ell = \sum_{i:i \neq k} t_{i,\ell} b^i,$$

or

$$\sum_{i:i \neq k} t_{i,\ell} b^i - 1a^\ell = 0,$$

so $\{b^1, \dots, b^{k-1}, a^\ell, b^{k+1}, \dots, b^m\}$ is dependent.

For the converse, assume $t_{k,\ell} \neq 0$, and that

$$\begin{aligned} 0 &= \alpha a^\ell + \sum_{i:i \neq k} \beta_i b^i \\ &= \alpha \left( \sum_{i=1}^m t_{i,\ell} b^i \right) + \sum_{i:i \neq k} \beta_i b^i \\ &= \alpha t_{k,\ell} b^k + \sum_{i:i \neq k} (\alpha t_{i,\ell} + \beta_i) b^i. \end{aligned}$$

Since $\{b^1,\dots,b^m\}$ is independent by hypothesis, we must have (i) $\alpha t_{k,\ell}=0$ and (ii) $\alpha t_{i,\ell}+\beta_i=0$ for $i\neq k$. Since $t_{k,\ell}\neq 0$, (i) implies that $\alpha=0$. But then (ii) implies that each $\beta_i=0$, too, which shows that the set $\{b^1,\dots,b^{k-1},a^\ell,b^{k+1},\dots,b^m\}$ is linearly independent.

To show that this set spans $\mathscr{A}$, and to verify the *tableau*, we must show that for each $j\neq\ell$,

$$a^j=\sum_{i:i\neq k} t'_{i,j}b^i+t'_{k,j}a^\ell.$$

But the right-hand side is just

$$\begin{aligned}
&=\sum_{i:i\neq k}\underbrace{\left(t_{i,j}-\frac{t_{i,\ell}}{t_{k,\ell}}t_{k,j}\right)}_{t'_{i,j}}b^i+\underbrace{\frac{t_{k,j}}{t_{k,\ell}}}_{t'_{k,j}}\underbrace{\sum_{i=1}^m t_{i,\ell}b^i}_{a^\ell}\\
&=\sum_{i=1}^m t_{i,j}b^i\\
&=a^j,
\end{aligned}$$

which completes the proof. ∎

Thus whenever $t_{k,\ell}\neq 0$, we can replace $b^k$ by $a^\ell$, and get a valid new *tableau*. We call this the **replacement operation** and the entry $t_{k,\ell}$, the **pivot**. Note that one replacement operation is actually $m$ elementary row operations.

Here are some observations.

- If at some point, an entire row of the *tableau* becomes 0, then any replacement operation leaves the row unchanged. This means that the dimension of the span of $\mathscr{A}$ is less than $m$, and that row may be omitted.

- We can use this method to select a basis from $\mathscr{A}$. Replace the standard basis with elements of $\mathscr{A}$ until no additional replacements can be made. By construction, the set $\mathscr{B}$ of elements of $\mathscr{A}$ appearing in the left-hand margin of the *tableau* will constitute a linearly independent set. If no more replacements can be made, then each row $i$ associated with a vector not in $\mathscr{A}$ must have $t_{i,j}=0$, for $j\notin\mathscr{B}$ (otherwise we could make another replacement with $t_{i,j}$ as pivot.) Thus $\mathscr{B}$ must be a basis for $\mathscr{A}$. See the examples in Sections 12.4 and 12.5.

- Note that the elementary row operations used preserve the field to which the coefficients belong. In particular, if the original coefficients belong to the field of rational numbers, the coefficients after a replacement operation also belong to the field of rational numbers.

## 4 More on *tableaux*

An important feature of *tableaux* is given in the following proposition.

**3 Proposition** *Let $b^1,\dots,b^m$ be a basis for $R^{\mathrm{m}}$ and let $a^1,\dots,a^n$ be vectors in $R^{\mathrm{m}}$. Consider the* *following* tableau.

| | $a^1$ | $\dots$ | $a^j$ | $\dots$ | $a^n$ | $e^1$ | | $\dots$ | | $e^m$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $b^1$ | $t_{1,1}$ | $\dots$ | $t_{1,j}$ | $\dots$ | $t_{1,n}$ | $y_{1,1}$ | $\dots$ | $\dots$ | $\dots$ | $y_{1,m}$ |
| $\vdots$ | $\vdots$ | | $\vdots$ | | $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ | | $\vdots$ |
| $b^i$ | $t_{i,1}$ | $\dots$ | $\boxed{t_{i,j}}$ | $\dots$ | $t_{i,n}$ | $y_{i,1}$ | $\dots$ | $\dots$ | $\dots$ | $y_{i,m}$ |
| $\vdots$ | $\vdots$ | | $\vdots$ | | $\vdots$ | $\vdots$ | | $\vdots$ | $\ddots$ | $\vdots$ |
| $b^m$ | $t_{m,1}$ | $\dots$ | $t_{m,j}$ | $\dots$ | $t_{m,n}$ | $y_{m,1}$ | $\dots$ | $\dots$ | $\dots$ | $y_{m,m}$ |

(5)

*That is, for each* $j$*,*

$$a^j = \sum_{i=1}^{m} t_{i,j} b^i \tag{6}$$

*and*

$$e^j = \sum_{i=1}^{m} y_{i,j} b^i. \tag{7}$$

*Let* $y^i$ *be the (row) vector made from the last* $m$ *elements of the* $i^{\text{th}}$ *row. Then*

$$y^i \cdot a^j = t_{i,j}. \tag{8}$$

*Proof*: Let $B$ be the $m \times m$ matrix whose $j^{\text{th}}$ column is $b^j$, let $A$ be the $m \times n$ matrix with column $j$ equal to $a^j$, let $T$ be the $m \times n$ matrix with $(i, j)$ element $t_{i,j}$, and let $Y$ be the $m \times m$ matrix with $(i, j)$ element $y_{i,j}$ (that is, $y^i$ is the $i^{\text{th}}$ row of $Y$). Then (6) is just

$$A = BT$$

where and (7) is just

$$I = BY.$$

Thus $Y = B^{-1}$, so

$$YA = B^{-1}(BT) = (B^{-1}B)T = T,$$

which is equivalent to (8). ∎

## 5 The Fredholm Alternative

**4 Theorem (Fredholm Alternative)** *Exactly one of the two following alternatives holds.*

$$\exists x \quad Ax = b. \tag{9}$$

$$\exists y \quad yA = 0 \text{ and } y \cdot b > 0. \tag{10}$$

*Moreover, if* $A$ *and* $b$ *have all rational entries, then* $x$ *or* $y$ *may be taken to have rational entries.*

*Proof*: We prove the theorem based on the Replacement Lemma 2, and simultaneously compute $x$ or $y$. Let $A$ be $m \times n$ with columns $A^1, \dots, A^n$ in $R^{\text{m}}$. Then $x \in R^{\text{n}}$ and $b \in R^{\text{m}}$. Begin with this *tableau.*

| | $A^1$ | $\dots$ | $A^n$ | $b$ | $e^1$ | $\dots$ | $e^m$ |
|---|---|---|---|---|---|---|---|
| $e^1$ | $\alpha_{1,1}$ | $\dots$ | $\alpha_{1,n}$ | $\beta_1$ | 1 | | 0 |
| $\vdots$ | $\vdots$ | | $\vdots$ | $\vdots$ | | $\ddots$ | |
| $e^m$ | $\alpha_{m,1}$ | $\dots$ | $\alpha_{m,n}$ | $\beta_m$ | 0 | | 1 |

Here $\alpha_{i,j}$ is the $i^{\text{th}}$ row, $j^{\text{th}}$ column element of $A$ and $\beta_i$ is the $i^{\text{th}}$ coordinate of $b$ with respect to the standard ordered basis. Now use the replacement operation to replace as many non-column vectors as possible in the left-hand margin basis. Say that we have replaced $p$ members of the standard basis with columns of $A$. Interchange rows and columns as necessary to bring the *tableau* into this form:

$$
\begin{array}{c||ccc|ccc|c|ccccc}
 & A^{j_1} & \dots & A^{j_p} & A^{j_{p+1}} & \dots & A^{j_n} & b & e^1 & \dots & e^k & \dots & e^m \\
\hline
A^{j_1} & 1 & & 0 & t_{1,p+1} & \dots & t_{1,n} & \xi_1 & y_{1,1} & \dots & y_{1,k} & \dots & y_{1,m} \\
\vdots & & \ddots & & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots \\
A^{j_p} & 0 & & 1 & t_{p,p+1} & \dots & t_{p,n} & \xi_p & y_{p,1} & \dots & y_{p,k} & \dots & y_{p,m} \\
\hline
e^{i_1} & 0 & \dots & 0 & 0 & \dots & 0 & \xi_{p+1} & y_{p+1,1} & \dots & y_{p+1,k} & \dots & y_{p+1,m} \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots \\
e^{i_r} & 0 & \dots & 0 & 0 & \dots & 0 & \boxed{\xi_{p+r}} & y_{p+r,1} & \dots & y_{p+r,k} & \dots & y_{p+r,m} \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots \\
e^{i_{m-p}} & 0 & \dots & 0 & 0 & \dots & 0 & \xi_m & y_{m,1} & \dots & y_{m,k} & \dots & y_{m,m}
\end{array}
$$

The $p \times p$ block in the upper left is an identity matrix, with an $(m-p) \times p$ block of zeroes below it. This comes from the fact that the representation of columns of $A$ in the left-hand margin basis puts coefficient 1 on the basis element and 0 elsewhere. The $(m-p) \times (m-p)$ block to the right is zero since no additional replacements can be made. The middle column indicates that

$$b = \sum_{k=1}^{p} \xi_k A^{j_k} + \sum_{r=1}^{m-p} \xi_{p+r} e^{i_r}.$$

If $\xi_{p+1} = \cdots = \xi_m = 0$ (which must be true if $p = m$), then $b$ is a linear combination only of columns of $A$, so alternative (9) holds, and we have found a solution. (We may have to rearrange the order of the coordinates of $x$.)

The Replacement Lemma 2 guarantees that $A^{j_1}, \dots, A^{j_p}, e^{i_1}, \dots, e^{i_{m-p}}$ is a basis for $R^{\mathrm{m}}$. So if some $\xi_k$ is not zero for $m \geqslant k > p$, then Proposition 3 implies that the corresponding $y^k$ row vector satisfies $y^k \cdot b = \xi_k \neq 0$, and $y^k \cdot A^j = 0$ for all $j$. Multiplying by $-1$ if necessary, $y_k$ satisfies alternative (10).

As for the rationality of $x$ and $y$, if all the elements of $A$ are rational, then all the elements of the original *tableau* are rational, and the results of pivot operation are all rational, so the final *tableau* is rational. ∎

**5 Remark** As an aside, observe that $A^{j_1}, \dots, A^{j_p}$ is a basis for the column space of $A$, and $y^{p+1}, \dots, y^m$ is a basis for its orthogonal complement.

**6 Remark** Another corollary is that if all the columns of $A$ are used in the basis, the matrix $Y$ is the inverse of $A$. This is the well-known result that the Gauss–Jordan method can be used to invert a matrix.

## 6 The simplex method

We are now ready to apply the replacement operation to linear programming. Dantzig [3] draws a distinction between the **simplex *method*** and the **simplex *algorithm***. The simplex method consists of two **phases**, each of which uses the simplex algorithm. The simplex algorithm is a rule for choosing pivots for successive replacement operations until a stopping condition is reached.

For concreteness, consider the following linear programming problem. Let $A$ be an $m \times n$ matrix, let $b$ belong to $R^m$, and $p$ belong to $R^n$. The primal program is:

$$\underset{x \in R^n}{\text{maximize}}\ p \cdot x$$

subject to

$$\begin{aligned} Ax &= b \\ x &\geqq 0 \end{aligned}$$

The dual program is:

$$\underset{y \in R^m}{\text{minimize}}\ b \cdot y$$

subject to

$$yA \geqq p$$

Notice that there are no nonnegativity constraints on $y$.)

A vector $x$ is **feasible** for the primal if $Ax = b$ and $x \geqq 0$, and it is **optimal** if it is feasible and attains the maximum. **Phase 1** of the simplex method uses the simplex algorithm to find a feasible vector, or else proves that none exists. **Phase 2** starts with a feasible vector, and uses the simplex algorithm to find an optimal vector. Paradoxically, Phase 1 uses Phase 2, so we start with that. Phase 1 is covered in Section 8.

## 6.1 The simplex *tableau* and Phase 2

The matrix $A$ is $m \times n$, so each column is a vector in $R^m$. The linear span of the columns is called the **column space** of $A$. The dimension of the column space is called the **rank** of $A$. For the time being, assume:

**7 Assumption (Rank Assumption)** *The column space of the $m \times n$ matrix $A$ has dimension $m$.*

Under the Rank Assumption, every basis for the column space of $A$ has $m$ elements, so we must have $n \geqslant m$. In fact we can find a basis (usually more than one) consisting only of columns of $A$.

Assume that we have somehow found (in Phase 1) a feasible solution $x = (x_1, \dots, x_n) \geqq 0$ of $Ax = b$ that depends on a basis $\{A^{c_1}, \dots, A^{c_m}\}$ of $m$ columns of $A$. That is,

$$b = \sum_{j=1}^{n} x_j A^j = \sum_{i=1}^{m} x_{c_i} A^{c_i},$$

where

$$x_{c_i} \geqslant 0,\ i = 1, \dots, m, \quad \text{and} \quad x_j = 0 \text{ for } j \notin \{c_1, \dots, c_m\}.$$

By the Rank Assumption, every column $A^j$ is a unique linear combination of the basis columns $\{A^{c_1}, \dots, A^{c_m}\}$, say

$$A^j = \sum_{i=1}^{m} t_{i,j} A^{c_i}, \quad j = 1, \dots, n.$$

Given this uniqueness, the basis determines $x$, and so determines $p \cdot x$. Thus:

> The linear programming problem can be thought of as finding the optimal basis out of the columns of $A$. The simplex algorithm is a rule for replacing columns in the basis, one at a time, until the optimal basis is found.

