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Notes on Comparative Statics, the Old-Fashioned Way

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"Comparative statics" analysis tells us how the "equilibrium" values of endogenous variables $x_1, \ldots, x_n$ (the things we want to solve for) change as a function of the exogenous parameters $p_1, \ldots, p_m$. (As such it is hardly unique to economics.) Typically we can write the equilibrium conditions of our model as the zero of a system of equations in the endogenous variables and the exogenous parameters:

$$\begin{aligned} F^1(x_1, \ldots, x_n; p_1, \ldots, p_m) &= 0 \\ &\vdots \\ F^n(x_1, \ldots, x_n; p_1, \ldots, p_m) &= 0 \end{aligned} \tag{E}$$

This implicitly defines x as a function of p, which we will explicitly denote $x = \xi(p)$, or

$$(x_1, \ldots, x_n) = \Big(\xi^1(p_1, \ldots, p_m), \ldots, \xi^n(p_1, \ldots, p_m)\Big).$$

This explicit function, if it exists, satisfies the implicit definition

$$F\Big(\xi(p); p\Big) = 0 \tag{I}$$

for at least a rectangle of values of p. The Implicit Function Theorem (see, e.g., Apostol [1, Theorem 7-6, p. 147], Rudin [6, Theorem 9.28, p. 224], or my notes [5]) tells us when such an explicit function exists. (Basically, an explicit function exists whenever it is possible to solve for all its partial derivatives.)

Ideally, we would like to be able to solve for the function ξ. But even if we can't solve for ξ analytically, it is possible that we may be able to at least solve for the partial derivatives

$$\frac{\partial \xi^i}{\partial p_j}.$$

The partial derivatives

Setting $G(p) = F\big(\xi(p); p\big)$ and differentiating G^i with respect to p_j in equation (I) yields

$$\frac{\partial G^i}{\partial p_j} = \sum_{k=1}^{n} \frac{\partial F^i}{\partial x_k}\frac{\partial \xi^k}{\partial p_j} + \frac{\partial F^i}{\partial p_j} = 0 \tag{1}$$

for each $i = 1, \ldots, n$, $j = 1, \ldots, m$. In matrix terms we have

$$\begin{bmatrix} \frac{\partial F^1}{\partial x_1} & \cdots & \frac{\partial F^1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial F^n}{\partial x_1} & \cdots & \frac{\partial F^n}{\partial x_n} \end{bmatrix} \begin{bmatrix} \frac{\partial \xi^1}{\partial p_1} & \cdots & \frac{\partial \xi^1}{\partial p_m} \\ \vdots & & \vdots \\ \frac{\partial \xi^n}{\partial p_1} & \cdots & \frac{\partial \xi^n}{\partial p_m} \end{bmatrix} + \begin{bmatrix} \frac{\partial F^1}{\partial p_1} & \cdots & \frac{\partial F^1}{\partial p_m} \\ \vdots & & \vdots \\ \frac{\partial F^n}{\partial p_1} & \cdots & \frac{\partial F^n}{\partial p_m} \end{bmatrix} = 0.$$

Provided $\begin{bmatrix} \frac{\partial F^1}{\partial x_1} & \cdots & \frac{\partial F^1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial F^n}{\partial x_1} & \cdots & \frac{\partial F^n}{\partial x_n} \end{bmatrix}$ is invertible (the hypothesis of the Implicit Function Theorem) we can solve this:

$$\begin{bmatrix} \frac{\partial \xi^1}{\partial p_1} & \cdots & \frac{\partial \xi^1}{\partial p_m} \\ \vdots & & \vdots \\ \frac{\partial \xi^n}{\partial p_1} & \cdots & \frac{\partial \xi^n}{\partial p_m} \end{bmatrix} = - \begin{bmatrix} \frac{\partial F^1}{\partial x_1} & \cdots & \frac{\partial F^1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial F^n}{\partial x_1} & \cdots & \frac{\partial F^n}{\partial x_n} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial F^1}{\partial p_1} & \cdots & \frac{\partial F^1}{\partial p_m} \\ \vdots & & \vdots \\ \frac{\partial F^n}{\partial p_1} & \cdots & \frac{\partial F^n}{\partial p_m} \end{bmatrix} \tag{2}$$

The old-fashioned derivation (see, e.g., Samuelson's *Foundations* [7, pp. 10–14]) of this same result runs like this: "Totally differentiate" the i^{th} row of the system of equations (E) to get

$$\sum_{k=1}^{n} \frac{\partial F^i}{\partial x_k}\, dx_k + \sum_{\ell} \frac{\partial F^i}{\partial p_\ell}\, dp_\ell = 0$$

for all i. Now set all dp_ℓ's equal to zero except p_j, and divide by dp_j to get

$$\sum_{k} \frac{\partial F^i}{\partial x_k}\frac{dx_k}{dp_j} + \frac{\partial F^i}{\partial p_j} = 0$$

for all i and j, which is the same as equation (1). For further information on total differentials and how to manipulate them, see Apostol [1, Chapter 6].

