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Introduction to Point-Set Topology

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Abstract

These notes are gathered from several of my other handouts, and are a terse introduction to the topological concepts used in economic theory. For further study I recommend Willard [4] and Wilanksy [3]. You may also be interested in my on-line notes on metric spaces [2].

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1 Topological spaces

You should know that the collection of open subsets of $\boldsymbol{R}^{\mathrm{m}}$ is closed under finite intersections and arbitrary unions. Use that as the motivation for the following definition.

1 Definition *A* ***topology*** τ *on a nonempty set* X *is a family of subsets of* X*, called* ***open sets*** *satisfying*

1. $\varnothing \in \tau$ *and* $X \in \tau$.

2. *The family τ is closed under finite intersections. That is, if $U_1, \dots, U_m$ belong to τ, then $\bigcap_{i=1}^m U_i$ belongs to τ.*

3. *The family τ is closed under arbitrary unions. That is, if U_α, $\alpha \in A$, belong to τ, then $\bigcup_{\alpha \in A} U_\alpha$ belongs to τ.*

The pair (X, τ) is a ***topological space****.*

The topology τ is a ***Hausdorff*** *topology if for every two distinct points x, y in X there are disjoint open sets U, V with $x \in U$ and $y \in V$.*

The collection of open sets in $\boldsymbol{R}^{\mathrm{m}}$ is a Hausdorff topology. A property of X that can be expressed in terms of its topology is called a topological property.

2 Relative topologies

2 Definition (Relative topology) *If (X, τ) is a topological space and $A \subset X$, then (A, τ_A) is a topological space with its* ***relative topology****, where $\tau_A = \{G \cap A : G \in \tau\}$.*

Not that if τ is a Hausdorff topology, then τ_A is also a Hausdorff topology.

3 Neighborhoods, interiors, closed sets, closures

3 Definition *The set A is a* ***neighborhood*** *of x if there is an open set U satisfying $x \in U \subset A$. We also say that x is an* ***interior point*** *of A.*

The ***interior of A****, denoted $\operatorname{int} A$, is the set of interior points of A.*

4 Lemma *A set is open if and only it is a neighborhood of each of it points.*

Proof: Clearly an open set is a neighborhood of each of its points. So assume the set G is a neighborhood of each of it points. That is, for each $x \in G$ there is an open set U_x satisfying $x \in U_x \subset G$. Then $G = \bigcup_{x \in G} U_x$ is open, being a union open sets. ∎

5 Exercise The interior of any set A is open (possibly empty), and is indeed the largest open set included in A. □

6 Definition *A set is* ***closed*** *if its complement is open.*

The ***closure*** *of a set A, denoted $\overline{A}$ or $\operatorname{cl} A$, is the intersection of all the closed sets that include A.*

7 Exercise The union of finitely many closed sets is closed and the intersection of an arbitrary family of closed sets is closed. □

8 Exercise The closure of A is the smallest closed set that includes A. □

9 Lemma *A point x is not in $\overline{A}$, that is, $x \in (\overline{A})^c$, if and only if there is an open neighborhood U of x disjoint from A.*

Proof: ($\Longleftarrow$) If $x \in U$, where U is open and $U \cap A = \varnothing$, then the complement U^c is a closed set including A^c, so by definition $\overline{A^c} \subset U^c$. Thus $x \notin \overline{A^c}$.

($\Longrightarrow$) Since $\overline{A}$ is closed, if $x \notin \overline{A}$, then $(\overline{A})^c$ is an open neighborhood of x disjoint from $\overline{A}$, so a fortiori disjoint from A. ∎

10 Definition *The* ***boundary*** *of a set A, denoted ∂A, is $\overline{A} \cap \overline{A^c}$.*

11 Corollary $\partial A = \overline{A} \setminus \operatorname{int} A$.

Proof: By Lemma 9, $\operatorname{int} A = (\overline{A^c})^c$. Thus $\overline{A} \setminus \operatorname{int} A = \overline{A} \cap \overline{A^c} = \partial A$. ∎

4 Bases

12 Definition *A family $\mathcal{G}$ of open sets is a* ***base*** *(or basis) for the topology τ if every open set in τ is a union of sets from $\mathcal{G}$. A* ***neighborhood base at x*** *is a collection $\mathcal{N}$ of neighborhoods of x such that for every neighborhood G of x there is a neighborhood U of x belong to $\mathcal{N}$ satisfying $x \in U \subset G$.*

In a metric space, the collection of open balls $\{B_\varepsilon(x) : \varepsilon > 0,\ x \in X\}$ is base for the metric topology, and $\{B_{1/n}(x) : n > 0\}$ is a neighborhood base at x.

