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Notes on the Theory of Linear Programming

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1 The saddlepoint theorem for linear programming

The material for this handout is based largely on the beautifully written book by David Gale [2].

A **maximum linear program in standard inequality form**[1] is a constrained maximization problem of the form

$$\underset{x}{\text{maximize}}\ p \cdot x$$

subject to

$$xA \leqq q \tag{1}$$
$$x \geqq 0 \tag{2}$$

where x and p belong to $\boldsymbol{R}^{\mathrm{n}}$, q belongs to $\boldsymbol{R}^{\mathrm{m}}$, and A is $n \times m$. The program is **feasible** if there is some x satisfying the constraints (1) and (2). Every maximum linear program in standard inequality form has a **dual program**, which is the minimization problem:

$$\underset{y}{\text{minimize}}\ q \cdot y \tag{3}$$

subject to

$$Ay \geqq p \tag{4}$$
$$y \geqq 0. \tag{5}$$

The original maximum linear program may be called the **primal** program to distinguish it from the dual.

Let us start by examining the Lagrangean for the primal program. Write (1) as $q_j - (xA)_j \geqslant 0$, $j = 1, \ldots, m$, and let y_j denote the Lagrange multiplier for this constraint. Incorporate (2) by setting the domain $X = \boldsymbol{R}^{\mathrm{n}}_{+}$. The **Lagrangean** is then

$$\mathcal{L}(x, y) = p \cdot x + q \cdot y - xAy. \tag{6}$$

[1]Gale [2] refers to this as the "standard form" of a linear program. However Dantzig [1] uses the term standard form in a different fashion.

Treating the dual as the problem of maximizing $-q \cdot y$ subject to $Ay - p \geqq 0$, and using x to denote the vector of Lagrange multipliers, the Lagrangean for the dual is:

$$q \cdot y - xAy + x \cdot p,$$

which is just (6) again. Consequently, by the easy half of the Saddlepoint Theorem, if $(\bar{x}, \bar{y})$ is a saddlepoint of $L(x, y) = p \cdot x + q \cdot y - xAy$ over $\boldsymbol{R}^{\mathrm{n}} \times \boldsymbol{R}^{\mathrm{m}}$, then $\bar{x}$ is optimal for the primal program and $\bar{y}$ is optimal (minimal) for the dual program. In particular, if there is a saddlepoint, then both programs are feasible. If we knew that both programs satisfied Slater's Condition, then the Saddlepoint Theorem would assert that any pair of optimal solutions would be a saddlepoint of the Lagrangean. *Remarkably, for the linear programming case, we do not need Slater's Condition.*

1 Saddlepoint Theorem for Linear Programming *The following are equivalent.*

1. *The Lagrangean*
$$\mathcal{L}(x, y) = p \cdot x + q \cdot y - xAy$$
has a saddlepoint over $\boldsymbol{R}^{\mathrm{n}}_{+} \times \boldsymbol{R}^{\mathrm{m}}_{+}$.

2. *The primal has an optimal solution.*

3. *The dual has an optimal solution.*

4. *Both the primal and dual are feasible.*

Also, the following are equivalent.

a. $(\bar{x}, \bar{y})$ *is a saddlepoint of the Lagrangean.*

b. $\bar{x}$ *is optimal for the primal and* $\bar{y}$ *is optimal for the dual.*

c. $\bar{x}$ *is feasible for the primal program and* $\bar{y}$ *is feasible for the dual, and* $p \cdot \bar{x} = q \cdot \bar{y}$.

The proof is broken down into a series of lemmas. The first of which is both easy and useful.

2 Lemma (Optimality Criterion for LP) *If x is feasible for the primal program and y is feasible for its dual, then $p \cdot x \leqslant q \cdot y$.*

If in addition $p \cdot x = q \cdot y$, then x is optimal and y is optimal for the dual program, and $p \cdot x = q \cdot y = xAy$.