Using Cramer's Rule (e.g., Apostol [2, pp. 93–94]), we see then that

$$\frac{dx_i}{dp_j} = \frac{\partial \xi^i}{\partial p_j} = -\frac{\begin{vmatrix} \frac{\partial F^1}{\partial x_1} & \cdots & \frac{\partial F^1}{\partial x_{i-1}} & \frac{\partial F^1}{\partial p_j} & \frac{\partial F^1}{\partial x_{i+1}} & \cdots & \frac{\partial F^1}{\partial x_n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \frac{\partial F^n}{\partial x_1} & \cdots & \frac{\partial F^n}{\partial x_{i-1}} & \frac{\partial F^n}{\partial p_j} & \frac{\partial F^n}{\partial x_{i+1}} & \cdots & \frac{\partial F^n}{\partial x_n} \end{vmatrix}}{\begin{vmatrix} \frac{\partial F^1}{\partial x_1} & \cdots & \frac{\partial F^1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial F^n}{\partial x_1} & \cdots & \frac{\partial F^n}{\partial x_n} \end{vmatrix}}.$$

Or, letting Δ denote the determinant of $\begin{bmatrix} \frac{\partial F^1}{\partial x_1} & \cdots & \frac{\partial F^1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial F^n}{\partial x_1} & \cdots & \frac{\partial F^n}{\partial x_n} \end{bmatrix}$, and letting $\Delta_{i,j}$ denote the determinant of the matrix formed by deleting its i^{th} row and j^{th} column, we can expand the determinant in the numerator along the i^{th} column to get:

$$\frac{\partial \xi^i}{\partial p_j} = -\sum_{k=1}^{n} (-1)^{i+k} \frac{\partial F^k}{\partial p_j} \frac{\Delta_{k,i}}{\Delta}.$$

Lagrangeans

In this section we consider the special case where the equilibrium conditions are the first order conditions of a maximization problem. Specifically we consider the problem:

$$\operatorname*{maximize}_{x} f(x;p) \quad \text{subject to the constraint} \quad g(x;p) = 0.$$

The Lagrangean for this is

$$L(x,\lambda;p) = f(x;p) + \lambda g(x;p).$$

If the assumptions of the Lagrange Multiplier Theorem are satisfied, then the first order conditions are

$$\begin{aligned} \frac{\partial L(x,\lambda;p)}{\partial x_1} &= 0 \\ &\vdots \\ \frac{\partial L(x,\lambda;p)}{\partial x_n} &= 0 \\ g(x;p) &= 0. \end{aligned} \tag{FOC}$$

This is a little bit messy because there are $n+1$ (not n) endogenous variables $x_1, \dots, x_n$ and λ. This means that we have to treat λ as x_{n+1} when trying to apply the previous formulae.

We shall assume that the strong second order conditions are satisfied, that is,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{\partial^2 L}{\partial x_i \partial x_j} \right) v_i v_j < 0 \tag{SSOC}$$

for every $v = (v_1, \dots, v_n) \neq 0$ satisfying

$$\sum_{i=1}^{n} \frac{\partial g}{\partial x_i} v_i = 0.$$

(The necessary condition is that (SSOC) holds with weak inequality, see, e.g, my notes on constrained maximization [3].) The strong second order conditions are equivalent to the following condition on the NW principal minors of the bordered Hessian. (See, e.g, my notes on quadratic forms [4].)

$$(-1)^k \begin{vmatrix} \frac{\partial^2 L}{\partial x_1^2} & \cdots & \frac{\partial^2 L}{\partial x_1 \partial x_k} & \frac{\partial g}{\partial x_1} \\ \vdots & & \vdots & \vdots \\ \frac{\partial^2 L}{\partial x_k \partial x_1} & \cdots & \frac{\partial^2 L}{\partial x_k^2} & \frac{\partial g}{\partial x_k} \\ \frac{\partial g}{\partial x_1} & \cdots & \frac{\partial g}{\partial x_k} & 0 \end{vmatrix} > 0, \qquad k = 2, \dots, n.$$

In particular, for $k = n$, this implies that the matrix

$$\begin{bmatrix} \frac{\partial^2 L}{\partial x_1^2} & \cdots & \frac{\partial^2 L}{\partial x_1 \partial x_n} & \frac{\partial g}{\partial x_1} \\ \vdots & & \vdots & \vdots \\ \frac{\partial^2 L}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 L}{\partial x_n^2} & \frac{\partial g}{\partial x_n} \\ \frac{\partial g}{\partial x_1} & \cdots & \frac{\partial g}{\partial x_n} & 0 \end{bmatrix}$$

is nonsingular, and hence invertible.