There is a little gap in my argument here. I have not shown that if the LP has an optimum, then I can find an optimum that depends only on a linearly independent set of columns. I should add this to my theory notes, but for now I leave the proof to you, or see Gale [8, Theorem 3.3, p. 84].

### 6.2 Replacement operations on the simplex *tableau*

The idea behind the simplex algorithm is to choose the replacement column so that at each stage $p \cdot x$ increases. In order to do this, we must examine how the *tableau* changes when we change the basis.

Start with the following *tableau*.

| | $A^1$ | $\dots$ | $A^j$ | $\dots$ | $A^n$ | $b$ |
|---|---|---|---|---|---|---|
| $A^{c_1}$ | $t_{1,1}$ | $\dots$ | $t_{1,j}$ | $\dots$ | $t_{1,n}$ | $x_{c_1}$ |
| $\vdots$ | $\vdots$ | | $\vdots$ | | $\vdots$ | $\dots$ |
| $A^{c_k}$ | $t_{k,1}$ | $\dots$ | $t_{k,j}$ | $\dots$ | $t_{k,n}$ | $x_{c_k}$ |
| $\vdots$ | $\vdots$ | | $\vdots$ | | $\vdots$ | $\dots$ |
| $A^{c_m}$ | $t_{m,1}$ | $\dots$ | $t_{m,j}$ | $\dots$ | $t_{m,n}$ | $x_{c_m}$ |

Bear with me while we see what happens when we pivot on $t_{k,\ell}$ in order to replace $A^{c_k}$ by $A^\ell$. This replacement will yield the new *tableau*

| | $A^1$ | $\dots$ | $A^\ell$ | $\dots$ | $A^n$ | $b$ |
|---|---|---|---|---|---|---|
| $A^{c'_1}$ | $t'_{1,1}$ | $\dots$ | $t'_{1,\ell}$ | $\dots$ | $t'_{1,n}$ | $x'_{c'_1}$ |
| $\vdots$ | $\vdots$ | | $\vdots$ | | $\vdots$ | $\dots$ |
| $A^{c'_{k-1}}$ | $t'_{k-1,1}$ | $\dots$ | $t'_{k-1,\ell}$ | $\dots$ | $t'_{k-1,n}$ | $x'_{c'_{k-1}}$ |
| $A^\ell = A^{c'_k}$ | $t'_{k,1}$ | $\dots$ | $t'_{k,\ell}$ | $\dots$ | $t'_{k,n}$ | $x'_\ell = x'_{c'_k}$ |
| $A^{c'_{k+1}}$ | $t'_{k+1,1}$ | $\dots$ | $t'_{k+1,\ell}$ | $\dots$ | $t'_{k+1,n}$ | $x'_{c'_{k+1}}$ |
| $\vdots$ | $\vdots$ | | $\vdots$ | | $\vdots$ | $\dots$ |
| $A^{c'_m}$ | $t'_{m,1}$ | $\dots$ | $t'_{m,\ell}$ | $\dots$ | $t'_{m,n}$ | $x'_{c'_m}$ |

where

$$c'_k = \ell \text{ and } c'_i = c_i \text{ for } i \neq k; \tag{11}$$

the new $\ell^{\text{th}}$ column has $t'_{i,\ell} = 0$ for $i \neq k$, and $t'_{k,\ell} = 1$; the new $k^{\text{th}}$ row has

$$t'_{k,j} = \frac{t_{k,j}}{t_{k,\ell}} \quad j = 1, \dots, n \tag{12}$$

and

$$x'_{c_k} = x'_\ell = \frac{x_{c_k}}{t_{k,\ell}}; \tag{13}$$

the new $i^{\text{th}}$ row for $i \neq k$ has

$$t'_{i,j} = t_{i,j} - \frac{t_{i,\ell}}{t_{k,\ell}} t_{k,j} \quad j = 1, \ldots, n \tag{14}$$

(note that this implies $t'_{i,\ell} = 0$) and

$$x'_{c_i} = x_{c_i} - \frac{t_{i,\ell}}{t_{k,\ell}} x_{c_k}. \tag{15}$$

We can now compute what happens to $p \cdot x$ when $A^{c_k}$ is replaced by $A^{\ell}$. Initially

$$p \cdot x = \sum_{i=1}^{m} p_{c_i} x_{c_i}.$$

After the replacement,

$$\begin{aligned} p \cdot x' &= \sum_{i=1}^{m} p_{c_i} x'_{c_i} \\ &= p_\ell \underbrace{\frac{x_{ck}}{t_{k,\ell}}}_{x'_\ell = x'_{c_k}} + \sum_{\substack{i=1 \\ i \neq k}}^{m} p_{c_i} \underbrace{\left( x_{c_i} - \frac{t_{i,\ell}}{t_{k,\ell}} x_{c_k} \right)}_{x'_{c_i}}. \end{aligned}$$

The difference is

$$\begin{aligned} p \cdot x' - p \cdot x &= p_\ell \frac{x_{ck}}{t_{k,\ell}} - p_{c_k} x_{c_k} - \sum_{\substack{i=1 \\ i \neq k}}^{m} p_{c_i} \frac{t_{i,\ell}}{t_{k,\ell}} x_{c_k} \\ &= \frac{x_{c_k}}{t_{k,\ell}} \left( p_\ell - \sum_{i=1}^{m} p_{c_i} t_{i,\ell} \right) \end{aligned} \tag{16}$$

This suggest the following definition. Given a *tableau*, define

$$\pi_j = \sum_{i=1}^{m} t_{i,j} p_{c_i}, \qquad j = 1, \ldots, n. \tag{17}$$

The interpretation of $\pi_j$ is this: the $j^{\text{th}}$ column $A^j$ is a linear combination $\sum_{i=1}^{m} t_{i,j} A^{c_i}$ of the basis columns $A^{c_1}, \ldots, A^{c_m}$. The value of the linear combination is $\pi_j = \sum_{i=1}^{m} t_{i,j} p_{c_i}$. The value of column $j$ is $p_j$. By (16), we have:

$$\boxed{p \cdot x' > p \cdot x \text{ if and only if } \frac{x_{c_k}}{t_{k,\ell}} > 0 \text{ and } p_\ell > \pi_\ell.}$$

## 6.3 Adding a criterion row

Let us keep track of changes in $p \cdot x$ by adding a **criterion row** to the bottom of the *tableau.*

> The $j^{\text{th}}$ column of the criterion row is $\pi_j - p_j$, for $j = 1, \ldots, n$ and the last column is $p \cdot x = \sum_{i=1}^{m} p_{c_i} x_{c_i}$.

The *tableau* now looks like this.

| | $A^1$ | $\ldots$ | $A^\ell$ | $\ldots$ | $A^n$ | $b$ |
|---|---|---|---|---|---|---|
| $A^{c_1}$ | $t_{1,1}$ | $\ldots$ | $t_{1,\ell}$ | $\ldots$ | $t_{1,n}$ | $x_{c_1}$ |
| $\vdots$ | $\vdots$ | | $\vdots$ | | $\vdots$ | $\ldots$ |
| $A^{c_k}$ | $t_{k,1}$ | $\ldots$ | $t_{k,\ell}$ | $\ldots$ | $t_{k,n}$ | $x_{c_k}$ |
| $\vdots$ | $\vdots$ | | $\vdots$ | | $\vdots$ | $\ldots$ |
| $A^{c_m}$ | $t_{m,1}$ | $\ldots$ | $t_{m,\ell}$ | $\ldots$ | $t_{m,n}$ | $x_{c_m}$ |
| | $\pi_1 - p_1$ | $\ldots$ | $\pi_\ell - p_\ell$ | $\ldots$ | $\pi_n - p_n$ | $p \cdot x$ |

Using (17), after a replacement operation where $A^{c_k}$ is replaced by $A^\ell$, the new criterion row must be computed:

$$\begin{aligned}
\pi'_j - p_j &= \sum_{i=1}^m t'_{i,j} p_{c'_i} - p_j \\
&= \sum_{\substack{i=1 \\ i \neq k}}^m t'_{i,j} p_{c_i} + t'_{k,j} p_\ell - p_j \quad \text{(only } c_k \text{ has changed; } c'_k = \ell) \\
&= \sum_{\substack{i=1 \\ i \neq k}}^m \left( t_{i,j} - \frac{t_{i,\ell}}{t_{k,\ell}} t_{k,j} \right) p_{c_i} + \frac{t_{k,j}}{t_{k,\ell}} p_\ell - p_j \\
&= \sum_{i=1}^m \left( t_{i,j} - \frac{t_{i,\ell}}{t_{k,\ell}} t_{k,j} \right) p_{c_i} - \underbrace{\left( t_{k,j} - \frac{t_{k,\ell}}{t_{k,\ell}} t_{k,j} \right)}_{=0} p_{c_k} + \frac{t_{k,j}}{t_{k,\ell}} p_\ell - p_j \\
&= \underbrace{\sum_{i=1}^m t_{i,j} p_{c_i}}_{\pi_j} - \frac{t_{k,j}}{t_{k,\ell}} \underbrace{\sum_{i=1}^m t_{i,\ell} p_{c_i}}_{\pi_\ell} + \frac{t_{k,j}}{t_{k,\ell}} p_\ell - p_j \\
&= (\pi_j - p_j) - \frac{t_{k,j}}{t_{k,\ell}} (\pi_\ell - p_\ell). 
\end{aligned} \tag{18}$$

Finally, by (16)

$$p \cdot x' = p \cdot x - \frac{x_{c_k}}{t_{k,\ell}} (\pi_\ell - p_\ell). \tag{19}$$

Comparing equations (18) and (19) to (14), we see that:

> *The updated criterion row is also computed from the* tableau *the same way as any other row!*

Equation (19) also explains why we use $\pi - p$ in the criterion row rather than $p - \pi$.

## 6.4 Choosing the pivot

We want to choose the pivot for the replacement operation to do two things:

1. Make $x' \geqq 0$, and

2. Increase $p \cdot x$.

So when is

$$x'_{c'_i} \geqslant 0?$$

We already assumed that $x_{c_i} \geqslant 0$ for each $i = 1, \dots, m$. Thus

$$x'_j = x'_{c'_k} = \frac{x_{c_k}}{t_{k,j}} \geqslant 0 \text{ if and only if the pivot } t_{k,j} > 0.$$

Thus in order to make $x' \geqq 0$, we need to ***choose the pivot so that*** $t_{k,j} > 0$. But that is not all. Having chosen $t_{k,j} > 0$, for $i \neq k$,

$$x'_{c'_i} = x'_{c_i} = x_{c_i} - \frac{t_{i,j}}{t_{k,j}} x_{c_k} \geqslant 0 \iff x_{c_i} \geqslant \frac{t_{i,j}}{t_{k,j}} x_{c_k}.$$

Now if $t_{i,j} \leqslant 0$, the right-hand side is nonpositive, so there is no problem. But if $t_{i,j} > 0$, then

$$x_{c_i} \geqslant \frac{t_{i,j}}{t_{k,j}} x_{c_k} \iff \frac{x_{c_i}}{t_{i,j}} \geqslant \frac{x_{c_k}}{t_{k,j}}.$$

In other words, to make sure each new $x'_{c'_i} \geqslant 0$, we have to ***choose*** $k$ ***so that*** $\frac{x_{c_k}}{t_{k,j}} \leqslant \frac{x_{c_i}}{t_{i,j}}$ ***for all*** $i$ ***such that*** $t_{i,j} > 0$.

Having done this, from (16), it follows that if we ***choose the column*** $j$ ***so that*** $\pi_j - p_j < 0$, then $p \cdot x' \geqslant p \cdot x$ and $p \cdot x' > p \cdot x$ provided $x_{c_k} > 0$.

## 6.5 The simplex algorithm made explicit

Thus the simplex algorithm is this (but there are many variations):

**The naïve simplex algorithm**

**Step 1.** Choose the pivot column $j$ so that

$$\begin{aligned} \pi_j - p_j &< 0 && \text{for maximization} \\ \pi_j - p_j &> 0 && \text{for minimization.} \end{aligned}$$

If more than one $j$ has this property, the choice is not crucial, and should be made for convenience.

**Step 2.** Choose the pivot row $k$ so that

$$t_{k,j} > 0,$$

and

$$\frac{x_{c_k}}{t_{k,j}} \leqslant \frac{x_{c_i}}{t_{i,j}} \text{ for all } i \text{ such that } t_{i,j} > 0.$$

**Step 3.** Perform the replacement operation with pivot $t_{k,j}$ on the *tableau*.

**Step 4.** Repeat Steps 1–3 until a stopping condition is reached. The stopping conditions are: (i) Step 1 cannot be carried out, or (ii) Step 2 cannot be carried out.

If Step 1 cannot be carried out, then the current $x$ is optimal, and $p \cdot x$ (the criterion row entry in the $b$ column) is the optimal value. (See Proposition 10 below.)

If Step 2 cannot be carried out, the problem has no optimum, that is, $p \cdot x$ is unbounded. (See Proposition 11 below.)

This is the algorithm in a nutshell, but there are several remaining issues:

1. How does one get an initial *tableau*?

   This is answered in section 8.