Proof: Suppose x satisfies (1) and $y \geqq 0$. Then $xAy \leqslant q \cdot y$. Likewise if y satisfies (4) and $x \geqq 0$, then $xAy \geqslant x \cdot p$. Combining these pieces proves the lemma. ∎

If by some means we have found an optimum for the primal and the dual, then the Optimality Criterion allows a simple way to prove that it is an optimum. Most other optimization techniques do not include a proof that the outcome is optimal.

The remaining gap is to show that if $\bar{x}$ is optimal, then the dual has an optimal solution $\bar{y}$ and that $p \cdot \bar{x} = q \cdot \bar{y}$ (instead of $p \cdot \bar{x} < q \cdot \bar{y}$). This brings us to the following.

3 Fundamental Duality Theorem of LP *If both a maximum linear program in standard inequality form and its dual are feasible, then both have optimal solutions, and the values of the two programs are the same. If one of the programs is infeasible, neither has an optimum.*

Proof: (Gale [2]) Start by assuming both programs are feasible. We already know that if x and y are feasible for the primal and dual respectively, then $p \cdot x \leqslant q \cdot y$. Thus it suffices to find a solution $(x, y) \geqq 0$ to the inequalities

$$\begin{aligned} xA &\leqq q \\ -yA' &\leqq -p \\ y \cdot q - x \cdot p &\leqslant 0, \end{aligned}$$

or, in matrix form

$$[x, y] \begin{bmatrix} A & 0 & -p \\ 0 & -A' & q \end{bmatrix} \leqq [q, -p, 0]. \tag{7}$$

Either these inequalities have a solution, or else by a Theorem of the Alternative (see, e.g., Gale [2, Theorem 2.9, p. 49]), there is a nonnegative vector $\begin{bmatrix} u \\ v \\ \alpha \end{bmatrix} \geqq 0$, where $u \in \boldsymbol{R}^{\mathrm{m}}_{+}$, $v \in \boldsymbol{R}^{\mathrm{n}}_{+}$, and $\alpha \in \boldsymbol{R}_{+}$, satisfying

$$\begin{bmatrix} A & 0 & -p \\ 0 & -A & q \end{bmatrix} \begin{bmatrix} u \\ v \\ \alpha \end{bmatrix} \geqq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{8}$$

and

$$[q, -p, 0] \begin{bmatrix} u \\ v \\ \alpha \end{bmatrix} < 0. \tag{9}$$

We shall show that this latter set of inequalities does not have a solution: Suppose by way of contradiction that (8) and (9) have a nonnegative solution. Rewriting (8), we have

$$Au \geqq \alpha p \tag{10}$$

and

$$vA \leqq \alpha q, \tag{11}$$

while (9) becomes

$$q \cdot u < p \cdot v. \tag{12}$$

Let $\bar{x} \geqq 0$ be feasible for the primal, that is, $\bar{x}A \leqq q$. Then

$$\bar{x}Au \leqslant q \cdot u \tag{13}$$

since $u \geqq 0$. Similarly let $\bar{y} \geqq 0$ be feasible for the dual, that is, $A\bar{y} \geqq p$. Then

$$vA\bar{y} \geqslant v \cdot p \tag{14}$$

since $v \geqq 0$.

We next show that $\alpha \neq 0$. For suppose $\alpha = 0$. Then (10) becomes $Au \geqq 0$, which implies

$$\bar{x}Au \geqslant 0,$$

since $\bar{x} \geqq 0$. Also (11) implies

$$vA\bar{y} \leqslant 0,$$

since $\bar{y} \geqq 0$. Combining this with (13) and (14) yields

$$q \cdot u \geqslant \bar{x}Au \geqslant 0 \geqslant vA\bar{y} \geqslant v \cdot p,$$

which contradicts (12).

This shows that $\alpha > 0$, so we may without loss of generality assume $\alpha = 1$. In this case, (10) becomes $Au \geqq p$ and (11) becomes $vA \leqq q$, which imply that v is feasible for the primal program and u is feasible for the dual. Therefore, by Lemma 2, $q \cdot u \geqslant p \cdot v$, which again contradicts (12). This contradiction shows that if both programs are feasible, then both have optimal solutions and both programs have the same value.

