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Production Possibility Frontier

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This is a very simple model of the production possibilities of an economy, which was formulated by Abba P. Lerner [1]. There are m outputs $y_1, \dots, y_m$ and n productive factors $x_1, \dots, x_n$. Each output is produced according to the production function $y_i = f^i(x^i)$. There are no intermediate goods, no joint production, and only one (industry) production function for each output. The supplies of each productive factor in the economy are fixed at levels $\bar{x}_1, \dots, \bar{x}_n$.

Assume that for each i, the production function satisfies

$$f^i \colon \boldsymbol{R}^{\mathrm{n}}_+ \to \boldsymbol{R} \text{ is continuous, } C^2 \text{ on } \boldsymbol{R}^{\mathrm{n}}_{++}, \ \nabla f^i \gg 0 \text{ on } \boldsymbol{R}^{\mathrm{n}}_{++},$$

and that the Hessian matrix

$$\left[f^i_{kj} \right]_{\substack{k=1,\dots,n \\ j=1,\dots,n}} \text{ is negative definite on the subspace orthogonal to } \nabla f^i.$$

(Throughout this note, subscripts on the production functions denote partial derivatives.) These assumptions guarantee that all the second order conditions hold as strict inequalities.

1 Production possibility frontier

The **production possibility set** (PPS) is

$$\left\{ y \in \boldsymbol{R}^{\mathrm{m}} : 0 \leqslant y^i \leqslant f^i(x^i), \ i = 1, \dots, m, \text{ and } \sum_{i=1}^n x^i \leqq \bar{x} \right\}.$$

Note that the PPS is compact since the f^is are continuous and the PPS is the continuous image of the compact set

$$\left\{ (x^1, \dots, x^m) \in \boldsymbol{R}^{\mathrm{nm}} : x^i \geqq 0, \ i = 1, \dots, n, \text{ and } \sum_{i=1}^m x^i \leqq \bar{x} \right\}.$$

(This implicitly assumes free disposal of factors and outputs, but this is not crucial.)

The **production possibility frontier** (PPF) is the outer boundary of the PPS. The production possibility frontier can be described by the following maximization problem. (Here each $x_i \in \boldsymbol{R}^{\mathrm{n}}_+$ and x^i_j denotes the quantity of factor j used to produce good i.)

$$\underset{x^1,\dots,x^m}{\text{maximize}} \ f^1(x^1) \quad \text{subject to}$$

$$\begin{aligned} f^i(x^i) &= \eta_i & i &= 2, \dots, m \\ \sum_{i=1}^n x^i_j &= \bar{x}_j & j &= 1, \dots, n \\ x^i &\geqq 0 & i &= 1, \dots, m. \end{aligned}$$

The Lagrangean for this maximization is:

$$L(x, \lambda, \mu; \eta, \bar{x}) =$$
$$f^1(x_1^1, \ldots, x_n^1) + \sum_{i=1}^{m} \lambda_i (f^i(x_1^i, \ldots, x_n^i) - \eta_i) + \sum_{j=1}^{n} \mu_j \left(\bar{x}_j - \sum_{i=1}^{m} x_j^i \right).$$

Q: Are the gradients of the constraints (wrt x) linearly independent? The answer is yes.

$$\begin{array}{c|ccccccc}
 & \lambda_2 & \cdots & \cdots & \lambda_m & \mu_1 & \cdots & \mu_n \\
\hline
x_1^1 & 0 & \cdots & \cdots & 0 & -1 & & 0 \\
\vdots & \vdots & & & \vdots & & \ddots & \\
x_n^1 & 0 & \cdots & \cdots & 0 & 0 & & -1 \\
x_1^2 & f_1^2 & 0 & \cdots & 0 & -1 & & 0 \\
\vdots & \vdots & \vdots & & \vdots & & \ddots & \\
x_n^2 & f_n^2 & 0 & \cdots & 0 & 0 & & -1 \\
\vdots & & \ddots & & & \vdots & & \vdots \\
\vdots & & & \ddots & & \vdots & & \vdots \\
x_1^m & 0 & \cdots & 0 & f_1^m & -1 & & 0 \\
\vdots & \vdots & & \vdots & & & \ddots & \\
x_n^m & 0 & \cdots & 0 & f_n^m & 0 & & -1
\end{array}$$

Figure 1. The columns are the gradients of the constraints.

To see this it might help to consult Figure 1. Suppose $\lambda_2, \ldots, \lambda_m, \mu_1, \ldots, \mu_n$ yield a linear combination of the gradients that adds up to the zero vector. Then clearly $\mu_1 = \cdots = \mu_n = 0$. Thus since each $f_j^i > 0$, we get $\lambda_i = 0$, for all i.

Thus by the Lagrange Multiplier Theorem the first order conditions are (assuming each $x_j^i > 0$):

$$\lambda_i f_j^i - \mu_j = 0 \qquad \begin{array}{l} i = 1, \ldots, m \\ j = 1, \ldots, n \end{array}$$

where for symmetry we define $\lambda_1 = 1$. This implies

$$\lambda_i = \frac{f_j^1}{f_j^i}$$

for any factor $j = 1, \ldots, n$.

Let $y_1(w, \bar{x})$ be the optimal value function. Then by the Envelope Theorem, the slope of the PPF satisfies

$$\begin{aligned}\frac{\partial y_1}{\partial \eta_i} &= \frac{\partial L}{\partial \eta_i} \\ &= -\lambda_i \\ &= -\frac{f_j^1}{f_j^i}\end{aligned}$$

for any $j = 1, \ldots, n$. In other words, λ_i is the marginal opportunity cost of a unit of y_i in terms of y_1.

Also note that

$$\frac{f_j^i}{f_k^i} = \frac{\mu_j}{\mu_k},$$

which is independent of i. That is, in every industry the slopes of the isoquants are the same.