2. Must the algorithm stop?

   The answer is generically yes. But it may cycle and never terminate. This appears not to be common, but there is a simple modification, called the lexicographic simplex algorithm that is guaranteed to stop and not to cycle. This is discussed in section 11.

3. What happens if the algorithm stops?

   This is answered in section 7. Briefly, it stops at an optimum if there is one, or else it stops and gives a proof that no optimum exists.

4. How many steps until it stops?

   This is beyond the scope of these notes, but let me say, for those of you who care, that generically the simplex algorithm stops in polynomial time, but there are exceptional cases.

# 7 The stopping conditions

I first turn to the question of whether the simplex algorithm ever stops. A sufficient condition for stopping is the following.

**8 Assumption (Nondegeneracy)** *The $m \times n$ matrix $A$ has rank $m$, and the vector $b$ cannot be written as a linear combination of fewer than $m$ columns of $A$.*

**9 Proposition** *Under the Nondegeneracy Assumption 8, after each replacement operation in the simplex algorithm, the value $p \cdot x'$ is strictly greater (for a maximization problem) than the previous value $p \cdot x$. Therefore, no basis is repeated. Since there are finitely many bases, the algorithm must stop in a finite number of steps.*

*Proof*: By equations (16–17),

$$p \cdot x' - p \cdot x = \frac{x_{c_k}}{t_{k,j}}(p_j - \pi_j).$$

But we chose $k, j$ so that $\pi_j - p_j < 0$, and $t_{k,j} > 0$. In addition, $x_{c_k} \geqslant 0$ for all $k$. Nondegeneracy implies that in fact $x_{c_k} > 0$ for all $k$. Thus $p \cdot x' > p \cdot x$. ∎

The lexicographic simplex algorithm described in section 11 will always stop, even in the degenerate case, see Gale [8, Chapter 4, section 7, pp. 123–128] or Dantzig [3, pp. 234–235]. The remainder of the section is devoted to examining the two states in which the algorithm can stop.

**10 Proposition (Gale [8, Theorem 4.2, p. 109])** *Under the Rank Assumption 7, if the algorithm reaches a* tableau *with $\pi_j - p_j \geqslant 0$ for all $j = 1, \dots, n$, then $x$ is optimal for a maximization problem; and if the algorithm reaches a* tableau *with $\pi_j - p_j \leqslant 0$ for all $j = 1, \dots, n$, then $x$ is optimal for a minimization problem.*

*Proof*: The proof makes use of the dual program

$$\underset{y \in R^m}{\text{minimize}} \; b \cdot y$$

subject to

$$yA \geqq p$$

Given the *tableau*

| | $A^1$ | $\dots$ | $A^j$ | $\dots$ | $A^n$ | $b$ |
|---|---|---|---|---|---|---|
| $A^{c_1}$ | $t_{1,1}$ | $\dots$ | $t_{1,j}$ | $\dots$ | $t_{1,n}$ | $x_{c_1}$ |
| $\vdots$ | $\vdots$ | | $\vdots$ | | $\vdots$ | $\vdots$ |
| $A^{c_k}$ | $t_{k,1}$ | $\dots$ | $t_{k,j}$ | $\dots$ | $t_{k,n}$ | $x_{c_k}$ |
| $\vdots$ | $\vdots$ | | $\vdots$ | | $\vdots$ | $\vdots$ |
| $A^{c_m}$ | $t_{m,1}$ | $\dots$ | $t_{m,j}$ | $\dots$ | $t_{m,n}$ | $x_{c_m}$ |
| $\pi - p$ | $\pi_1 - p_1$ | $\dots$ | $\pi_j - p_j$ | $\dots$ | $\pi_n - p_n$ | $p \cdot x$ |

we know that $A^{c_1}, \dots, A^{c_m}$ are linearly independent. Therefore the $m$ equations

$$y \cdot A^{c_i} = p_i, \quad i = 1, \dots, m$$

have a solution $y$. For $j \notin \{c_1, \dots, c_m\}$, we have from the *tableau* that

$$A^j = \sum_{i=1}^{m} t_{i,j} A^{c_i}$$

so

$$y \cdot A^j = \sum_{i=1}^m t_{i,j} y \cdot A^{c_i} = \sum_{i=1}^m t_{i,j} p_i = \pi_j \geqslant p_j.$$

That is,

$$yA \geqq p$$

so $y$ is feasible for the dual. (Remember there are no nonnegativity constraints on $y$.)

Now remember that $x$ is given by $x_j = 0$ for $j \notin \{c_1, \ldots, c_m\}$. Thus

$$p \cdot x = \sum_{i=1}^m p_{c_i} x_{c_i} = \sum_{i=1}^m (y \cdot A^{c_i}) x_{c_i} = y \cdot \sum_{i=1}^m A^{c_i} x_{c_i} = y \cdot b,$$

where the last equality comes from the $b$ column of the *tableau*. Now recall that $p \cdot x = b \cdot y$ implies that $x$ is optimal for the primal and $y$ is optimal for the dual. ∎

**11 Proposition** *If the algorithm stops with $\pi_j - p_j < 0$, but $t_{i,j} \leqslant 0$ for all $i = 1, \ldots, m$, then the primal has no optimum. That is, $p \cdot x$ is unbounded.*

*Proof*: From the *tableau*

$$A^j = \sum_{i=1}^m t_{i,j} A^{c_i} \quad \text{and} \quad b = \sum_{i=1}^m x_{c_i} A^{c_i}.$$

Thus for any $\lambda > 0$,

$$b = \sum_{i=1}^m x_{c_i} A^{c_i} + \lambda \left( A^j - \sum_{i=1}^m t_{i,j} A^{c_i} \right) = \lambda A^j + \sum_{i=1}^m (x_{c_i} - \lambda t_{i,j}) A^{c_i}. \tag{20}$$

So define $x'$ by

$$x'_j = \lambda, \; x'_{c_i} = x_{c_i} - \lambda t_{i,j}, \; i = 1, \ldots, m, \text{ and } x_k = 0 \text{ otherwise.}$$

By (20), $Ax' = b$ and $x' \geqq 0$ since each $t_{i,j} \leqslant 0$. But

$$p \cdot x' = \lambda p_j + \sum_{i=1}^m p_{c_i}(x_{c_i} - \lambda t_{i,j}) = \sum_{i=1}^m p_{c_i} x_{c_i} + \lambda (p_j - \sum_{i=1}^m p_{c_i} t_{i,j}) = p \cdot x + \lambda (p_j - \pi_j).$$

Since $\lambda > 0$ is arbitrary and $p_j - \pi_j > 0$, we see that $p \cdot x'$ is unbounded above. Thus no optimum exists. ∎

The following corollary deals minimization problems, where the pivot is chosen to satisfy $\pi_j - p_j > 0$.

**12 Corollary** *If the algorithm stops with $\pi_j - p_j > 0$, but $t_{i,j} \leqslant 0$ for all $i = 1, \ldots, m$, then $p \cdot x$ is unbounded below.*

## 8 Phase 1: Finding a starting point

In order to get started with Phase 2, we need to find a nonnegative $x$ with $Ax = b$.

**Case 1: $b \geqq 0$.**

We can reduce this to an ancillary LP, namely:

$$\underset{z\in R^{m}}{\text{minimize}}\ 1\cdot z$$

subject to

$$\begin{aligned} Ax+z &= b \\ x &\geqq 0 \\ z &\geqq 0 \end{aligned}$$

This LP has one important property—Phase 1 is trivial. Indeed

$$x=0, \quad z=b,$$

is a feasible nonnegative solution. Applying Phase 2 to the ancillary problem solves Phase 1.

**Case 2: $b \ngeqq 0$.**

If $b \ngeqq 0$, setting $z=b$ does not give a nonnegative feasible starting point. But we can fix that as follows. If $b_i<0$, multiply the $i^{\text{th}}$ constraint by $-1$. Then the constraints become

$$DAx+z=Db,$$

where $D$ is the diagonal matrix with $d_{ii}=1$ if $b_i \geqslant 0$ and $d_{ii}=-1$ if $b_i<0$, so that the right-hand side constants satisfy $Db \geqq 0$. We now use the simplex algorithm to solve the ancillary problem

$$\underset{z\in R^{m}}{\text{minimize}}\ 1\cdot z$$

subject to $x \geqq 0$, $z \geqq 0$, and

$$DAx+z=Db.$$

Phase 1 is also trivial for this LP:

$$x=0, \quad z=Db,$$

is a feasible nonnegative solution.

Note that while the solution to the primal remains the same under this transformation, the solution to the dual does not. If $y$ is the solution to the unmodified dual, then $Dy$ is the solution to the modified dual. That is, the solution to the original dual is obtained from the solution to the modified dual by flipping the sign of $y_i$ whenever $b_i<0$.

## 8.1 Infeasibility

Phase 1 consists of the application of the simplex algorithm as described in Phase 2 to this ancillary problem, starting as described above. If the optimum $(\bar{x},\bar{z})$ of the ancillary problem has $\bar{z}=0$, then $\bar{x}$ is feasible for the primal. But if the optimal $\bar{z}\neq 0$ then the primal has no feasible solution.

Note that if all we want to do is find some solution to a system of inequalities, we can stop at the end of Phase 1.

### 8.2 Inequality constraints

Often linear programs are not given with equality constraints, but with inequality constraints, typically like this:

$$\underset{x \in R^n}{\text{maximize}} \ p \cdot x$$

subject to

$$\begin{aligned} Ax &\leqq b \\ x &\geqq 0 \end{aligned}$$

***For some kinds of inequality constraints, Phase 1 is trivial.*** If all $m$ constraints are inequality constraints, introduce slack variables $z_1, \ldots, z_m \geqslant 0$. Let $A_i$ denote the $i^{\text{th}}$ row of $A$. There are four cases, depending on the sense of the inequality and the sign of $b_i$.

$$\begin{array}{llllll} \text{Replace} & A_i \cdot x \leqslant b_i & \text{where} & b_i \geqslant 0 & \text{with} & A_i \cdot x + z_i = b_i. \\ \text{Replace} & A_i \cdot x \geqslant -b_i & \text{where} & b_i \geqslant 0 & \text{with} & A_i \cdot x - z_i = -b_i. \end{array}$$

Then an initial feasible solution is given by

$$x = 0, \quad z = b.$$

On the other hand, if we have one of these cases, then Phase 1 is non trivial, and we have to introduce auxiliary variables $u$:

$$\begin{array}{llllll} \text{Replace} & A_i \cdot x \leqslant -b_i & \text{where} & b_i \geqslant 0 & \text{with} & -A_i \cdot x - z_i + u_i = b_i. \\ \text{Replace} & A_i \cdot x \geqslant b_i & \text{where} & b_i \geqslant 0 & \text{with} & A_i \cdot x - z_i + u_i = b_i. \end{array}$$

Then an initial feasible solution is given by

$$x = 0, \quad z = 0, \quad u = b$$

but now we must minimize $1 \cdot u$ in order to find a feasible solution of the original problem, where $u = 0$.

## 9 A worked example

The first example illustrates how a problem involving inequalities can combine Phases 1 and 2.

$$\text{maximize} \quad 2x_1 + 4x_2 + x_3 + x_4$$

subject to $x_1 \geqslant 0, \ldots, x_4 \geqslant 0$, and

$$\begin{array}{ccccccccc} 2x_1 & + & x_2 & & & & & \leqslant & 3 \\ & & x_2 & + & 4x_3 & + & x_4 & \leqslant & 3 \\ x_1 & + & 3x_2 & & & + & x_4 & \leqslant & 4 \end{array}$$

To convert this to a problem with equalities, introduce three slack variables $z_1$, $z_2$, and $z_3$, and write the problem as

$$\text{maximize} \quad 2x_1 + 4x_2 + x_3 + x_4 + 0z_1 + 0z_2 + 0z_3$$

subject to $x_1 \geqslant 0, \ldots, x_4 \geqslant 0$, $z_1, z_2, z_3 \geqslant 0$, and

$$\begin{array}{ccccccccccccccc}
2x_1 & + & x_2 & & & & & + & z_1 & & & & & = & 3 \\
& & x_2 & + & 4x_3 & + & x_4 & & & + & z_2 & & & = & 3 \\
x_1 & + & 3x_2 & & & + & x_4 & & & & & + & z_3 & = & 4
\end{array}$$

Since the right-hand side is already nonnegative there is no need to multiply any rows by $-1$. Moreover, the right-hand side provides a ready made feasible vector: $x_1 = \cdots = x_4 = 0$, $z_1 = 3$, $z_2 = 3$, $z_3 = 4$. The columns corresponding to these three slack variables are simply the three unit coordinate vectors. This makes it especially easy to create a starting *tableau* with these three vectors as the basis in the left-hand margin. But since the three slack variables are in a sense artificial, it is customary to segregate the columns corresponding to them. Finally, note that by introducing three new variables, we must extend the $p$ vector to include three zero components. This makes the computation of the criterion row especially easy. Here then is the initial *tableau*.

| $p_{c_i}$ | | $a^1$ | $a^2$ | $a^3$ | $a^4$ | $e^1$ | $e^2$ | $e^3$ | $b$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| Initial *tableau* | | | | | | | | | | |
| 0 | $e^1$ | 2 | 1 | 0 | 0 | 1 | 0 | 0 | 3 | 3 |
| 0 | $e^2$ | 0 | 1 | 4 | 1 | 0 | 1 | 0 | 3 | 3 |
| 0 | $e^3$ | 1 | $\boxed{3}$ | 0 | 1 | 0 | 0 | 1 | 4 | $1\frac{1}{3}$ |
| | | $-2$ | $-4$ | $-1$ | $-1$ | 0 | 0 | 0 | 0 | |

Notice that the *tableau* is obtained by filling the matrix inequality with an identity matrix to the right. The criterion row $\pi - p$ is just $-p$, as everything is expressed as a linear combination of $e^1, e^2, e^3$, which have zero prices associated with them. To help you keep track, I have placed the "prices" $p_{c_i}$ associated with each row in the far left margin.