If either program is infeasible, then certainly it cannot have an optimal solution. So suppose that the primal program is infeasible, but the dual is feasible. That is, $xA \leqq q$ has no nonnegative solution, so by the theorem of the alternative again, there is a nonnegative y satisfying $Ay \geqq 0$ and $q \cdot y < 0$. Let z be any feasible nonnegative solution to the dual. Then $z + \alpha y$ is feasible for any $\alpha \geqslant 0$, and $q \cdot (z + \alpha y) = q \cdot z + \alpha q \cdot y \to -\infty$ as $\alpha \to \infty$. Therefore no optimal solution exists for the dual.

A similar argument works if the dual is infeasible, but the primal is feasible. ∎

Proof of Theorem 1: By the easy half of the Saddlepoint Theorem, if $(\bar{x}, \bar{y})$ is a saddlepoint of the Lagrangean, then $\bar{x}$ is optimal for the primal and $\bar{y}$ is optimal for the dual. Thus (1) $\implies$ (2) & (3) and $(a) \implies (b)$. By the Optimality Criterion 2 if $\bar{x}$ is optimal for the primal and $\bar{y}$ is optimal for the dual, then $(\bar{x}, \bar{y})$ is a saddlepoint of the Lagrangean. It follows that $(b) \implies (c) \implies (a)$ and (2) & (3) $\implies$ (1).

Clearly (2) & (3) $\implies$ (4). The Fundamental Duality Theorem 3 implies (2) $\iff$ (3) and that (4) $\implies$ (2) & (3).

This finishes the proof of the Saddlepoint Theorem for Linear Programing. ∎

Gale refers to the following result as the **Equilibrium Theorem**. It is also known as the **Complementary Slackness Theorem**. It is a simple consequence of the Optimality Criterion (Lemma 2).

4 Complementary Slackness Theorem *Suppose x and y are feasible for the primal and dual respectively. They are optimal if and only if both*

$$(xA)_j < q_j \implies y_j = 0 \tag{15}$$

and

$$(Ay)_i > p_i \implies x_i = 0. \tag{16}$$

Proof: Suppose x and y are feasible for the primal and dual respectively. From $xA \leqq q$ and $y \geqq 0$, we have $xAy \leqslant q \cdot y$ with equality if and only if (15) holds. Similarly, (16) holds if and only if $xAy = p \cdot x$. The conclusion now follows from the Optimality Criterion 2, which says that x and y are optimal if and only if $p \cdot x = q \cdot y = xAy$. ∎

You may suspect that it is possible to combine linear constraints with more general concave constraints that satisfy Slater's Condition. This is indeed the case as Uzawa [4] has shown. (See also Moore [3].)

2 Other formulations

Not every linear program comes to us already in standard inequality form, nor is the inequality form always the easiest to work with. There are other forms, some of which have names, and all of which can be translated into one another. In fact, we just translated a standard minimum inequality form into a standard maximum inequality form above. Each of these forms also has a dual, and the program and its dual satisfy the Fundamental Duality Theorem of LP 3. That is, if both a linear program (in any form) and its dual are feasible, then both have optimal solutions, and the values of the two programs are the same. If one of the programs is infeasible, neither has a solution. Table 1 summarizes these forms and their dual programs.

2.1 The primal is the dual of the dual

Remember that the dual of a maximum linear program in standard inequality form is a linear program of the form

$$\underset{y}{\text{minimize}} \ \ q \cdot y$$

subject to

$$\begin{aligned} Ay &\geqq p \\ y &\geqq 0 \end{aligned}$$

where x and p belong to $\boldsymbol{R}^{\mathrm{n}}$, q belongs to $\boldsymbol{R}^{\mathrm{m}}$, and A is $n \times m$. Let us call this a **minimum linear program in standard inequality form**. Now the dual program itself can be rewritten as the following maximum LP in standard inequality form:

$$\underset{y}{\text{maximize}} \ \ -q \cdot y$$

subject to

$$\begin{aligned} y(-A') &\leqq -p \\ y &\geqq 0, \end{aligned}$$

where A' is the transpose of A. The dual of this program is:

$$\underset{x}{\text{minimize}} \ \ -p \cdot x$$

subject to

$$\begin{aligned} -A'x &\geqq -p \\ x &\geqq 0, \end{aligned}$$

or

$$\underset{x}{\text{maximize}} \; p \cdot x$$

subject to

$$\begin{aligned} xA &\leqq p \\ x &\geqq 0, \end{aligned}$$

which is the primal. Thus the dual of a minimum LP in standard inequality form is a maximum LP in standard inequality form, and vice-versa. Moreover the dual of the dual is the primal.