Now return attention to the system of equations (FOC). For $i = 1, \dots, n$ define

$$F^i(x_1, \dots, x_n, \lambda; p_1, \dots, p_m) = \frac{\partial L(x, \lambda; p)}{\partial x_i} = \frac{\partial f(x; p)}{\partial x_i} + \lambda \frac{\partial g(x; p)}{\partial x_i}$$

and define

$$F^{n+1}(x_1, \dots, x_n, \lambda; p_1, \dots, p_m) = g(x, p).$$

The first order conditions take the form $F(x, \lambda; p) = 0$.

Now we consider the partials of each F^i with respect to the x_j's and λ:

$$\frac{\partial F^i}{\partial x_j} = \frac{\partial^2 L}{\partial x_i \partial x_j} \qquad \begin{array}{l} i = 1, \dots, n \\ j = 1, \dots, n \end{array}$$

$$\frac{\partial F^i}{\partial \lambda} = \frac{\partial g}{\partial x_i} \qquad i = 1, \dots, n,$$

$$\frac{\partial F^{n+1}}{\partial x_j} = \frac{\partial g}{\partial x_i} \qquad j = 1, \dots, n,$$

and

$$\frac{\partial F^{n+1}}{\partial \lambda} = 0.$$

In other words,

$$\begin{bmatrix} \frac{\partial F^1}{\partial x^1} & \cdots & \frac{\partial F^1}{\partial x^n} & \frac{\partial F^1}{\partial \lambda} \\ \vdots & & \vdots & \vdots \\ \frac{\partial F^n}{\partial x^1} & \cdots & \frac{\partial F^n}{\partial x_n} & \frac{\partial F^n}{\partial \lambda} \\ \frac{\partial F^{n+1}}{\partial x_1} & \cdots & \frac{\partial F^{n+1}}{\partial x_n} & \frac{\partial F^{n+1}}{\partial \lambda} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 L}{\partial x_1^2} & \cdots & \frac{\partial^2 L}{\partial x_1 \partial x_n} & \frac{\partial g}{\partial x_1} \\ \vdots & & \vdots & \vdots \\ \frac{\partial^2 L}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 L}{\partial x_n^2} & \frac{\partial g}{\partial x_n} \\ \frac{\partial g}{\partial x_1} & \cdots & \frac{\partial g}{\partial x_n} & 0 \end{bmatrix}$$

is nonsingular, and hence invertible.

Thus the strong second order conditions imply the Jacobian conditions for the Implicit Function Theorem, which guarantees the differentiability of x and λ as functions of the parameters. To translate equation (2) we note that

$$\frac{\partial F^i}{\partial p_j} = \frac{\partial^2 L}{\partial x_i \partial p_j}, \qquad \begin{array}{l} i = 1, \dots, n \\ j = 1, \dots, m \end{array}$$

and

$$\frac{\partial F^{n+1}}{\partial p_j} = \frac{\partial g}{\partial p_j} \qquad j = 1, \dots, m.$$

equation (2) becomes

$$\begin{bmatrix} \frac{\partial x^1}{\partial p_1} & \cdots & \frac{\partial x^1}{\partial p_m} \\ \vdots & & \vdots \\ \frac{\partial x^n}{\partial p_1} & \cdots & \frac{\partial x^n}{\partial p_m} \\ \frac{\partial \lambda}{\partial p_1} & \cdots & \frac{\partial \lambda}{\partial p_m} \end{bmatrix} =$$

$$- \begin{bmatrix} \frac{\partial^2 L}{\partial x_1^2} & \cdots & \frac{\partial^2 L}{\partial x_1 \partial x_n} & \frac{\partial g}{\partial x_1} \\ \vdots & & \vdots & \vdots \\ \frac{\partial^2 L}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 L}{\partial x_n^2} & \frac{\partial g}{\partial x_n} \\ \frac{\partial g}{\partial x_1} & \cdots & \frac{\partial g}{\partial x_n} & 0 \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial^2 L}{\partial x_1 \partial p_1} & \cdots & \frac{\partial^2 L}{\partial x_1 \partial p_m} \\ \vdots & & \vdots \\ \frac{\partial^2 L}{\partial x_n \partial p_1} & \cdots & \frac{\partial^2 L}{\partial x_n \partial p_m} \\ \frac{\partial g}{\partial p_1} & \cdots & \frac{\partial g}{\partial p_m} \end{bmatrix} \tag{3}$$