Since we are maximizing, we look for a criterion row entry that is strictly negative. We may as well choose the most negative column, but that is not essential. It corresponds to the column $a^2$. Now to choose the row, look at the ratios of the $b$ column entries (the current $x, z$) to the positive $a^2$ entries, and choose the smallest ratio. For convenience I have put these ratios in the right-hand margin. In this case the smallest is $1\frac{1}{3} < 3$. Thus we want to replace $e^3$ by $a^2$, as is indicated by the rectangle around the pivot above.

The new *tableau* is given below, and the next pivot is indicated.

| $p_{c_i}$ | | $a^1$ | $a^2$ | $a^3$ | $a^4$ | $e^1$ | $e^2$ | $e^3$ | $b$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| Replace $e^3$ by $a^2$: | | | | | | | | | | |
| 0 | $e^1$ | $1\frac{2}{3}$ | 0 | 0 | $-\frac{1}{3}$ | 1 | 0 | $-\frac{1}{3}$ | $1\frac{2}{3}$ | |
| 0 | $e^2$ | $-\frac{1}{3}$ | 0 | $\boxed{4}$ | $\frac{2}{3}$ | 0 | 1 | $-\frac{1}{3}$ | $1\frac{2}{3}$ | $\frac{5}{12}$ |
| 4 | $a^2$ | $\frac{1}{3}$ | 1 | 0 | $\frac{1}{3}$ | 0 | 0 | $\frac{1}{3}$ | $1\frac{1}{3}$ | |
| | | $-\frac{2}{3}$ | 0 | $-1$ | $\frac{1}{3}$ | 0 | 0 | $1\frac{1}{3}$ | $5\frac{1}{3}$ | |
| Replace $e^2$ by $a^3$: | | | | | | | | | | |
| 0 | $e^1$ | $\boxed{1\frac{2}{3}}$ | 0 | 0 | $-\frac{1}{3}$ | 1 | 0 | $-\frac{1}{3}$ | $1\frac{2}{3}$ | 1 |
| 1 | $a^3$ | $-\frac{1}{12}$ | 0 | 1 | $\frac{1}{6}$ | 0 | $\frac{1}{4}$ | $-\frac{1}{12}$ | $\frac{5}{12}$ | |
| 4 | $a^2$ | $\frac{1}{3}$ | 1 | 0 | $\frac{1}{3}$ | 0 | 0 | $\frac{1}{3}$ | $1\frac{1}{3}$ | 4 |
| | | $-\frac{3}{4}$ | 0 | 0 | $\frac{1}{2}$ | 0 | $\frac{1}{4}$ | $1\frac{1}{4}$ | $5\frac{3}{4}$ | |
| Replace $e^1$ by $a^1$: | | | | | | | | | | |
| 2 | $a^1$ | 1 | 0 | 0 | $-\frac{1}{5}$ | $\frac{3}{5}$ | 0 | $-\frac{1}{5}$ | 1 | |
| 1 | $a^3$ | 0 | 0 | 1 | $\frac{3}{20}$ | $\frac{1}{20}$ | $\frac{1}{4}$ | $-\frac{1}{10}$ | $\frac{1}{2}$ | |
| 4 | $a^2$ | 0 | 1 | 0 | $\frac{2}{5}$ | $-\frac{1}{5}$ | 0 | $\frac{2}{5}$ | 1 | |
| | | 0 | 0 | 0 | $\frac{7}{20}$ | $\frac{9}{20}$ | $\frac{1}{4}$ | $1\frac{1}{10}$ | $6\frac{1}{2}$ | |

The algorithm stops here because the criterion row has no more negative entries. Note that we have replaced all the unit coordinate vectors by columns of $A$.

**Warning!** The solution can now be read off from column $b$, but remember what those numbers are—the coefficients on the corresponding left-hand basis element, and that basis is in no particular order, so read them with care! If the basis element in the left-hand column of row $i$ is $a^c$, then the right-hand column value (under $b$) is $x_c$, the $c^{\text{th}}$ coordinate of $\bar{x}$, not $\bar{x}_i$, the $i^{\text{th}}$ coordinate! If the basis element in the left-hand column of row $i$ is $e^c$, then the right-hand column value (under $b$) is $z_c$, the $c^{\text{th}}$ coordinate of $\bar{z}$, a slack variable.

The solution we have found is

$$\bar{x}_1 = 1,\ \bar{x}_2 = 1,\ \bar{x}_3 = \tfrac{1}{2},\ \bar{x}_4 = 0,$$

and the value $p \cdot \bar{x}$ is $6\frac{1}{2}$.

Let me just verify that this satisfies the constraints:

$$\begin{array}{rcccl} 2(1)+1(1)+0(\frac{1}{2})+0(0) & = & 2+1+0+0 & = & 3 \\ 0(1)+1(1)+4(\frac{1}{2})+1(0) & = & 0+1+2+0 & = & 3 \\ 1(1)+3(1)+0(\frac{1}{2})+1(0) & = & 1+3+0+0 & = & 4 \end{array}$$

Now you either have to redo these calculations yourself or put your faith in the computer program that I wrote to produce these *tableau*x. I don't recommend the latter, as I am a notoriously poor programmer. But you don't need to do the former either. Remember that I told you that it is enough to find a solution to the dual that yields the same value. And here is the surprise I have been saving:

***The criterion row entries under the unit vectors comprise a solution to the dual program.***

That is,

$$\bar{y}_1 = \frac{9}{20}, \quad \bar{y}_2 = \frac{1}{4}, \quad \bar{y}_3 = 1\frac{1}{10},$$

solves the dual problem, which is

$$\text{minimize} \quad 3y_1 + 3y_2 + 4y_3$$

subject to

$$\begin{array}{rcrcrcl}
2y_1 & & & + & y_3 & \geqslant & 2 \\
y_1 & + & y_2 & + & 3y_3 & \geqslant & 4 \\
 & + & 4y_2 & & & \geqslant & 1 \\
 & & y_2 & + & y_3 & \geqslant & 1
\end{array}$$

Now it is easy to verify that

$$b \cdot \bar{y} = 3(\tfrac{9}{20}) + 3(\tfrac{1}{4}) + 4(1\tfrac{1}{10}) = 1\tfrac{7}{20} + \tfrac{3}{4} + 4\tfrac{2}{5} = 6\tfrac{1}{2}.$$

has the same value as primal, and I leave it to you to verify the feasibility. But I can tell you right now that the first three inequalities will be satisfied as equalities (since the dual variables $\bar{x}_1, \bar{x}_2, \bar{x}_3$ are strictly positive), and the fourth inequality is likely strict (as $\bar{x}_4 = 0$).

I changed my mind. Here is the verification that $\bar{y}$ is feasible for the dual:

$$\begin{array}{rcrcrcl}
2(\frac{9}{20}) + 0(\frac{1}{4}) + 1(1\frac{1}{10}) & = & \frac{9}{10} + 0 + 1\frac{1}{10} & = & 2 & = & 2 \\
1(\frac{9}{20}) + 1(\frac{1}{4}) + 3(1\frac{1}{10}) & = & \frac{9}{20} + \frac{1}{4} + 3\frac{3}{10} & = & 4 & = & 4 \\
0(\frac{9}{20}) + 4(\frac{1}{4}) + 0(1\frac{1}{10}) & = & 0 + 1 + \ \ 0 & = & 1 & = & 1 \\
0(\frac{9}{20}) + 1(\frac{1}{4}) + 1(1\frac{1}{10}) & = & 0 + \frac{1}{4} + 1\frac{1}{10} & = & 1\frac{7}{20} & > & 1
\end{array}$$

Now either this is an incredibly contrived example, or there is something magical I haven't yet told you about the simplex algorithm. It's the latter.

# 10 The simplex algorithm solves the dual program too

The simplex algorithm applied to the following sort of problem also computes a solution to the dual program.

$$\underset{x \in R^n}{\text{maximize}} \ p \cdot x$$

subject to

$$\begin{array}{rcl}
Ax & = & b \\
x & \geqq & 0
\end{array}$$

The dual program is

$$\underset{y \in R^m}{\text{minimize}} \ b \cdot y$$

subject to

$$yA \geqq p$$

As we saw in the last section, the initial *tableau* can be written

| | $A^1$ | $\dots$ | $\dots$ | $A^n$ | $e^1$ | $\dots$ | $\dots$ | $e^m$ | $b$ |
|---|---|---|---|---|---|---|---|---|---|
| $e^1$ | $a_{1,1}$ | $\dots$ | $\dots$ | $a_{1,n}$ | $1$ | $0$ | $\dots$ | $0$ | $b_1$ |
| $\vdots$ | $\vdots$ | | | $\vdots$ | $0$ | $\ddots$ | | $\vdots$ | $\vdots$ |
| $\vdots$ | $\vdots$ | | | $\vdots$ | $\vdots$ | | $\ddots$ | $0$ | $\vdots$ |
| $e^m$ | $a_{m,1}$ | $\dots$ | $\dots$ | $a_{m,n}$ | $0$ | $\dots$ | $0$ | $1$ | $b_m$ |
| $\pi - p$ | $-p_1$ | $\dots$ | $\dots$ | $-p_n$ | $0$ | $\dots$ | $\dots$ | $0$ | $0$ |

Assume now that the simplex algorithm enables us to replace all the coordinate vectors with columns of $A$. Without loss of generality, by rearranging the rows and columns of $A$ if necessary, we can assume the algorithm stops in the following configuration, which has the property that $c_i = i$ for $i = 1, \dots, m$.

| | $A^1$ | $\dots$ | $\dots$ | $A^m$ | $A^{m+1}$ | $\dots$ | $A^n$ | $e^1$ | $\dots$ | $e^m$ | $b$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $A^1$ | $1$ | $0$ | $\dots$ | $0$ | $t_{1,m+1}$ | $\dots$ | $t_{1,n}$ | $s_{1,1}$ | $\dots$ | $s_{1,m}$ | $x_1$ |
| $\vdots$ | $0$ | $\ddots$ | | $\vdots$ | $\vdots$ | | $\vdots$ | $\vdots$ | | $\vdots$ | $\vdots$ |
| $\vdots$ | $\vdots$ | | $\ddots$ | $0$ | $\vdots$ | | $\vdots$ | $\vdots$ | | $\vdots$ | $\vdots$ |
| $A^m$ | $0$ | $\dots$ | $0$ | $1$ | $t_{m,m+1}$ | $\dots$ | $t_{m,n}$ | $s_{m,1}$ | $\dots$ | $s_{m,m}$ | $x_m$ |
| $\pi - p$ | $0$ | $\dots$ | $\dots$ | $0$ | $\pi_{m+1} - p_{m+1}$ | $\dots$ | $\pi_n - p_n$ | $y_1$ | $\dots$ | $y_m$ | $p \cdot x$ |

There are three key observations to make here.

1. The block $[s_{i,j}]_{\substack{j=1\dots m\\ i=1\dots m}}$ is the inverse of the block $A_{m,m} = [a_{i,j}]_{\substack{j=1\dots m\\ i=1\dots m}}$. (Recall the use of the Gauss–Jordan method for inverting a matrix.)
2. By construction of the criterion row, the $y_k$s satisfy
$$y_k = \sum_{i=1}^{m} s_{i,k} p_i \qquad k = 1, \dots, m.$$
3. For $j > m$, we have $\pi_j \geqslant p_j$. (Otherwise the algorithm would not stop here with an optimal $x$.)

Thus, as in the proof of Proposition 10, for $j = 1, \dots, m$ we have

$$y \cdot A^j = \sum_{k=1}^{m} y_k a_{k,j} = \sum_{k=1}^{m} \left( \sum_{i=1}^{m} s_{i,k} p_i \right) a_{k,j} = \sum_{i=1}^{m} p_i \sum_{k=1}^{m} \left( s_{i,k} a_{k,j} \right) = \sum_{i=1}^{m} p_i \delta_{i,j} = p_j,$$

where the penultimate equality follows because $[s_{i,j}]$ is the inverse of $A_{m,m}$. For $j > m$,

$$y \cdot A^j = y \cdot \sum_{i=1}^{m} t_{i,j} A^i = \sum_{i=1}^{m} t_{i,j} y \cdot A^i = \sum_{i=1}^{m} t_{i,j} p_i = \pi_j \geqslant p_j,$$

by the third observation. In other words,

$$yA \geqq p.$$

In addition,

$$b \cdot y = \left( \sum_{i=1}^{m} x_i A^i \right) \cdot y = \sum_{i=1}^{m} x_i (A^i \cdot y) = \sum_{i=1}^{m} x_i p_i = p \cdot x$$

since $x_j = 0$ for $j > m$. Thus $p \cdot x = b \cdot y$, so $y$ is optimal.