2.2 The general form

Let us start with a linear program in **general maximum form**, which allows for both linear inequalities and equations, and optional sign constraints on the components of x.

$$\underset{x}{\text{maximize}} \; p \cdot x = \sum_{j=1}^{n} p_j x_j$$

subject to

$$\begin{aligned} x \cdot A^j &\leqslant q_j & j \in J_L \\ x \cdot A^j &= q_j & j \in J_E \\ x \cdot A^j &\geqslant q_j & j \in J_G \\ x_i &\leqslant 0 & i \in I_N \\ x_i & \text{ free} & i \in I_F \\ x_i &\geqslant 0 & i \in I_P \end{aligned}$$

where $J_L \cup J_E \cup J_G = \{1, \ldots, n\}$, and $I_N \cup I_F \cup I_P = \{1, \ldots, m\}$.

We can translate this into standard inequality maximum form as follows. Start by rewriting all the constraints as $\leqslant$ inequalities,

$$\begin{aligned} x \cdot A^j &\leqslant q_j & j \in J_L \\ x \cdot A^j &\leqslant q_j & j \in J_E \\ x \cdot (-A^j) &\leqslant -q_j & j \in J_E \\ x \cdot (-A^j) &\leqslant -q_j & j \in J_G. \end{aligned}$$

Next replace x by $u - v$, where both $\geqq 0$ and $v \geqq 0$. This palces no sign restrictions on the components of $u - v$. To capture the requirement that $x_i \leqslant 0$ for $i \in I_N$, we require $u \cdot e^i \leqslant 0$, where e^i is the i^{th} unit coordinate vector in $\boldsymbol{R}^{\mathrm{n}}$. (Do you see why this works?) Similalry $x_i \geqslant 0$ corresponds to $v \cdot e^i \leqslant 0$. Thus our rewritten problem is

$$\underset{u,v}{\text{maximize}} \ p \cdot (u - v)$$

subject to

$$\begin{aligned}
(u-v)\cdot A^j &\leqslant & q_j \quad & j \in J_L \\
(u-v)\cdot A^j &\leqslant & q_j \quad & j \in J_E \\
(u-v)\cdot(-A^j) &\leqslant & -q_j \quad & j \in J_E \\
(u-v)\cdot(-A^j) &\leqslant & -q_j \quad & j \in J_G \\
u \cdot e^i &\leqslant & 0 \quad & i \in I_N \\
v \cdot e^i &\leqslant & 0 \quad & i \in I_P \\
u &\geqq & 0 & \\
v &\geqq & 0. &
\end{aligned}$$

Or in matrix form

$$\underset{u,v}{\text{maximize}} \ (p, -p) \cdot (u, v)$$

subject to

$$(u,v)\begin{pmatrix} A^L & A^E & -A^E & -A^G & I^N & 0^P \\ -A^L & -A^E & A^E & A^G & 0^N & I^P \end{pmatrix} \geqq (q_L, q_E, -q_E, -q_G, 0_N, 0_P),$$

where A^L, A^E, and A^G are m-rowed matrices whose columns are A^j for $j \in J_M$, $j \in J_E$, and $j \in_G$ respectively, the columns of I^N and I^P are unit coordinate vectors e^i for $i \in N$ and $i \in P$ respectively, and q_L, q_E, and q_P have components q_j for $j \in L$, $j \in E$, and $j \in G$ respectively. The zeros are of the dimension they need to be.