Given a nonempty family $\mathcal{A}$ of subsets of X there is a smallest topology $\tau_{\mathcal{A}}$ on X that includes $\mathcal{A}$, called the **topology generated by $\mathcal{A}$**. It consists of arbitrary unions of finite intersections of members of $\mathcal{A}$. If $\mathcal{A}$ is closed under finite intersections, then $\mathcal{A}$ is a base for the topology $\tau_{\mathcal{A}}$.

5 Product topology

13 Definition *If X and Y are topological spaces, the collection sets of the form $U \times V$, where U is an open set in X and V is an open set in Y, is closed under finite intersections, so it is a base for the topology it generates on $X \times Y$, called the* ***product topology****.*

6 Continuous functions

14 Definition *Let X and Y be topological spaces and let $f \colon X \to Y$. Then f is* ***continuous*** *if the inverse image of open sets are open. That is, if U is an open subset of Y, then $f^{-1}(U)$ is an open subset of X.*

This corresponds to the usual ε-δ definition of continuity that you are familiar with.

15 Lemma *A function $f \colon X \to Y$ is continuous if and only if the inverse image of every closed set is closed.*

16 Lemma *If $f \colon X \to Y$ is continuous, then for every $A \subset X$, we have $f(\overline{A}) \subset \overline{f(A)}$.*

Proof: Since f is continuous and $\overline{f(A)}$ is closed, $f^{-1}\big(\overline{f(A)}\big)$ is a closed set that clearly includes A, and so includes its closure $\overline{A}$. That is, $\overline{A} \subset f^{-1}\big(\overline{f(A)}\big)$, so $f(\overline{A}) \subset f\Big(f^{-1}\big(\overline{f(A)}\big)\Big) = \overline{f(A)}$. ∎

7 Homeomorphisms

17 Definition *Let X and Y be topological spaces. A function $f\colon X \to Y$ is a* ***homeomorphism*** *if it is a bijection (one-to-one and onto), is continuous, and its inverse is continuous.*

If f is homeomorphism $U \leftrightarrow f(U)$ is a one-to-one correspondence between the topologies of X and Y. Thus X and Y have the same topological properties. They can in effect be viewed as the same topological space, where f simply renames the points.

8 Compactness

Let K be a subset of a topological space. A family $\mathcal{A}$ of sets is a **cover** of K if

$$K \subset \bigcup_{A \in \mathcal{A}} A.$$

If each set in the cover $\mathcal{A}$ is open, then $\mathcal{A}$ is an **open cover** of K. A family $\mathcal{B}$ of sets is a **subcover** of $\mathcal{A}$ if $\mathcal{B} \subset \mathcal{A}$ and $K \subset \cup_{A \in \mathcal{B}} A$.

For example, let K be a subset of $\boldsymbol{R}$, and for each $x \in K$, let $\varepsilon_x > 0$. Then the family $\mathcal{A} = \{(x - \varepsilon_x, x + \varepsilon_x) : x \in K\}$ of open intervals is a open cover of K.

18 Definition *A set K in a topological space X is* ***compact*** *if for every family $\mathcal{G}$ of open sets satisfying $K \subset \cup \mathcal{G}$ (an* ***open cover*** *of K), there is a finite subfamily $\{G_1, \ldots, G_k\} \subset \mathcal{G}$ with $K \subset \bigcup_{i=1}^k G_i$ (a* ***finite subcover*** *of K).*

19 Lemma *If (X, τ) is a topological space and $K \subset A \subset X$, then K is a compact subset of (A, τ_A) if and only if it is a compact subset of (X, τ).*

Proof: Assume K is a compact subset of (X, τ). Let $\mathcal{G}$ be a τ_A-open cover of K in A. For each $G \in \mathcal{G}$ there is some $U_G \in \tau$ with $G = U_G \cap A$. Then $\{U_G : G \in \mathcal{G}\}$ is a τ-open cover of K in X, so it has a finite subcover $U_{G_1}, \ldots, U_{G_k}$. But then $G_1, \ldots, G_k$ is a finite subcover of K in A.

The converse is similar. ∎

There is an equivalent characterization of compact sets that is sometimes more convenient. A family $\mathcal{A}$ of sets has the **finite intersection property** if every finite subset $\{A_1, \ldots, A_n\}$ of $\mathcal{A}$ has a nonempty intersection, $\bigcap_{i=1}^n A_i \neq \varnothing$.