### 10.1 Solving the dual with inequality constraints

The same technique also solves the dual for problems of the form

$$\underset{x \in \mathbf{R}^{\mathrm{n}}}{\text{maximize}} \; p \cdot x$$

subject to

$$\begin{aligned} Ax &\leqq b \\ x &\geqq 0 \end{aligned}$$

The dual program is

$$\underset{y \in \mathbf{R}^{\mathrm{m}}}{\text{minimize}} \; b \cdot y$$

subject to $y \geqq 0$ and

$$yA \geqq p$$

Instead, we introduce a vector $z$ of slack variables and solve the following problem:

$$\underset{x \in \mathbf{R}^{\mathrm{n}},\ z \in \mathbf{R}^{\mathrm{m}}}{\text{maximize}} \; p \cdot x + 0 \cdot z$$

subject to

$$\begin{aligned} Ax + Iz &= b \\ x &\geqq 0 \\ z &\geqq 0 \end{aligned}$$

The dual program is

$$\underset{y \in \mathbf{R}^{\mathrm{m}}}{\text{minimize}} \; b \cdot y$$

subject to $y \geqq 0$ and

$$y[A, I] \geqq [p, 0]$$

with no inequality constraints on $y$.

Our algorithm applied to this problem produces vectors $\bar{x}$, $\bar{z}$, and $\bar{y}$ that satisfies $b \cdot \bar{y} = p \cdot \bar{x}(+0 \cdot \bar{z})$, and $\bar{y}[A, I] \geqq [p, 0]$. But this implies $\bar{y}A \geqq p$ and $\bar{y} \geqq 0$, so the computed solution $\bar{y}$ to the dual of the equality case also solves the dual for the inequality case.

## 11 Degeneracy, cycling, and the lexicographic simplex algorithm

Proposition 9 shows that the simplex algorithm must stop if the linear program is nondegenerate. But verification of nondegeneracy is difficult. This is unfortunate, as the next example shows that the naïve simplex algorithm can cycle and never stop in the degenerate case. However there is a simple modification, the lexicographic simplex algorithm, that will stop even in the degenerate case.

## 11.1 A cycling example

The first example of cycling in the simplex algorithm is due to Hoffman [9]. Beale [1] constructed the following simpler example of cycling. (See also [3, pp. 228–230].) The problem is to

$$\text{maximize} \quad \tfrac{3}{4}x_1 - 150x_2 + \tfrac{1}{50}x_3 - 6x_4$$

subject to $x \geqq 0$, and

$$\begin{array}{rcl} \tfrac{1}{4}x_1 - 60x_2 - \tfrac{1}{25}x_3 + 9x_4 & \leqslant & 0 \\ \tfrac{1}{2}x_1 - 90x_2 - \tfrac{1}{50}x_3 + 3x_4 & \leqslant & 0 \\ + \ x_3 \qquad & \leqslant & 1 \end{array}$$

Introducing slack variables $z$ and setting them to the right-hand side constants leads directly to the next *tableau*.

| $p_{c_i}$ | | $a^1$ | $a^2$ | $a^3$ | $a^4$ | $e^1$ | $e^2$ | $e^3$ | $b$ | |
|---|---|---|---|---|---|---|---|---|---|---|

Initial *tableau*

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | $e^1$ | $\boxed{\frac{1}{4}}$ | $-60$ | $-\frac{1}{25}$ | 9 | 1 | 0 | 0 | 0 | 0 |
| 0 | $e^2$ | $\frac{1}{2}$ | $-90$ | $-\frac{1}{50}$ | 3 | 0 | 1 | 0 | 0 | 0 |
| 0 | $e^3$ | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | |
| | | $-\frac{3}{4}$ | 150 | $-\frac{1}{50}$ | 6 | 0 | 0 | 0 | 0 | |

Replace $e^1$ by $a^1$:

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\frac{3}{4}$ | $a^1$ | 1 | $-240$ | $-\frac{4}{25}$ | 36 | 4 | 0 | 0 | 0 | |
| 0 | $e^2$ | 0 | $\boxed{30}$ | $\frac{3}{50}$ | $-15$ | $-2$ | 1 | 0 | 0 | 0 |
| 0 | $e^3$ | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | |
| | | 0 | $-30$ | $-\frac{7}{50}$ | 33 | 3 | 0 | 0 | 0 | |

Replace $e^2$ by $a^2$:

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\frac{3}{4}$ | $a^1$ | 1 | 0 | $\boxed{\frac{8}{25}}$ | $-84$ | $-12$ | 8 | 0 | 0 | 0 |
| $-150$ | $a^2$ | 0 | 1 | $\frac{1}{500}$ | $-\frac{1}{2}$ | $-\frac{1}{15}$ | $\frac{1}{30}$ | 0 | 0 | 0 |
| 0 | $e^3$ | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 1 |
| | | 0 | 0 | $-\frac{2}{25}$ | 18 | 1 | 1 | 0 | 0 | |

Replace $a^1$ by $a^3$:

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\frac{1}{50}$ | $a^3$ | $3\frac{1}{8}$ | 0 | 1 | $-262\frac{1}{2}$ | $-37\frac{1}{2}$ | 25 | 0 | 0 | |
| $-150$ | $a^2$ | $-\frac{1}{160}$ | 1 | 0 | $\boxed{\frac{1}{40}}$ | $\frac{1}{120}$ | $-\frac{1}{60}$ | 0 | 0 | 0 |
| 0 | $e^3$ | $-3\frac{1}{8}$ | 0 | 0 | $262\frac{1}{2}$ | $37\frac{1}{2}$ | $-25$ | 1 | 1 | $\frac{2}{525}$ |
| | | $\frac{1}{4}$ | 0 | 0 | $-3$ | $-2$ | 3 | 0 | 0 | |

Replace $a^2$ by $a^4$:

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\frac{1}{50}$ | $a^3$ | $-62\frac{1}{2}$ | 10500 | 1 | 0 | $\boxed{50}$ | $-150$ | 0 | 0 | 0 |
| $-6$ | $a^4$ | $-\frac{1}{4}$ | 40 | 0 | 1 | $\frac{1}{3}$ | $-\frac{2}{3}$ | 0 | 0 | 0 |
| 0 | $e^3$ | $62\frac{1}{2}$ | $-10500$ | 0 | 0 | $-50$ | 150 | 1 | 1 | |
| | | $-\frac{1}{2}$ | 120 | 0 | 0 | $-1$ | 1 | 0 | 0 | |

Replace $a^3$ by $e^1$:

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | $e^1$ | $-1\frac{1}{4}$ | 210 | $\frac{1}{50}$ | 0 | 1 | $-3$ | 0 | 0 | |
| $-6$ | $a^4$ | $\frac{1}{6}$ | $-30$ | $-\frac{1}{150}$ | 1 | 0 | $\boxed{\frac{1}{3}}$ | 0 | 0 | 0 |
| 0 | $e^3$ | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | |
| | | $-1\frac{3}{4}$ | 330 | $\frac{1}{50}$ | 0 | 0 | $-2$ | 0 | 0 | |

Replace $a^4$ by $e^2$:

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | $e^1$ | $\frac{1}{4}$ | $-60$ | $-\frac{1}{25}$ | 9 | 1 | 0 | 0 | 0 | |
| 0 | $e^2$ | $\frac{1}{2}$ | $-90$ | $-\frac{1}{50}$ | 3 | 0 | 1 | 0 | 0 | |
| 0 | $e^3$ | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | |
| | | $-\frac{3}{4}$ | 150 | $-\frac{1}{50}$ | 6 | 0 | 0 | 0 | 0 | |

The algorithm was implemented to choose the pivot column with the most negative value in the criterion row, and when more than one row minimized the ratio, the first row to do so was selected for the pivot. As you can see, the seventh *tableau* is the same as the first, so the algorithm is doomed to repeat itself.

A peculiar (and nongeneric) feature of this problem is that the *tableau* always gives a choice of two pivot rows, and the minimum ratio is always zero. Indeed the proof of Proposition 9 shows that a zero ratio is necessary for cycling.

## 11.2 The lexicographic simplex algorithm

Dantzig, Orden, and Wolfe [4] provide a pivot choice rule that will not cycle. Their rule for choosing the pivot row is lexicographic. To use it, we need to use an extended *tableau* with an identity matrix spliced in to the left of the $b$ column. (You will probably want this anyway to compute the solution to the dual.) Here is a typical *tableau*:

| | $A^1$ | $\dots$ | $A^j$ | $\dots$ | $A^n$ | $e^1$ | $\dots$ | $e^m$ | $b$ |
|---|---|---|---|---|---|---|---|---|---|
| $A^{c_1}$ | $t_{1,1}$ | $\dots$ | $t_{1,j}$ | $\dots$ | $t_{1,n}$ | $s_{1,1}$ | $\dots$ | $s_{1,m}$ | $x_{c_1}$ |
| $\vdots$ | $\vdots$ | | $\vdots$ | | $\vdots$ | $\vdots$ | | $\vdots$ | $\vdots$ |
| $A^{c_k}$ | $t_{k,1}$ | $\dots$ | $t_{k,j}$ | $\dots$ | $t_{k,n}$ | $s_{k,1}$ | $\dots$ | $s_{k,m}$ | $x_{c_k}$ |
| $\vdots$ | $\vdots$ | | $\vdots$ | | $\vdots$ | $\vdots$ | | $\vdots$ | $\vdots$ |
| $A^{c_m}$ | $t_{m,1}$ | $\dots$ | $t_{m,j}$ | $\dots$ | $t_{m,n}$ | $s_{m,1}$ | $\dots$ | $s_{m,m}$ | $x_{c_m}$ |
| $\pi - p$ | $\pi_1 - p_1$ | $\dots$ | $\pi_j - p_j$ | $\dots$ | $\pi_n - p_n$ | $y_1$ | $\dots$ | $y_m$ | $p \cdot x$ |

The rule for choosing the column $k$ is this

Lexicographic rule

Choose the pivot row $k$ so that

$$t_{k,j} > 0$$

and the vector

$$q_k = \left( \frac{x_{c_k}}{t_{k,j}}, \frac{s_{k,1}}{t_{k,j}}, \dots, \frac{s_{k,m}}{t_{k,j}} \right)$$

is lexicographically minimal in $\{q_i : t_{i,j} > 0\}$.

This differs from our previous rule, which only looked at the first component of these vectors. The proof that this rule works is not hard, and may be found in Gale [8, Chapter 4, section 7, pp. 123–128] or Dantzig [3, pp. 234–235]. In practice, it appears that cycling is not a problem. Charnes [2] deals with the problem by slightly perturbing $b$.

## 11.3 Lexicographic simplex example

Here is the lexicographic simplex method applied to Beale's example. I have placed the entire $q_i$ vector in the right-hand margin. (This is not computationally efficient—if you have tens of thousands of variables, you don't want to compute these extra ratios unless you need them all to break ties.)

| $p_{c_i}$ | | $a^1$ | $a^2$ | $a^3$ | $a^4$ | $e^1$ | $e^2$ | $e^3$ | $b$ | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Initial *tableau* | | | | | | | | | | | | | |
| 0 | $e^1$ | $\frac{1}{4}$ | $-60$ | $-\frac{1}{25}$ | 9 | 1 | 0 | 0 | 0 | 0 | 4 | 0 | 0 |
| 0 | $e^2$ | $\boxed{\frac{1}{2}}$ | $-90$ | $-\frac{1}{50}$ | 3 | 0 | 1 | 0 | 0 | 0 | 0 | 2 | 0 |
| 0 | $e^3$ | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | | | | |
| | | $-\frac{3}{4}$ | 150 | $-\frac{1}{50}$ | 6 | 0 | 0 | 0 | 0 | | | | |
| Replace $e^2$ by $a^1$: | | | | | | | | | | | | | |
| 0 | $e^1$ | 0 | $-15$ | $-\frac{3}{100}$ | $7\frac{1}{2}$ | 1 | $-\frac{1}{2}$ | 0 | 0 | | | | |
| $\frac{3}{4}$ | $a^1$ | 1 | $-180$ | $-\frac{1}{25}$ | 6 | 0 | 2 | 0 | 0 | | | | |
| 0 | $e^3$ | 0 | 0 | $\boxed{1}$ | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 1 |
| | | 0 | 15 | $-\frac{1}{20}$ | $10\frac{1}{2}$ | 0 | $1\frac{1}{2}$ | 0 | 0 | | | | |
| Replace $e^3$ by $a^3$: | | | | | | | | | | | | | |
| 0 | $e^1$ | 0 | $-15$ | 0 | $7\frac{1}{2}$ | 1 | $-\frac{1}{2}$ | $\frac{3}{100}$ | $\frac{3}{100}$ | | | | |
| $\frac{3}{4}$ | $a^1$ | 1 | $-180$ | 0 | 6 | 0 | 2 | $\frac{1}{25}$ | $\frac{1}{25}$ | | | | |
| $\frac{1}{50}$ | $a^3$ | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | | | | |
| | | 0 | 15 | 0 | $10\frac{1}{2}$ | 0 | $1\frac{1}{2}$ | $\frac{1}{20}$ | $\frac{1}{20}$ | | | | |

There is no pivot column, so the current basis is optimal. A solution is

$$\boxed{x = \left(\tfrac{1}{25},\quad 0,\quad 1,\quad 0\right)}$$

Verify that $x$ satisfies the constraints:

$$\begin{aligned}
\tfrac{1}{4}(\tfrac{1}{25}) - 60(0) - \tfrac{1}{25}(1) + 9(0) &= \tfrac{1}{100} + 0 - \tfrac{1}{25} + 0 &= -\tfrac{3}{100} &< 0\\
\tfrac{1}{2}(\tfrac{1}{25}) - 90(0) - \tfrac{1}{50}(1) + 3(0) &= \tfrac{1}{50} + 0 - \tfrac{1}{50} + 0 &= 0 &= 0\\
0(\tfrac{1}{25}) + 0(0) + 1(1) + 0(0) &= 0 + 0 + 1 + 0 &= 1 &= 1\ .
\end{aligned}$$

