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Investment and Risk Aversion

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There are two assets, a safe asset that returns $(1 + r_0)$ for each dollar invested and risky asset that returns $(1 + \boldsymbol{r})$ for each dollar invested, where $\boldsymbol{r}$ is a nondegenerate random variable.

If his wealth is $\hat{w}$, an expected utility maximizing investor will choose the amount x to invest in the risky asset to maximize

$$\boldsymbol{E}\, u\big((\hat{w} - x)(1 + r_0) + x(1 + \boldsymbol{r})\big).$$

The difference $\boldsymbol{r} - r_0$ is the excess of $\boldsymbol{r}$ over the safe return, so for convenience, let us call it $\boldsymbol{q}$, i.e., $\boldsymbol{q} = \boldsymbol{r} - r_0$, and set $w = (1 + r_0)\hat{w}$. Thus x is chosen to maximize

$$\boldsymbol{E}\, u(w + x\boldsymbol{q}),$$

which is a prettier problem.

There are some questions that are frequently glossed over in the literature. One is whether we want to restrict x to lie in the interval $[0, w]$. If so, we have to worry about boundary conditions. We also have to worry whether $w + x\boldsymbol{q}$ lies in the domain of the utility function with probability one. For instance, a utility function that is commonly studied is the logarithmic utility $u(w) = \ln w$ (where $u(0) = -\infty$ is allowed). If we make the limited liability assumption that $1 + \boldsymbol{r} \geqslant 0$ a.s., and also restrict x to $[0, w]$, then we have no problems in that regard. On the other hand, we may actually want to allow borrowing $(x > w)$ and/or short selling $(x < 0)$. In that case, we probably need to have the utility defined on the whole real line, which rules out the logarithmic utility, among others.

In what follows, I shall assume that utilities are defined on an interval D of the real line, are continuous strictly increasing functions on D that are twice continuously differentiable, with strictly positive derivatives everywhere on the interior of D, and that a solution exists and is interior to the domain.

The first order necessary condition for an interior maximum[1] is

$$\boldsymbol{E}\, u'(w + x^*\boldsymbol{q})\boldsymbol{q} = 0. \tag{$\star$}$$

Observe that ($\star$) has a solution only if $\boldsymbol{q} < 0$ with positive probability, which makes perfect economic sense. (Otherwise there would be an arbitrage opportunity: borrow at r_0 and invest at $\boldsymbol{r}$, earning a riskless profit.)

The second order necessary condition is

$$\boldsymbol{E}\, u''(w + x^*\boldsymbol{q})\boldsymbol{q}^2 \leqslant 0.$$

If u is concave, then $u'' \leqslant 0$, so the second order condition is automatically satisfied. I may also assume that the strong second order condition

$$\boldsymbol{E}\, u''(w + x^*\boldsymbol{q})\boldsymbol{q}^2 < 0$$

holds at a particular solution. This is usually necessary to make the solution a differentiable function of the parameters.

1 A trivial lemma

Lemma 1 *Let f be a nondecreasing real function on an interval I, let x belong to I, and let $\alpha > 0$. Then for any v for which $x + \alpha v \in I$, we have*

$$f(x + \alpha v)v \geqslant f(x)v.$$

This equality is reversed if $\alpha < 0$ or if f is nonincreasing. The inequality is strict provided $v \neq 0$ and f is not constant on the interval from x to $x + \alpha v$.

Proof: We prove the claim for $\alpha > 0$, the others are obvious from its proof. There are two interesting cases: $v > 0$ and $v < 0$. When $v > 0$, then the monotonicity of f implies $f(x + \alpha v) \geqslant f(x)$, so $f(x + \alpha v)v \geqslant f(x)v$. And when $v < 0$, then $f(x+\alpha v) \leqslant f(x)$, but multiplying by the negative quantity v reverses the inequality, so again $f(x + \alpha v)v \geqslant f(x)v$. ∎

2 Decreasing risk aversion

A natural comparative statics question is: What happens to x^* as a function of w?

[1] See, e.g., Hildreth [2, 3] and Hildreth and Tesfatsion [4], or my on-line notes [1], for technical details on sufficient conditions to be able to differentiate under an expectation.

Proposition 2 *Assume u is C^2 and $u' > 0$, and define the Arrow–Pratt coefficient of risk aversion $r(w) = \dfrac{-u''(w)}{u'(w)}$. Fix w_0, and assume that x_0^* satisfies the strong second order condition. Then there is a neighborhood of w_0 on which x^* is a C^1 function of w.*

Moreover, if r is decreasing at w_0, then x^* is increasing at w_0 if x_0^* is positive and decreasing if x_0^* is negative. If, on the other hand, r is increasing at w_0, then $x^*(w)$ is decreasing when x_0^* is positive and increasing when x_0^* is negative.