20 Theorem *A set K is compact if and only if every family of closed subsets of K having the finite intersection property has a nonempty intersection.*

Proof: Start with this observation: Let $\mathcal{A}$ be an arbitrary family of subsets of K, and define $\overline{\mathcal{A}} = \{K \setminus A : A \in \mathcal{A}\}$. By de Morgan's Laws $\bigcap_{A \in \mathcal{A}} A = \varnothing$ if and only if $K = \bigcup_{B \in \overline{\mathcal{A}}} B$. That is, $\mathcal{A}$ has an empty intersection if and only if $\overline{\mathcal{A}}$ covers K.

($\Longrightarrow$) Assume K is compact and let $\mathcal{F}$ be a family of closed subsets of K. Then $\overline{\mathcal{F}}$ is a family of relatively open sets of K. If $\mathcal{F}$ has the finite intersection property, by the above observation, no finite subset of $\overline{\mathcal{F}}$ can cover K. Since K is compact, this implies that $\overline{\mathcal{F}}$ itself cannot cover K. But then by the observation $\mathcal{F}$ has nonempty intersection.

($\Longleftarrow$) Assume that every family of closed subsets of K having the finite intersection property has a nonempty intersection, and let $\mathcal{G}$ be an open cover of K. Then $\overline{\mathcal{G}}$ is a family of closed having an empty intersection. Thus $\overline{\mathcal{G}}$ cannot have the fintiie intersection property, so there is a finite subfamily $\overline{\mathcal{G}_0}$ of $\overline{\mathcal{G}}$ with empty intersection. But then $\overline{\overline{\mathcal{G}_0}}$ is a finite subfamily of $\mathcal{G}$ that covers K. Thus K is compact. ∎

21 Lemma *A closed subset of a compact set is compact.*

Proof: Let K be compact and $F \subset K$ be closed. Let $\mathcal{G}$ be an open cover of F. Then $\mathcal{G} \cup \{F^c\}$ is an open cover of K. Let $\{G_1, \dots, G_k, F^c\}$ be a finite subcover of K. Then $\{G_1, \dots, G_k\}$ is a finite subcover of F. ∎

22 Lemma *A compact subset of a Hausdorff space is closed.*

Proof: Let K be compact, and let $x \notin K$. Then by the Hausdorff property, for each $y \in K$ there are disjoint open sets U_y and V_y with $y \in U_y$ and $x \in V_y$. By compactness there are $y_1, \dots, y_k$ with $K \subset \bigcup_{i=1}^k U_{y_i} = U$. Then $V = \bigcap_{i=1}^k V_{y_i}$ is an open set satisfying $x \in V \subset U^c \subset K^c$. That is, K^c is a neighborhood of x. Since x is an arbitrary member of K^c, we see that K^c is open (Lemma 4), so K is closed. ∎

23 Lemma *Let $f\colon X \to Y$ be continuous. If K is a compact subset of X, then $f(K)$ is a compact subset of Y.*

Proof: Let $\mathcal{G}$ be an open cover of $f(K)$. Then $\{f^{-1}(G) : G \in \mathcal{G}\}$ is an open cover of K. Let $\{f^{-1}(G_1), \dots, f^{-1}(G_k)\}$ be a finite subcover of K. Then $\{G_1, \dots, G_k\}$ is a finite subcover of $f(K)$. ∎

24 Lemma *Let $f\colon X \to Y$ be one-to-one and continuous, where Y is a Hausdorff space and X is compact. The $f\colon X \to f(X)$ is a homeomorphism, where $f(X)$ has its relative topology as a subset of Y.*

Proof: We need to show that the function $f^{-1}\colon f(X) \to X$ is continuous. So let G be any open subset of X. We must show that $(f^{-1})^{-1}(G) = f(G)$ is open in $f(X)$. Now G^c is a closed subset of X, and thus compact. Therefore $f(G^c)$ is compact, and since Y is Hausdorff, so is $f(X)$, so $f(G^c)$ is a closed subset of Y. Now $f(X) \cap f(G^c)^c = f(G)$, so $f(G)$ is open in $f(X)$. ∎

25 Weierstrass's Theorem *If K is compact and $f\colon K \to \boldsymbol{R}$ is continuous, then there exists a point x^* in K that maximizes f. That is, $(\forall x \in K)\ [\, f(x^*) \geqslant f(x)\,]$.*