Thus a solution is

$$\boxed{x = \left(\tfrac{1}{25},\quad 0,\quad 1,\quad 0\right)}$$

Check the value of $p \cdot x$:

$$\tfrac{3}{4}(\tfrac{1}{25}) - 150(0) + \tfrac{1}{50}(1) - 6(0) = \tfrac{3}{100} + 0 + \tfrac{1}{50} + 0 = \tfrac{1}{20}.$$

A solution to the dual is

$$\boxed{y = \left(0,\quad 1\tfrac{1}{2},\quad \tfrac{1}{20}\right).}$$

Recall that the dual problem is

$$\text{minimize} \quad y_3$$

subject to $y \geqq 0$ and

$$\begin{aligned}
\tfrac{1}{4}y_1 + \tfrac{1}{2}y_2 &\geqslant 3/4 \\
-60y_1 - 90y_2 &\geqslant -150 \\
-\tfrac{1}{25}y_1 - \tfrac{1}{50}y_2 + y_3 &\geqslant 1/50 \\
9y_1 + 3y_2 &\geqslant -6\,.
\end{aligned}$$

Check that the value of the dual is

$$0(0) + 0(1\tfrac{1}{2}) + 1(\tfrac{1}{20}) = 0 + 0 + \tfrac{1}{20} = \tfrac{1}{20}.$$

Now verify the feasibility of the dual.

$$\begin{array}{rcccccc}
\frac{1}{4}(0) + \frac{1}{2}(1\frac{1}{2}) + 0(\frac{1}{20}) & = & 0 + \frac{3}{4} + 0 & = & \frac{3}{4} & = & \frac{3}{4} \\
-60(0) - 90(1\frac{1}{2}) + 0(\frac{1}{20}) & = & 0 - 135 + 0 & = & -135 & > & -150 \\
-\frac{1}{25}(0) - \frac{1}{50}(1\frac{1}{2}) + 1(\frac{1}{20}) & = & 0 - \frac{3}{100} + \frac{1}{20} & = & \frac{1}{50} & = & \frac{1}{50} \\
9(0) + 3(1\frac{1}{2}) + 0(\frac{1}{20}) & = & 0 + 4\frac{1}{2} + 0 & = & 4\frac{1}{2} & > & -6\ .
\end{array}$$

# 12 More worked examples

Just as a picture is worth a thousand words, a good example is worth several pages of dense notation.

## 12.1 Minimization with equality constraints

Consider the following problem.

$$\text{minimize} \quad x_1 + 6x_2 - 7x_3 + x_4 + 5x_5$$

subject to $x \geqq 0$, and

$$\begin{aligned}
5x_1 - 4x_2 + 13x_3 - 2x_4 + x_5 &= 20 \\
x_1 - x_2 + 5x_3 - x_4 + x_5 &= 8
\end{aligned}$$

Since the constraints take the form of equalities, no slack variables are necessary, but there is no obvious starting point. So in Phase I, we introduce nonnegative artificial variables $u_1$ and $u_2$, and proceed to solve the ancillary problem

$$\text{minimize} \quad u_1 + u_2$$

subject to $x \geqq 0$, $u \geqq 0$, and

$$\begin{aligned}
5x_1 - 4x_2 + 13x_3 - 2x_4 + x_5 + u_1 \phantom{{}+u_2} &= 20 \\
x_1 - x_2 + 5x_3 - x_4 + x_5 \phantom{{}+u_1} + u_2 &= 8
\end{aligned}$$

Since we require that $u \geqq 0$, the minimum of $u_1 + u_2 \geqslant 0$, with equality only if $u_1 = u_2 = 0$. Thus if the solution to this LP has value zero, we will have succeeded in finding a feasible solution to the original problem. The virtue of this ancillary problem is that there is an obvious starting point: set $x = 0$, and setting $u = (20, 8)$ (that is, set $u$ to the right-hand side). The criterion row is based on the artificial price vector indicated in the left margin of the *tableau*, and is searched for positive entries. Here is the initial *tableau*.

| $p_{c_i}$ | | $a^1$ | $a^2$ | $a^3$ | $a^4$ | $a^5$ | $e^1$ | $e^2$ | $b$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| Initial *tableau* | | | | | | | | | | |
| 1 | $u^1$ | 5 | $-4$ | $\boxed{13}$ | $-2$ | 1 | 1 | 0 | 20 | $1\frac{7}{13}$ |
| 1 | $u^2$ | 1 | $-1$ | 5 | $-1$ | 1 | 0 | 1 | 8 | $1\frac{3}{5}$ |
| | | 6 | $-5$ | 18 | $-3$ | 2 | 1 | 1 | 28 | |
| Replace $u^1$ by $a^3$: | | | | | | | | | | |
| 0 | $a^3$ | $\frac{5}{13}$ | $-\frac{4}{13}$ | 1 | $-\frac{2}{13}$ | $\frac{1}{13}$ | $\frac{1}{13}$ | 0 | $1\frac{7}{13}$ | 20 |
| 1 | $u^2$ | $-\frac{12}{13}$ | $\frac{7}{13}$ | 0 | $-\frac{3}{13}$ | $\boxed{\frac{8}{13}}$ | $-\frac{5}{13}$ | 1 | $\frac{4}{13}$ | $\frac{1}{2}$ |
| | | $-\frac{12}{13}$ | $\frac{7}{13}$ | 0 | $-\frac{3}{13}$ | $\frac{8}{13}$ | $-\frac{5}{13}$ | 1 | $\frac{4}{13}$ | |
| Replace $u^2$ by $a^5$: | | | | | | | | | | |
| 0 | $a^3$ | $\frac{1}{2}$ | $-\frac{3}{8}$ | 1 | $-\frac{1}{8}$ | 0 | $\frac{1}{8}$ | $-\frac{1}{8}$ | $1\frac{1}{2}$ | |
| 0 | $a^5$ | $-1\frac{1}{2}$ | $\frac{7}{8}$ | 0 | $-\frac{3}{8}$ | 1 | $-\frac{5}{8}$ | $1\frac{5}{8}$ | $\frac{1}{2}$ | |
| | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | |

According to this, the value (found in the lower right-hand corner) is zero, so we have indeed found a feasible solution to the original problem, namely

$$x = \big(0,\ \ 0,\ \ 1\tfrac{1}{2},\ \ 0,\ \ \tfrac{1}{2}\big).$$

I leave it to you to check that $x$ does indeed satisfy the constraints.

In Phase II, we now proceed with the original minimization problem. To do so, we must recalculate the $\pi - p$ criterion row, and search for positive entries. Here is the new initial *tableau*.

| $p_{c_i}$ | | $a^1$ | $a^2$ | $a^3$ | $a^4$ | $a^5$ | $e^1$ | $e^2$ | $b$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| Initial *tableau* | | | | | | | | | | |
| $-7$ | $a^3$ | $\frac{1}{2}$ | $-\frac{3}{8}$ | 1 | $-\frac{1}{8}$ | 0 | $\frac{1}{8}$ | $-\frac{1}{8}$ | $1\frac{1}{2}$ | |
| 5 | $a^5$ | $-1\frac{1}{2}$ | $\boxed{\frac{7}{8}}$ | 0 | $-\frac{3}{8}$ | 1 | $-\frac{5}{8}$ | $1\frac{5}{8}$ | $\frac{1}{2}$ | $\frac{4}{7}$ |
| | | $-12$ | 1 | 0 | $-2$ | 0 | $-4$ | 9 | $-8$ | |
| Replace $a^5$ by $a^2$: | | | | | | | | | | |
| $-7$ | $a^3$ | $-\frac{1}{7}$ | 0 | 1 | $-\frac{2}{7}$ | $\frac{3}{7}$ | $-\frac{1}{7}$ | $\frac{4}{7}$ | $1\frac{5}{7}$ | |
| 6 | $a^2$ | $-1\frac{5}{7}$ | 1 | 0 | $-\frac{3}{7}$ | $1\frac{1}{7}$ | $-\frac{5}{7}$ | $1\frac{6}{7}$ | $\frac{4}{7}$ | |
| | | $-10\frac{2}{7}$ | 0 | 0 | $-1\frac{4}{7}$ | $-1\frac{1}{7}$ | $-3\frac{2}{7}$ | $7\frac{1}{7}$ | $-8\frac{4}{7}$ | |

Notice that in Phase II, I never pivot on a column corresponding to the artificial variables (look at that nice fat 9 in the criterion row of the first *tableau*), because they may not be used in a bona fide solution. Why, then you might ask, do I keep them in the *tableau*? The answer is that they compute the solution to the dual.

We can read a solution from the final *tableau* above:

$$\boxed{x = \big(0,\ \ \tfrac{4}{7},\ \ 1\tfrac{5}{7},\ \ 0,\ \ 0\big)}$$

Let me verify that the constraints are satisfied:

$$\begin{array}{rcrcrcr} 5(0)-4(\frac{4}{7})+13(1\frac{5}{7})-2(0)+1(0) & = & 0-2\frac{2}{7}+22\frac{2}{7}+0+0 & = & 20 & = & 20 \\ 1(0)-1(\frac{4}{7})+5(1\frac{5}{7})-1(0)+1(0) & = & 0-\frac{4}{7}+8\frac{4}{7}+0+0 & = & 8 & = & 8 \end{array}$$

$$\boxed{\text{The value is } -8\tfrac{4}{7}.}$$

We can also read off a solution to the dual:

$$\boxed{y = \left(-3\tfrac{2}{7},\ \ 7\tfrac{1}{7}\right).}$$

Recall that the dual problem is

$$\text{maximize} \quad 20y_1 + 8y_2$$

subject to

$$\begin{array}{rcr} 5y_1 + \ y_2 & \leqslant & 1 \\ -4y_1 - \ y_2 & \leqslant & 6 \\ 13y_1 + 5y_2 & \leqslant & -7 \\ -2y_1 - \ y_2 & \leqslant & 1 \\ y_1 + \ y_2 & \leqslant & 5 \end{array}$$

Verify the feasibility of the dual.

$$\begin{array}{rcrcrcr} 5(-3\frac{2}{7})+1(7\frac{1}{7}) & = & -16\frac{3}{7}+7\frac{1}{7} & = & -9\frac{2}{7} & < & 1 \\ -4(-3\frac{2}{7})-1(7\frac{1}{7}) & = & 13\frac{1}{7}-7\frac{1}{7} & = & 6 & = & 6 \\ 13(-3\frac{2}{7})+5(7\frac{1}{7}) & = & -42\frac{5}{7}+35\frac{5}{7} & = & -7 & = & -7 \\ -2(-3\frac{2}{7})-1(7\frac{1}{7}) & = & 6\frac{4}{7}-7\frac{1}{7} & = & -\frac{4}{7} & < & 1 \\ 1(-3\frac{2}{7})+1(7\frac{1}{7}) & = & -3\frac{2}{7}+7\frac{1}{7} & = & 3\frac{6}{7} & < & 5 \end{array}$$

## 12.2 An example with a negative right-hand side constant

Consider the problem

$$\text{maximize} \quad 2x_1 - 3x_2 + \ x_3 + \ x_4$$

subject to $x \geqq 0$, and

$$\begin{array}{rcr} x_1 + 2x_2 + \ x_3 + \ x_4 & = & 3 \\ x_1 - 2x_2 + 2x_3 + \ x_4 & = & -2 \\ 3x_1 - \ x_2 \qquad - \ x_4 & = & -1 \end{array}$$

Rewrite the constraints as

$$\begin{array}{rcr} x_1 + 2x_2 + \ x_3 + \ x_4 & = & 3 \\ -\ x_1 + 2x_2 - 2x_3 - \ x_4 & = & 2 \\ -3x_1 + \ x_2 \qquad + \ x_4 & = & 1 \end{array}$$

This has no effect on the primal, but the dual is different. This form has the virtue that the following ancillary problem has an obvious starting feasible point.

$$\text{minimize} \quad u_1 + u_2 + u_3$$

subject to $x \geqq 0$, $u \geqq 0$, and

$$\begin{aligned} x_1 + 2x_2 + x_3 + x_4 + u_1 \qquad\qquad &= 3 \\ -x_1 + 2x_2 - 2x_3 - x_4 \quad + u_2 \qquad &= 2 \\ -3x_1 + x_2 \quad + x_4 \qquad\quad + u_3 &= 1 \end{aligned}$$

A feasible starting point is given by setting $x = 0$, and setting $u = (3, 2, 1)$. Here is the initial *tableau*.