Proof: Now x_0^* satisfies the first order condition

$$\boldsymbol{E}\, u'(w_0 + x_0^* \boldsymbol{q})\boldsymbol{q} = 0.$$

By the strong second order condition, the Implicit Function Theorem implies that x^* is a C^1 function of w on an appropriate neighborhood of w_0. Thus differentiating the first order condition with respect to w gives

$$\boldsymbol{E}\, u''(w_0 + x_0^* \boldsymbol{q})\boldsymbol{q} \left(1 + \boldsymbol{q}\frac{dx^*(w_0)}{dw}\right) = 0$$

or

$$\frac{dx^*(w_0)}{dw} = -\frac{\boldsymbol{E}\, u''(w_0 + x_0^* \boldsymbol{q})\boldsymbol{q}}{\boldsymbol{E}\, u''(w + x_0^* \boldsymbol{q})\boldsymbol{q}^2}.$$

The strong second order condition implies that the denominator is negative so the sign of $\dfrac{dx^*(w_0)}{dw}$ is the sign of $\boldsymbol{E}\, u''(w_0 + x_0^* \boldsymbol{q})\boldsymbol{q}$.

Now suppose $r(w)$ is decreasing at w_0. Consider first the case $x_0^* > 0$. By Lemma 1,

$$r(w_0 + x_0^* \boldsymbol{q})\boldsymbol{q} \leqslant r(w_0)\boldsymbol{q}.$$

Therefore, recalling the definition of r and multiplying by the negative quantity $-u'(w_0 + x_0^* \boldsymbol{q})$, we have

$$u''(w_0 + x_0^* \boldsymbol{q})\boldsymbol{q} \geqslant -r(w_0)u'(w_0 + x_0^* \boldsymbol{q})\boldsymbol{q}.$$

Taking the expectation of both sides gives

$$\boldsymbol{E}\, u''(w_0 + x_0^* \boldsymbol{q})\boldsymbol{q} \geqslant -r(w_0)\, \boldsymbol{E}\, u'(w_0 + x_0^* \boldsymbol{q})\boldsymbol{q} = 0$$

where the equality follows from the first order condition ($\star$). Thus

$$\operatorname{sign} \frac{dx^*(w_0)}{dw} = \operatorname{sign} Eu''(w + x_0^* \boldsymbol{q})\boldsymbol{q} \geqslant 0$$

when r is decreasing at w_0. Similarly, $\dfrac{dx^*(w_0)}{dw} \leqslant 0$ when r is increasing at w_0.

These conclusions are reversed if $x_0^* < 0$. ∎

3 What if u is more risk averse than v?

Proposition 3 *Assume u is more risk averse than v. If v is risk averse or the two preferences are "sufficiently close" (in a sense to be made precise in the proof), then*

$$0 \leqslant x_u^* \leqslant x_v^* \qquad \text{or} \qquad x_v^* \leqslant x_u^* \leqslant 0.$$

That is, the more risk averse utility adopts the more conservative portfolio.

Proof: We prove only the case $x_u^* \geqslant 0$. The other follows *mutatis mutandis*. Write $u = G \circ v$, where G is strictly increasing and concave. Then ($\star$) becomes

$$\boldsymbol{E}\, G'\big(v(w + x_u^* \boldsymbol{q})\big) v'(w + x_u^* \boldsymbol{q})\boldsymbol{q} = 0.$$

Since G is concave, G' is nonincreasing, and thus so is $G' \circ v$. By Lemma 1,

$$G'\big(v(w + x_u^* \boldsymbol{q})\big)\boldsymbol{q} \leqslant G'\big(v(w)\big)\boldsymbol{q}.$$

Since $v' > 0$, we have

$$G'\big(v(w + x_u^* \boldsymbol{q})\big) v'(w + x_u^* \boldsymbol{q})\boldsymbol{q} \leqslant G'\big(v(w)\big) v'(w + x_u^* \boldsymbol{q})\boldsymbol{q},$$

and taking expectations yields

$$\underbrace{\boldsymbol{E}\, G'\big(v(w + x_u^* \boldsymbol{q})\big) v'(w + x_u^* \boldsymbol{q})\boldsymbol{q}}_{=0 \text{ by } (\star)} \leqslant G'\big(v(w)\big)\, \boldsymbol{E}\, v'(w + x_u^* \boldsymbol{q})\boldsymbol{q}.$$

That is,

$$\boldsymbol{E}\, v'(w + x_u^* \boldsymbol{q})\boldsymbol{q} \geqslant 0.$$

But the first order condition for x_v^* is

$$\boldsymbol{E}\, v'(w + x_v^* \boldsymbol{q})\boldsymbol{q} = 0.$$

Now set $h(x) = \boldsymbol{E}\, v'(w + x\boldsymbol{q})\boldsymbol{q}$. Then $h(x_u^*) \geqslant 0 = h(x_v^*)$. But $h'(x_v^*) = \boldsymbol{E}\, v''(w + x_v^* \boldsymbol{q})\boldsymbol{q}^2 \leqslant 0$ by the second order condition for x_v^*. If u and v are close enough so that $h'(x) \leqslant 0$ on the interval between x_v^* and x_u^*, then $x_u^* \leqslant x_v^*$. (If v is concave, then $h' \leqslant 0$ and no closeness assumption is needed.) ∎

References

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[3] ——— . 1974. Ventures, bets and initial prospects. In M. Balch, D. L. McFadden, and S. Wu, eds., *Decision Rules and Uncertainty: NSF–NBER Proceedings*, pages 99–131. Amsterdam: North Holland.

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