1.1 Second order conditions

While we're at it, let's check the second order conditions. The Hessian of the Lagrangean is the $mn \times mn$ block-diagonal matrix

$$H = \begin{bmatrix} \begin{matrix}\lambda_1 f_{11}^1 & \cdots & \lambda_1 f_{1n}^1 \\ \vdots & & \vdots \\ \lambda_1 f_{n1}^1 & \cdots & \lambda_1 f_{nn}^1\end{matrix} & & & \\ & \begin{matrix}\lambda_2 f_{11}^2 & \cdots & \lambda_2 f_{1n}^2 \\ \vdots & & \vdots \\ \lambda_2 f_{n1}^2 & \cdots & \lambda_2 f_{nn}^2\end{matrix} & & \\ & & \ddots & \\ & & & \begin{matrix}\lambda_m f_{11}^m & \cdots & \lambda_m f_{1n}^m \\ \vdots & & \vdots \\ \lambda_m f_{n1}^m & \cdots & \lambda_m f_{nn}^m\end{matrix} \end{bmatrix}$$

Let $v = (v^1, \ldots, v^m)$ belong to $\boldsymbol{R}^{\mathrm{n}m}$. The second order condition is that the quadratic form $v \cdot Hv$ is negative semidefinite on the subspace orthogonal to the gradients of the constraints. Again referring to Figure 1, it is straightforward to see that this means

$$\sum_{1=1}^{m} \left[\sum_{j=1}^{n} \sum_{k=1}^{n} \lambda_i f_{jk}^i v_j^i v_k^i \right] \leqslant 0$$

for all nonzero v satisfying

$$\nabla f^i \cdot v^i = \sum_{j=1}^{n} f_j^i v_j^i = 0, \qquad i = 2, \ldots, m,$$

and

$$\sum_{i=1}^{m} v_j^i = 0 \qquad j = 1, \dots, n.$$

What about the case $i = 1$? If we can show that $\nabla f^1 \cdot v^1 = 0$, then by our assumption on the gradients of the f^is, each $\lambda_i > 0$, so by the assumption on the Hessian of the f^is, each bracketed term is nonpositive, and at least one is strictly negative (since at least one $v^i \neq 0$).

To see that $\nabla f^1 \cdot v^1 = 0$, observe that for each j, $v_j^1 = -\sum_{i=2}^{m} v_j^i$. Thus

$$\begin{aligned}
\nabla f^1 \cdot v^1 &= \sum_{j=1}^{n} f_j^1 v_j^1 \\
&= -\sum_{j=1}^{n} f_j^1 \sum_{i=2}^{m} v_j^i \\
&= -\sum_{i=2}^{m} \left[\sum_{j=1}^{n} \lambda_i f_j^i v_j^i \right] \\
&= 0.
\end{aligned}$$

The penultimate equality follows from the first order condition that $\lambda_i f_j^i = \mu_j = f_j^1$ for all i.

2 Relation to cost minimization

Assume that each producer faces the same wages $w = (w_1, \dots, w_n)$ for the factors and minimizes costs. To ease notation in this section, I shall suppress the superscripts denoting the particular output.

The cost minimization problem is to

$$\text{minimize } w \cdot x \quad \text{subject to } y \leqslant f(x).$$

Form the Lagrangean

$$L(x, \gamma; w, y) = w \cdot x + \gamma(y - f(x)).$$

The value function is the cost function $c(w, y)$. By the Envelope Theorem, the marginal cost is

$$\text{MC} = \frac{\partial c}{\partial y} = \frac{\partial L}{\partial y} = \gamma.$$

We also have the first order conditions (check the gradient of the constraint):

$$w_j - \gamma f_j = 0, \qquad j = 1, \dots, n$$

assuming each $x_j > 0$. (Note that these implies $\gamma > 0$.) In other words,

$$f_j = \frac{w_j}{\text{MC}}$$

Now back to the PPF. If all firms face the same wages and minimize costs, then

$$\begin{aligned}\frac{\partial y_1}{\partial \eta_i} &= -\lambda_i \\ &= -\frac{f_j^1}{f_j^i} \\ &= -\frac{\frac{w_j}{\mathrm{MC}_1}}{\frac{w_j}{\mathrm{MC}\,i}} \\ &= -\frac{MC_i}{MC_1}.\end{aligned}$$

That is, the marginal opportunity cost of one unit of y_i expressed in terms of y_1 is exactly the ratio of the marginal cost of a unit of y_i (calculated in terms of wages) relative to the marginal cost of a unit of y_1. What this tells us is that marginal costs (derived from wages) indicate real opportunity costs!

2.1 Extensions

What if there are several production functions for each y_i? Call them $f^{i,1}, \dots, f^{i,p_1}$.

Then

$$\lambda_{i,k} f_j^{i,k} - \mu_j = 0,$$

and we proceed as before.

What if there are joint products? Describe feasibility as

$$T(y_1, \dots, y_m, x_1^1, \dots, x_n^1, \dots, x_1^m, \dots, x_n^m) \geqslant 0$$

where each $\frac{\partial T}{\partial y_i} < 0$ and each $\frac{\partial T}{\partial x_j^i} > 0$, and consider the Lagrangean

$$y_1 + \lambda T(y^1, \eta_2 \dots, \eta_m, x_1^1, \dots, x_n^1, \dots, x_1^m, \dots, x_n^m) + \sum_{j=1}^n \mu_j \left(\bar{x}_j - \sum_{i=1}^m x_j^i \right).$$

Also how do we deal seriously with the nonnegativity constraints?

References

[1] Lerner, A. P. 1934. The concept of monopoly and the measurement of monopoly power. *Review of Economic Studies* 1(3):157–175.