Initial *tableau*:

| | $a^1$ | $a^2$ | $a^3$ | $a^4$ | $e^1$ | $e^2$ | $e^3$ | $b$ | |
|---|---|---|---|---|---|---|---|---|---|
| $e^1$ | $1$ | $2$ | $1$ | $1$ | $1$ | $0$ | $0$ | $3$ | $1\frac{1}{2}$ |
| $e^2$ | $-1$ | $\boxed{2}$ | $-2$ | $-1$ | $0$ | $1$ | $0$ | $2$ | $1$ |
| $e^3$ | $-3$ | $1$ | $0$ | $1$ | $0$ | $0$ | $1$ | $1$ | $1$ |
| | $-3$ | $5$ | $-1$ | $1$ | $0$ | $0$ | $0$ | $6$ | |

Replace $e^2$ by $a^2$ to get:

| | $a^1$ | $a^2$ | $a^3$ | $a^4$ | $e^1$ | $e^2$ | $e^3$ | $b$ | |
|---|---|---|---|---|---|---|---|---|---|
| $e^1$ | $2$ | $0$ | $3$ | $2$ | $1$ | $-1$ | $0$ | $1$ | $\frac{1}{3}$ |
| $a^2$ | $-\frac{1}{2}$ | $1$ | $-1$ | $-\frac{1}{2}$ | $0$ | $\frac{1}{2}$ | $0$ | $1$ | |
| $e^3$ | $-2\frac{1}{2}$ | $0$ | $\boxed{1}$ | $1\frac{1}{2}$ | $0$ | $-\frac{1}{2}$ | $1$ | $0$ | $0$ |
| | $-\frac{1}{2}$ | $0$ | $4$ | $3\frac{1}{2}$ | $0$ | $-2\frac{1}{2}$ | $0$ | $1$ | |

Replace $e^3$ by $a^3$ to get:

| | $a^1$ | $a^2$ | $a^3$ | $a^4$ | $e^1$ | $e^2$ | $e^3$ | $b$ | |
|---|---|---|---|---|---|---|---|---|---|
| $e^1$ | $\boxed{9\frac{1}{2}}$ | $0$ | $0$ | $-2\frac{1}{2}$ | $1$ | $\frac{1}{2}$ | $-3$ | $1$ | $\frac{2}{19}$ |
| $a^2$ | $-3$ | $1$ | $0$ | $1$ | $0$ | $0$ | $1$ | $1$ | |
| $a^3$ | $-2\frac{1}{2}$ | $0$ | $1$ | $1\frac{1}{2}$ | $0$ | $-\frac{1}{2}$ | $1$ | $0$ | |
| | $9\frac{1}{2}$ | $0$ | $0$ | $-2\frac{1}{2}$ | $0$ | $-\frac{1}{2}$ | $-4$ | $1$ | |

Replace $e^1$ by $a^1$ to get:

| | $a^1$ | $a^2$ | $a^3$ | $a^4$ | $e^1$ | $e^2$ | $e^3$ | $b$ |
|---|---|---|---|---|---|---|---|---|
| $a^1$ | $1$ | $0$ | $0$ | $-\frac{5}{19}$ | $\frac{2}{19}$ | $\frac{1}{19}$ | $-\frac{6}{19}$ | $\frac{2}{19}$ |
| $a^2$ | $0$ | $1$ | $0$ | $\frac{4}{19}$ | $\frac{6}{19}$ | $\frac{3}{19}$ | $\frac{1}{19}$ | $1\frac{6}{19}$ |
| $a^3$ | $0$ | $0$ | $1$ | $\frac{16}{19}$ | $\frac{5}{19}$ | $-\frac{7}{19}$ | $\frac{4}{19}$ | $\frac{5}{19}$ |
| | $0$ | $0$ | $0$ | $0$ | $-1$ | $-1$ | $-1$ | $0$ |

Since the value is 0, we have found a feasible starting point for the original problem.

Now to maximize. But first we must recalculate the $\pi - p$ criterion row. Here is the new initial *tableau*.

Initial *tableau*:

| | $a^1$ | $a^2$ | $a^3$ | $a^4$ | $e^1$ | $e^2$ | $e^3$ | $b$ | |
|---|---|---|---|---|---|---|---|---|---|
| $a^1$ | $1$ | $0$ | $0$ | $-\frac{5}{19}$ | $\frac{2}{19}$ | $\frac{1}{19}$ | $-\frac{6}{19}$ | $\frac{2}{19}$ | |
| $a^2$ | $0$ | $1$ | $0$ | $\frac{4}{19}$ | $\frac{6}{19}$ | $\frac{3}{19}$ | $\frac{1}{19}$ | $1\frac{6}{19}$ | $6\frac{1}{4}$ |
| $a^3$ | $0$ | $0$ | $1$ | $\boxed{\frac{16}{19}}$ | $\frac{5}{19}$ | $-\frac{7}{19}$ | $\frac{4}{19}$ | $\frac{5}{19}$ | $\frac{5}{16}$ |
| | $0$ | $0$ | $0$ | $-1\frac{6}{19}$ | $-\frac{9}{19}$ | $-\frac{14}{19}$ | $-\frac{11}{19}$ | $-3\frac{9}{19}$ | |

Replace $a^3$ by $a^4$ to get:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $a^1$ | $1$ | $0$ | $\frac{5}{16}$ | $0$ | $\frac{3}{16}$ | $-\frac{1}{16}$ | $-\frac{1}{4}$ | $\frac{3}{16}$ |
| $a^2$ | $0$ | $1$ | $-\frac{1}{4}$ | $0$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $0$ | $1\frac{1}{4}$ |
| $a^4$ | $0$ | $0$ | $1\frac{3}{16}$ | $1$ | $\frac{5}{16}$ | $-\frac{7}{16}$ | $\frac{1}{4}$ | $\frac{5}{16}$ |
| | $0$ | $0$ | $1\frac{9}{16}$ | $0$ | $-\frac{1}{16}$ | $-1\frac{5}{16}$ | $-\frac{1}{4}$ | $-3\frac{1}{16}$ |

Thus a solution is

$$\boxed{x = \left(\tfrac{3}{16},\quad 1\tfrac{1}{4},\quad 0,\quad \tfrac{5}{16}\right)}$$

Verify the constraints are satisfied:

$$\begin{aligned}
1(\tfrac{3}{16}) + 2(1\tfrac{1}{4}) + 1(0) + 1(\tfrac{5}{16}) &= \tfrac{3}{16} + 2\tfrac{1}{2} + 0 + \tfrac{5}{16} &= 3 &= 3\\
1(\tfrac{3}{16}) - 2(1\tfrac{1}{4}) + 2(0) + 1(\tfrac{5}{16}) &= \tfrac{3}{16} - 2\tfrac{1}{2} + 0 + \tfrac{5}{16} &= -2 &= -2\\
3(\tfrac{3}{16}) - 1(1\tfrac{1}{4}) + 0(0) - 1(\tfrac{5}{16}) &= \tfrac{9}{16} - 1\tfrac{1}{4} + 0 - \tfrac{5}{16} &= -1 &= -1
\end{aligned}$$

$$\boxed{\text{The value is } -3\tfrac{1}{16}.}$$

According to the criterion row we see that a solution to the dual is $y = \left(-\frac{1}{16},\ -1\frac{5}{16},\ -\frac{1}{4}\right)$. But this is a solution to the modified dual, not the original dual. To convert it we must flip the signs on the components corresponding to negative right-hand sides in the original problem. These are the second and third components. Thus a solution to the original dual is

$$\boxed{y = \left(-\tfrac{1}{16},\quad 1\tfrac{5}{16},\quad \tfrac{1}{4}\right).}$$

Recall that the original dual problem is

$$\text{minimize} \quad 3y_1 - 2y_2 - y_3$$

subject to

$$\begin{aligned}
y_1 + y_2 + 3y_3 &\geqslant 2\\
2y_1 - 2y_2 - y_3 &\geqslant -3\\
y_1 + 2y_2 &\geqslant 1\\
y_1 + y_2 - y_3 &\geqslant 1
\end{aligned}$$

Check that the value of the dual solution is

$$3(-\tfrac{1}{16}) - 2(1\tfrac{5}{16}) - 1(\tfrac{1}{4}) = -\tfrac{3}{16} - 2\tfrac{5}{8} - \tfrac{1}{4} = -3\tfrac{1}{16}.$$

Now verify the feasibility of the dual solution for the original dual.

$$\begin{array}{rcrcrcr}
1(-\frac{1}{16})+1(1\frac{5}{16})+3(\frac{1}{4}) & = & -\frac{1}{16}+1\frac{5}{16}+\frac{3}{4} & = & 2 & = & 2\\
2(-\frac{1}{16})-2(1\frac{5}{16})-1(\frac{1}{4}) & = & -\frac{1}{8}-2\frac{5}{8}-\frac{1}{4} & = & -3 & = & -3\\
1(-\frac{1}{16})+2(1\frac{5}{16})+0(\frac{1}{4}) & = & -\frac{1}{16}+2\frac{5}{8}+0 & = & 2\frac{9}{16} & > & 1\\
1(-\frac{1}{16})+1(1\frac{5}{16})-1(\frac{1}{4}) & = & -\frac{1}{16}+1\frac{5}{16}-\frac{1}{4} & = & 1 & = & 1
\end{array}$$

## 12.3 A tricky point with negative right-hand side constants

If the constraints are inequality constraints and the right-hand side has negative values, simply adding slack variables does not immediately lead to a feasible point, so Phase 1 cannot be combined with Phase 2.

Change the constraints in the previous problem to inequalities.

$$\text{maximize} \quad 2x_1 - 3x_2 + x_3 + x_4$$

subject to $x \geqq 0$, and

$$\begin{array}{rcr}
x_1 + 2x_2 + x_3 + x_4 & \leqslant & 3\\
x_1 - 2x_2 + 2x_3 + x_4 & \leqslant & -2\\
3x_1 - x_2 \qquad - x_4 & \leqslant & -1
\end{array}$$

Add nonnegative slack variables to convert the constraints to equalities.

$$\begin{array}{rcr}
x_1 + 2x_2 + x_3 + x_4 + z_1 \qquad\qquad & = & 3\\
x_1 - 2x_2 + 2x_3 + x_4 \qquad + z_2 \qquad & = & -2\\
3x_1 - x_2 \qquad - x_4 \qquad\qquad + z_3 & = & -1
\end{array}$$

Now multiply the second and third equations by $-1$ to get

$$\begin{array}{rcr}
x_1 + 2x_2 + x_3 + x_4 + z_1 \qquad\qquad & = & 3\\
-x_1 + 2x_2 - 2x_3 - x_4 \qquad - z_2 \qquad & = & 2\\
-3x_1 + x_2 \qquad + x_4 \qquad\qquad - z_3 & = & 1
\end{array}$$

In this case setting $x = 0$ and $z = b$ does not give a feasible solution to the primal. To find a nonnegative feasible point, solve the ancillary problem

$$\text{minimize} \quad u_1 + u_2 + u_3$$

subject to $x \geqq 0$, $z \geqq 0$, $u \geqq 0$, and

$$\begin{array}{rcr}
x_1 + 2x_2 + x_3 + x_4 + z_1 \qquad\qquad + u_1 \qquad\qquad & = & 3\\
-x_1 + 2x_2 - 2x_3 - x_4 \qquad - z_2 \qquad\qquad + u_2 \qquad & = & 2\\
-3x_1 + x_2 \qquad + x_4 \qquad\qquad - z_3 \qquad\qquad + u_3 & = & 1
\end{array}$$

This problem has a trivial starting point, given by $x = 0$, $z = 0$, and $u = (3,2,1)$. Here is the initial *tableau*.

Initial *tableau*:

| | $a^1$ | $a^2$ | $a^3$ | $a^4$ | $e^1$ | $e^2$ | $e^3$ | $u^1$ | $u^2$ | $u^3$ | $b$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $u^1$ | $1$ | $2$ | $1$ | $1$ | $1$ | $0$ | $0$ | $1$ | $0$ | $0$ | $3$ | $1\frac{1}{2}$ |
| $u^2$ | $-1$ | $\boxed{2}$ | $-2$ | $-1$ | $0$ | $-1$ | $0$ | $0$ | $1$ | $0$ | $2$ | $1$ |
| $u^3$ | $-3$ | $1$ | $0$ | $1$ | $0$ | $0$ | $-1$ | $0$ | $0$ | $1$ | $1$ | $1$ |
| | $-3$ | $5$ | $-1$ | $1$ | $1$ | $-1$ | $-1$ | $0$ | $0$ | $0$ | $6$ | |

Replace $u^2$ by $a^2$ to get:

| | $a^1$ | $a^2$ | $a^3$ | $a^4$ | $e^1$ | $e^2$ | $e^3$ | $u^1$ | $u^2$ | $u^3$ | $b$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $u^1$ | $2$ | $0$ | $3$ | $2$ | $1$ | $1$ | $0$ | $1$ | $-1$ | $0$ | $1$ | $\frac{1}{3}$ |
| $a^2$ | $-\frac{1}{2}$ | $1$ | $-1$ | $-\frac{1}{2}$ | $0$ | $-\frac{1}{2}$ | $0$ | $0$ | $\frac{1}{2}$ | $0$ | $1$ | |
| $u^3$ | $-2\frac{1}{2}$ | $0$ | $\boxed{1}$ | $1\frac{1}{2}$ | $0$ | $\frac{1}{2}$ | $-1$ | $0$ | $-\frac{1}{2}$ | $1$ | $0$ | $0$ |
| | $-\frac{1}{2}$ | $0$ | $4$ | $3\frac{1}{2}$ | $1$ | $1\frac{1}{2}$ | $-1$ | $0$ | $-2\frac{1}{2}$ | $0$ | $1$ | |

Replace $u^3$ by $a^3$ to get:

| | $a^1$ | $a^2$ | $a^3$ | $a^4$ | $e^1$ | $e^2$ | $e^3$ | $u^1$ | $u^2$ | $u^3$ | $b$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $u^1$ | $\boxed{9\frac{1}{2}}$ | $0$ | $0$ | $-2\frac{1}{2}$ | $1$ | $-\frac{1}{2}$ | $3$ | $1$ | $\frac{1}{2}$ | $-3$ | $1$ | $\frac{2}{19}$ |
| $a^2$ | $-3$ | $1$ | $0$ | $1$ | $0$ | $0$ | $-1$ | $0$ | $0$ | $1$ | $1$ | |
| $a^3$ | $-2\frac{1}{2}$ | $0$ | $1$ | $1\frac{1}{2}$ | $0$ | $\frac{1}{2}$ | $-1$ | $0$ | $-\frac{1}{2}$ | $1$ | $0$ | |
| | $9\frac{1}{2}$ | $0$ | $0$ | $-2\frac{1}{2}$ | $1$ | $-\frac{1}{2}$ | $3$ | $0$ | $-\frac{1}{2}$ | $-4$ | $1$ | |

Replace $u^1$ by $a^1$ to get:

| | $a^1$ | $a^2$ | $a^3$ | $a^4$ | $e^1$ | $e^2$ | $e^3$ | $u^1$ | $u^2$ | $u^3$ | $b$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $a^1$ | $1$ | $0$ | $0$ | $-\frac{5}{19}$ | $\frac{2}{19}$ | $-\frac{1}{19}$ | $\frac{6}{19}$ | $\frac{2}{19}$ | $\frac{1}{19}$ | $-\frac{6}{19}$ | $\frac{2}{19}$ |
| $a^2$ | $0$ | $1$ | $0$ | $\frac{4}{19}$ | $\frac{6}{19}$ | $-\frac{3}{19}$ | $-\frac{1}{19}$ | $\frac{6}{19}$ | $\frac{3}{19}$ | $\frac{1}{19}$ | $1\frac{6}{19}$ |
| $a^3$ | $0$ | $0$ | $1$ | $\frac{16}{19}$ | $\frac{5}{19}$ | $\frac{7}{19}$ | $-\frac{4}{19}$ | $\frac{5}{19}$ | $-\frac{7}{19}$ | $\frac{4}{19}$ | $\frac{5}{19}$ |
| | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $-1$ | $-1$ | $-1$ | $0$ |

The value is 0, so we have found a feasible starting point for Phase 2. Now to recalculate the $\pi - p$ criterion row and maximize. Here is the new initial *tableau*.