The dual of this maximum problem in standard inequality form is thus the following:

$$\underset{z,w}{\text{minimize}} \ (q_L, q_E, -q_E, -q_G, 0_N, 0_P) \cdot (z_L, z_E, w_E, z_G, z_N, z_P)$$

subject to $z \geqq 0$, $w \geqq 0$, and

$$\begin{pmatrix} A^L & A^E & -A^E & -A^G & I^N & 0^P \\ -A^L & -A^E & A^E & A^G & 0^N & I^P \end{pmatrix} \begin{pmatrix} z_L \\ z_E \\ w_E \\ z_G \\ z_N \\ z_P \end{pmatrix} \geqq \begin{pmatrix} p \\ -p \end{pmatrix},$$

where z_L, z_E, w_E, z_G, z_N, z_P are of the appropriate dimensions.

Define $y \in \boldsymbol{R}^{\mathrm{m}}$ by

$$y_j = \begin{cases} z_j & j \in J_L \\ z_j - w_j & j \in J_E \\ -z_j & j \in J_G \end{cases}$$

so that

$$\begin{array}{cccc} y_j & \geqslant & 0 & j \in J_L \\ y_j & \text{unsigned} & & j \in J_E \\ y_j & \leqslant & 0 & j \in J_G \end{array}$$

and rewrite the dual as

$$\underset{y}{\text{minimize}}\ q \cdot y$$

subject to $z \geqq 0$, $w \geqq 0$, and

$$\begin{pmatrix} A \\ -A \end{pmatrix} y + \begin{pmatrix} I^N & 0 \\ 0 & I^P \end{pmatrix} \begin{pmatrix} z_N \\ z_P \end{pmatrix} \geqq \begin{pmatrix} p \\ -p \end{pmatrix},$$

where A is the $m \times n$ matrix of columns $A^1, \ldots, A^n$. What this says is

$$\begin{aligned} Ay + \sum_{i \in N} z_i e^i &\geqq p \\ Ay - \sum_{i \in P} z_i e^i &\leqslant p \end{aligned}$$

Since $z \geqq 0$, the dual can be written:

$$\underset{y}{\text{minimize}}\ q \cdot y$$

subject to

$$\begin{array}{cccc} y_j & \geqslant & 0 & j \in J_L \\ y_j & \text{free} & & j \in J_E \\ y_j & \leqslant & 0 & j \in J_G \\ A_i y & \geqslant & p_i & i \in I_N \\ A_i y & = & p_i & i \in I_F \\ A_i y & \leqslant & p_i & i \in I_P \end{array}$$

Recall that the variables in the dual are the Lagrange multipliers for the primal. Thus we see that, the Lagrange multipliers associated with the equality constraints ($j \in J_E$) are not a priori restricted in sign, while the multipliers for the $\leqslant$ inequality constraints ($i \in J_L$) are nonnegative, and the multipliers for the $\geqslant$ inequality constraints ($i \in J_G$) are nonpositive. Since the primal variables are the Lagrange multipliers for the dual program, the nonnegativity constraints ($i \in I_P$) on the primal correspond to $\leqslant$ inequality constraints in the dual, the nonpositivity constraints ($i \in I_N$) on the primal correspond to $\geqslant$ inequality constraints in the dual, and the unrestricted primal variable are associated with equality constraints in the dual.

2.3 Canonical (equality) form

There is one more useful form for linear programs, the **canonical** or **equality form**.[2] In it, all the constraints are equations, and all the variables are nonnegative. An LP is in **canonical maximum form** if it is written as:

$$\underset{x}{\text{maximize}}\ p \cdot x$$

subject to

$$\begin{aligned} xA &= q \\ x &\geqq 0 \end{aligned}$$

To transform an inequality form into the equality form, introduce slack variables $x \in \boldsymbol{R}^{\text{m}}$ and observe that

$$xA \leqq q \qquad \Longleftrightarrow \qquad xA + z = q,\ z \geqq 0.$$

It follows from the characterization of the dual to the general maximum problem that the dual program can be written as the decidedly non-equality minimum problem

$$\underset{y}{\text{minimize}}\ q \cdot y$$

subject to

$$Ay \geqq p$$

Note the lack of sign restrictions on y.