Proof: Since f is continuous, $F_\alpha = \{x \in K : f(x) \geqslant \alpha\} = f^{-1}\big([\alpha, \infty)\big)$ is closed for each $\alpha \in \boldsymbol{R}$, as inverse images of closed sets are closed for a continuous function. Let $A = \{f(x) : x \in K\}$ denote the range of f. Then $\mathcal{F} = \{f_\alpha : \alpha \in A\}$ is family of nonempty closed subsets of K having the finite intersection property.[1] Since K is compact, $M = \bigcap_{\alpha \in A} F_\alpha$ is nonempty. Let x^* belong to M and let x be any point in K. Set $\alpha = f(x)$. Then $x^* \in F_\alpha$, which means that $f(x^*) \geqslant \alpha = f(x)$. That is, x^* maximizes f over K. ∎

[1] Let $\alpha = \max\{\alpha_1, \dots, \alpha_n\}$. Then $F_\alpha = F_{\alpha_1} \cap \dots \cap F_{\alpha_n}$, and $F_\alpha \neq \varnothing$ since α belongs to A, the range of f.

Note that this proof works if we only assume that each F_α is closed, that is, that f is **upper semicontinuous**.

The following result is well known.

26 Heine–Borel–Lebesgue Theorem *A subset of $\boldsymbol{R}^{\mathrm{m}}$ is compact if and only if it is both closed and bounded in the Euclidean metric.*

This result is special. In general, a subset of a metric space may be closed and bounded without being compact. (Consider the coordinate vectors in ℓ_∞.)

9 Topological vector spaces

For a detailed discussion of topological vector spaces, see chapter five of the Hitchhiker's Guide [1]. But here are some of the results we will need.

27 Definition *A (real)* ***topological vector space*** *is a vector space X together with a topology τ where τ has the property that the mappings scalar multiplication and vector addition are continuous functions. That is, the mappings*

$$(\alpha, x) \mapsto \alpha x$$

from $\boldsymbol{R} \times X$ to X and

$$(x, y) \mapsto x + y$$

from $X \times X$ to X are continuous. (Where, of course, $\boldsymbol{R}$ has its usual topology, and $\boldsymbol{R} \times X$ and $X \times X$ have their product topologies.)

28 Lemma *If V is open, then $V + y$ is open.*

Proof: Since $f\colon x \mapsto x - y$ is continuous, $V + y = f^{-1}(V)$ is open. ∎

29 Lemma *If V is open, and $\alpha \neq 0$, then αV is open.*

Proof: Since $f\colon x \mapsto (1/\alpha)x$ is continuous, $\alpha V = f^{-1}(V)$ is open. ∎

30 Definition *A set C in a vector space is* ***circled*** *or* ***radial*** *if $\alpha C \subset C$ whenever $|\alpha| \leqslant 1$.*

31 Lemma *Let V be a neighborhood of zero. Then there is an open circled neighborhood U of zero included in V.*

Proof: The mapping $f\colon (\alpha, x) \mapsto \alpha x$ is continuous, and $f(0,0) = 0$, the inverse image $f^{-1}(V)$ is a neighborhood of 0. Thus there is an $\delta > 0$ and an open neighborhood W of 0 such that $(-\delta, \delta) \times W \subset f^{-1}(V)$. This implies that for any α with $|\alpha| < \delta$ and $x \in W$, we have $\alpha x \in V$. In other words $\alpha W \subset V$. Set

$$U = \bigcup_{\alpha : 0 < |\alpha| < \delta} \alpha W$$

Then $U \subset V$, U is circled, and U is open, being the union of the open sets αW. ∎

32 Lemma *Let $T\colon X \to Y$ be a linear transformation between topological vector spaces. Then T is continuous on X if it is continuous at 0.*

Proof: It suffices to prove that T is continuous at each point x. So let V be an open neighborhood of $T(x)$. Then $V - T(x)$ is an open neighborhood of 0. Since T is continuous at 0, the inverse image $T^{-1}\big(V - T(x)\big)$, is a neighborhood of 0, so $T^{-1}\big(V - T(x)\big) + x$ is a neighborhood of x. But by linearity, $T^{-1}\big(V - T(x)\big) + x = T^{-1}(V)$, and we are done. ∎

References

[1] C. D. Aliprantis and K. C. Border. 2006. *Infinite dimensional analysis: A hitchhiker's guide*, 3d. ed. Berlin: Springer–Verlag.

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[3] A. Wilansky. 1998. *Topology for analysis*. Mineola, NY: Dover. Unabridged republication of the work originally published by Ginn and Company, Waltham, MA in 1970.

[4] S. Willard. 1970. *General topology*. Reading, Massachusetts: Addison Wesley.