Initial *tableau*:

| | $a^1$ | $a^2$ | $a^3$ | $a^4$ | $e^1$ | $e^2$ | $e^3$ | $u^1$ | $u^2$ | $u^3$ | $b$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $a^1$ | $1$ | $0$ | $0$ | $-\frac{5}{19}$ | $\frac{2}{19}$ | $-\frac{1}{19}$ | $\frac{6}{19}$ | $\frac{2}{19}$ | $\frac{1}{19}$ | $-\frac{6}{19}$ | $\frac{2}{19}$ | |
| $a^2$ | $0$ | $1$ | $0$ | $\frac{4}{19}$ | $\frac{6}{19}$ | $-\frac{3}{19}$ | $-\frac{1}{19}$ | $\frac{6}{19}$ | $\frac{3}{19}$ | $\frac{1}{19}$ | $1\frac{6}{19}$ | $6\frac{1}{4}$ |
| $a^3$ | $0$ | $0$ | $1$ | $\boxed{\frac{16}{19}}$ | $\frac{5}{19}$ | $\frac{7}{19}$ | $-\frac{4}{19}$ | $\frac{5}{19}$ | $-\frac{7}{19}$ | $\frac{4}{19}$ | $\frac{5}{19}$ | $\frac{5}{16}$ |
| | $0$ | $0$ | $0$ | $-1\frac{6}{19}$ | $-\frac{9}{19}$ | $\frac{14}{19}$ | $\frac{11}{19}$ | $-\frac{9}{19}$ | $-\frac{14}{19}$ | $-\frac{11}{19}$ | $-3\frac{9}{19}$ | |

Replace $a^3$ by $a^4$ to get:

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $a^1$ | $1$ | $0$ | $\frac{5}{16}$ | $0$ | $\boxed{\frac{3}{16}}$ | $\frac{1}{16}$ | $\frac{1}{4}$ | $\frac{3}{16}$ | $-\frac{1}{16}$ | $-\frac{1}{4}$ | $\frac{3}{16}$ | $1$ |
| $a^2$ | $0$ | $1$ | $-\frac{1}{4}$ | $0$ | $\frac{1}{4}$ | $-\frac{1}{4}$ | $0$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $0$ | $1\frac{1}{4}$ | $5$ |
| $a^4$ | $0$ | $0$ | $1\frac{3}{16}$ | $1$ | $\frac{5}{16}$ | $\frac{7}{16}$ | $-\frac{1}{4}$ | $\frac{5}{16}$ | $-\frac{7}{16}$ | $\frac{1}{4}$ | $\frac{5}{16}$ | $1$ |
| | $0$ | $0$ | $1\frac{9}{16}$ | $0$ | $-\frac{1}{16}$ | $1\frac{5}{16}$ | $\frac{1}{4}$ | $-\frac{1}{16}$ | $-1\frac{5}{16}$ | $-\frac{1}{4}$ | $-3\frac{1}{16}$ | |

Replace $a^1$ by $e^1$ to get:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $e^1$ | $5\frac{1}{3}$ | $0$ | $1\frac{2}{3}$ | $0$ | $1$ | $\frac{1}{3}$ | $1\frac{1}{3}$ | $1$ | $-\frac{1}{3}$ | $-1\frac{1}{3}$ | $1$ |
| $a^2$ | $-1\frac{1}{3}$ | $1$ | $-\frac{2}{3}$ | $0$ | $0$ | $-\frac{1}{3}$ | $-\frac{1}{3}$ | $0$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $1$ |
| $a^4$ | $-1\frac{2}{3}$ | $0$ | $\frac{2}{3}$ | $1$ | $0$ | $\frac{1}{3}$ | $-\frac{2}{3}$ | $0$ | $-\frac{1}{3}$ | $\frac{2}{3}$ | $0$ |
| | $\frac{1}{3}$ | $0$ | $1\frac{2}{3}$ | $0$ | $0$ | $1\frac{1}{3}$ | $\frac{1}{3}$ | $0$ | $-1\frac{1}{3}$ | $-\frac{1}{3}$ | $-3$ |

Thus a solution is

$$\boxed{x = (0,\ \ 1,\ \ 0,\ \ 0)}$$

Verify the constraints are satisfied:

$$\begin{aligned}
1(0)+2(1)+1(0)+1(0) &= 0+2+0+0 = 2 < 3\\
1(0)-2(1)+2(0)+1(0) &= 0-2+0+0 = -2 = -2\\
3(0)-1(1)+0(0)-1(0) &= 0-1+0+0 = -1 = -1
\end{aligned}$$

$$\boxed{\text{The value is } -3.}$$

Note that by relaxing the constraints from equations in the previous section to inequalities, the value has increased.

A solution to the dual is

$$\boxed{y = \left(0,\ \ 1\tfrac{1}{3},\ \ \tfrac{1}{3}\right).}$$

This can be read off the criterion row in two places, under the slack variables, or by appropriate sign flips under the auxiliary variables. Recall that the dual problem is

$$\text{minimize} \quad 3y_1 - 2y_2 - \ y_3$$

subject to

$$\begin{aligned}
y_1 + \ y_2 + 3y_3 &\geqslant 2\\
2y_1 - 2y_2 - \ y_3 &\geqslant -3\\
y_1 + 2y_2 \qquad &\geqslant 1\\
y_1 + \ y_2 - \ y_3 &\geqslant 1
\end{aligned}$$

Check that the value of the dual is

$$3(0) - 2(1\tfrac{1}{3}) - 1(\tfrac{1}{3}) = 0 - 2\tfrac{2}{3} - \tfrac{1}{3} = -3.$$

Now verify the feasibility of the dual.

$$\begin{array}{rclclcr}
1(0)+1(1\frac{1}{3})+3(\frac{1}{3}) &=& 0+1\frac{1}{3}+1 &=& 2\frac{1}{3} &>& 2\\
2(0)-2(1\frac{1}{3})-1(\frac{1}{3}) &=& 0-2\frac{2}{3}-\frac{1}{3} &=& -3 &=& -3\\
1(0)+2(1\frac{1}{3})+0(\frac{1}{3}) &=& 0+2\frac{2}{3}+0 &=& 2\frac{2}{3} &>& 1\\
1(0)+1(1\frac{1}{3})-1(\frac{1}{3}) &=& 0+1\frac{1}{3}-\frac{1}{3} &=& 1 &=& 1
\end{array}$$

## 12.4 Finding a basis

The replacement operation can be used to find a basis for a set of columns vectors. To be picky, this does not really use the simplex method, but it is useful.

**13 Example** Find a basis for the column space of

$$\begin{bmatrix}
1 & 2 & 0 & 1 & 1 & 1\\
0 & 1 & 1 & 1 & 0 & 1\\
3 & 2 & -4 & 1 & 1 & 1\\
-2 & 0 & 4 & -2 & 2 & 1
\end{bmatrix}$$

Start with a basis of unit coordinate vectors.

Initial *tableau*:

| | $a^1$ | $a^2$ | $a^3$ | $a^4$ | $a^5$ | $a^6$ |
|---|---|---|---|---|---|---|
| $e^1$ | $\boxed{1}$ | 2 | 0 | 1 | 1 | 1 |
| $e^2$ | 0 | 1 | 1 | 1 | 0 | 1 |
| $e^3$ | 3 | 2 | −4 | 1 | 1 | 1 |
| $e^4$ | −2 | 0 | 4 | −2 | 2 | 1 |

Replace $e^1$ by $a^1$ to get:

| | $a^1$ | $a^2$ | $a^3$ | $a^4$ | $a^5$ | $a^6$ |
|---|---|---|---|---|---|---|
| $a^1$ | 1 | 2 | 0 | 1 | 1 | 1 |
| $e^2$ | 0 | $\boxed{1}$ | 1 | 1 | 0 | 1 |
| $e^3$ | 0 | −4 | −4 | −2 | −2 | −2 |
| $e^4$ | 0 | 4 | 4 | 0 | 4 | 3 |

Replace $e^2$ by $a^2$ to get:

| | | | | | | |
|---|---|---|---|---|---|---|
| $a^1$ | 1 | 0 | −2 | −1 | 1 | −1 |
| $a^2$ | 0 | 1 | 1 | 1 | 0 | 1 |
| $e^3$ | 0 | 0 | 0 | $\boxed{2}$ | −2 | 2 |
| $e^4$ | 0 | 0 | 0 | −4 | 4 | −1 |

Replace $e^3$ by $a^4$ to get:

| | | | | | | |
|---|---|---|---|---|---|---|
| $a^1$ | 1 | 0 | −2 | 0 | 0 | 0 |
| $a^2$ | 0 | 1 | 1 | 0 | 1 | 0 |
| $a^4$ | 0 | 0 | 0 | 1 | −1 | 1 |
| $e^4$ | 0 | 0 | 0 | 0 | 0 | $\boxed{3}$ |

Replace $e^4$ by $a^6$ to get:

| | | | | | | |
|---|---|---|---|---|---|---|
| $a^1$ | 1 | 0 | −2 | 0 | 0 | 0 |
| $a^2$ | 0 | 1 | 1 | 0 | 1 | 0 |
| $a^4$ | 0 | 0 | 0 | 1 | −1 | 0 |
| $a^6$ | 0 | 0 | 0 | 0 | 0 | 1 |

This asserts that $\{a^1, a^2, a^4, a^6\}$ is a basis for the column space. □

## 12.5 Finding a basis without the rank assumption

**14 Example** Find a basis for the column space of

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 3 & 2 & -4 & 1 & 1 & 1 \\ 4 & 5 & -3 & 3 & 2 & 3 \end{bmatrix}$$

Note that the last row is the sum of the first three rows, so the rows are not independent.

Start with a basis of unit coordinate vectors.

Initial *tableau*:

| | $a^1$ | $a^2$ | $a^3$ | $a^4$ | $a^5$ | $a^6$ |
|---|---|---|---|---|---|---|
| $e^1$ | $\boxed{1}$ | 2 | 0 | 1 | 1 | 1 |
| $e^2$ | 0 | 1 | 1 | 1 | 0 | 1 |
| $e^3$ | 3 | 2 | −4 | 1 | 1 | 1 |
| $e^4$ | 4 | 5 | −3 | 3 | 2 | 3 |

Replace $e^1$ by $a^1$ to get:

| | $a^1$ | $a^2$ | $a^3$ | $a^4$ | $a^5$ | $a^6$ |
|---|---|---|---|---|---|---|
| $a^1$ | 1 | 2 | 0 | 1 | 1 | 1 |
| $e^2$ | 0 | $\boxed{1}$ | 1 | 1 | 0 | 1 |
| $e^3$ | 0 | −4 | −4 | −2 | −2 | −2 |
| $e^4$ | 0 | −3 | −3 | −1 | −2 | −1 |

Replace $e^2$ by $a^2$ to get:

| | $a^1$ | $a^2$ | $a^3$ | $a^4$ | $a^5$ | $a^6$ |
|---|---|---|---|---|---|---|
| $a^1$ | 1 | 0 | −2 | −1 | 1 | −1 |
| $a^2$ | 0 | 1 | 1 | 1 | 0 | 1 |
| $e^3$ | 0 | 0 | 0 | $\boxed{2}$ | −2 | 2 |
| $e^4$ | 0 | 0 | 0 | 2 | −2 | 2 |

Replace $e^3$ by $a^4$ to get:

| | $a^1$ | $a^2$ | $a^3$ | $a^4$ | $a^5$ | $a^6$ |
|---|---|---|---|---|---|---|
| $a^1$ | 1 | 0 | −2 | 0 | 0 | 0 |
| $a^2$ | 0 | 1 | 1 | 0 | 1 | 0 |
| $a^4$ | 0 | 0 | 0 | 1 | −1 | 1 |
| $e^4$ | 0 | 0 | 0 | 0 | 0 | 0 |

Notice that we are unable to replace the unit coordinate vector $e^4$, but none of the columns of $A$ depend on it. That is, $\{a^1, a^2, a^4\}$ is a basis for the column space.

□

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