3 Linear equations as LPs

It is possible to recast the problem of solving linear equations and inequalities as LP problems. Consider the problem of finding a nonnegative solution to a system of equations. That is, find x such that

$$xA = q$$

$$x \geqq 0.$$

Consider the linear program in equality minimum form:

$$\underset{x,z}{\text{minimize}}\ \mathbf{1} \cdot z$$

subject to

$$\begin{aligned} xA + z &= q \\ [x, z] &\geqq 0 \end{aligned}$$

[2] The equality form is what Dantzig [1] calls the standard form, and what Gale [2] calls the canonical form. Dantzig uses the term canonical in a different fashion.

Primal program		**Dual program**
General maximum form		General minimum form
$\text{maximize}_x \ p \cdot x$ subject to $xA^j \leqslant q_j \quad j \in J_L$ $xA^j = q_j \quad j \in J_E$ $xA^j \geqslant q_j \quad j \in J_G$ $x_i \geqslant 0 \quad i \in I_N$ x_i free $\quad i \in I_F$ $x_i \leqslant 0 \quad i \in I_P$		$\text{minimize}_y \ q \cdot y$ subject to $y_j \geqslant 0 \quad j \in J_L$ y_j free $\quad j \in J_E$ $y_j \leqslant 0 \quad j \in J_G$ $A_i y \geqslant p_i \quad i \in I_N$ $A_i y = p_i \quad i \in I_F$ $A_i y \leqslant p_i \quad i \in I_P$
Standard maximum form		Standard minimum form
$\text{maximize}_x \ p \cdot x$ subject to $xA \leqq q$ $x \geqq 0$		$\text{minimize}_y \ q \cdot y$ subject to $y \geqq 0$ $Ay \geqq p$
Canonical maximum form		
$\text{maximize}_x \ p \cdot x$ subject to $xA = q$ $x \geqq 0$		$\text{minimize}_y \ q \cdot y$ subject to y free $Ay \geqq p$
Canonical minimum form		
$\text{minimize}_x \ p \cdot x$ subject to $Ax = q$ $x \geqq 0$		$\text{maximize}_y \ q \cdot y$ subject to y free $yA \leqq p$

Table 1. Selected forms of linear programs and their duals.

The characterization of the general form implies the other results. The primal constraints may also be expressed in terms of Ax rather than xA, in which case the dual is expressed in terms of yA instead of Ay.

Here $\mathbf{1}$ is the vector whose components are all 1. Without loss of generality we may assume $q \geqq 0$, for if $q_j < 0$ we may multiply A^j and q^j by -1 without affecting the solution set. Then note that this program is feasible, since $x = 0$, $z = q$ is a nonnegative feasible solution. Since we require $z \geqq 0$, we have $\mathbf{1} \cdot z \geqslant 0$ and $\mathbf{1} \cdot z = 0$ if and only if $z = 0$, in which case $xA = q$. Thus, if this linear program has value 0 if and only $xA = q$, $x \geqq 0$ has a solution, and any optimal (x, z) provides a nonnegative solution to the equation.

At this point you might be inclined to say "so what?" In another handout, I will describe the simplex algorithm, which is a special version of Gauss–Jordan elimination, that is a reasonably efficient and easily programmable method for solving linear programs. In other words, it also finds nonnegative solutions to linear equations when they exist.

References

[1] G. B. Dantzig. 1963. *Linear programming and extensions*. Princeton: Princeton University Press.

[2] D. Gale. 1989. *Theory of linear economic models*. Chicago: University of Chicago Press. Reprint of the 1960 edition published by McGraw-Hill.

[3] J. C. Moore. 1968. A note on point-set mappings. In J. P. Quirk and A. M. Zarley, eds., *Papers in Quantitative Economics, 1*, pages 129–137. Lawrence, Kansas: University of Kansas Press.

[4] H. Uzawa. 1958. The Kuhn–Tucker conditions in concave programming. In K. J. Arrow, L. Hurwicz, and H. Uzawa, eds., *Studies in Linear and Non-linear Programming*, number 2 in Stanford Mathematical Studies in the Social Sciences, chapter 3, pages 32–37. Stanford, California: Stanford University Press.