Utility maximization subject to a budget constraint

Now apply this to the case where

$$f(x; p, w) = u(x) \text{ and } g(x; p, w) = w - p \cdot x.$$

The $n+1$ endogenous variables are $x_1, \dots, x_n$ and λ. There are also $n+1$ parameters, $p_1, \dots, p_n$ and w. Moreover,

$$\frac{\partial^2 L}{\partial x_i \partial x_j} = \frac{\partial^2 u}{\partial x_i \partial x_j},$$

$$\frac{\partial g}{\partial x_i} = -\lambda p_i$$

$$\frac{\partial^2 L}{\partial x_i \partial p_j} = -\lambda \delta_{ij}, \qquad \frac{\partial^2 L}{\partial p_i \partial w} = 0;$$

$$\frac{\partial g}{\partial p_i} = -x_i, \qquad \frac{\partial g}{\partial w} = 1.$$

We can write (3) as

$$\begin{bmatrix} \frac{\partial x^1}{\partial p_1} & \cdots & \frac{\partial x^1}{\partial p_n} & \frac{\partial x^1}{\partial w} \\ \vdots & & \vdots & \vdots \\ \frac{\partial x^n}{\partial p_1} & \cdots & \frac{\partial x^n}{\partial p_n} & \frac{\partial x^n}{\partial w} \\ \frac{\partial \lambda}{\partial p_1} & \cdots & \frac{\partial \lambda}{\partial p_n} & \frac{\partial \lambda}{\partial w} \end{bmatrix} = - \begin{bmatrix} \frac{\partial^2 u}{\partial x_1^2} & \cdots & \frac{\partial^2 u}{\partial x_1 \partial x_n} & -\lambda p_1 \\ \vdots & & \vdots & \vdots \\ \frac{\partial^2 u}{\partial x_1 \partial x_n} & \cdots & \frac{\partial^2 u}{\partial x_n^2} & -\lambda p_n \\ -\lambda p_1 & \cdots & -\lambda p_n & 0 \end{bmatrix}^{-1} \begin{bmatrix} -\lambda & 0 & \cdots & 0 & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & -\lambda & 0 \\ -x_1 & \cdots & \cdots & -x_n & 1 \end{bmatrix}$$

Using the first order condition $\frac{\partial u}{\partial x_i} = \lambda p_i$ we have

$$\begin{bmatrix} \frac{\partial^2 u}{\partial x_1^2} & \cdots & \frac{\partial^2 u}{\partial x_1 \partial x_n} & -\lambda p_1 \\ \vdots & & \vdots & \vdots \\ \frac{\partial^2 u}{\partial x_1 \partial x_n} & \cdots & \frac{\partial^2 u}{\partial x_n^2} & -\lambda p_n \\ -\lambda p_1 & \cdots & -\lambda p_n & 0 \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 u}{\partial x_1^2} & \cdots & \frac{\partial^2 u}{\partial x_1 \partial x_n} & -\frac{\partial u}{\partial x_1} \\ \vdots & & \vdots & \vdots \\ \frac{\partial^2 u}{\partial x_1 \partial x_n} & \cdots & \frac{\partial^2 u}{\partial x_n^2} & -\frac{\partial u}{\partial x_n} \\ -\frac{\partial u}{\partial x_1} & \cdots & -\frac{\partial u}{\partial x_n} & 0 \end{bmatrix}$$

Since

$$\begin{vmatrix} \frac{\partial^2 u}{\partial x_1^2} & \cdots & \frac{\partial^2 u}{\partial x_1 \partial x_n} & -\frac{\partial u}{\partial x_1} \\ \vdots & & \vdots & \vdots \\ \frac{\partial^2 u}{\partial x_1 \partial x_n} & \cdots & \frac{\partial^2 u}{\partial x_n^2} & -\frac{\partial u}{\partial x_n} \\ -\frac{\partial u}{\partial x_1} & \cdots & -\frac{\partial u}{\partial x_n} & 0 \end{vmatrix} = (-1)^2 \begin{vmatrix} \frac{\partial^2 u}{\partial x_1^2} & \cdots & \frac{\partial^2 u}{\partial x_1 \partial x_n} & \frac{\partial u}{\partial x_1} \\ \vdots & & \vdots & \vdots \\ \frac{\partial^2 u}{\partial x_1 \partial x_n} & \cdots & \frac{\partial^2 u}{\partial x_n^2} & \frac{\partial u}{\partial x_n} \\ \frac{\partial u}{\partial x_1} & \cdots & \frac{\partial u}{\partial x_n} & 0 \end{vmatrix},$$

the strong second order conditions reduce to

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j} v_i v_j < 0$$

for every $v = (v_1, \dots, v_i) \neq 0$ satisfying

$$\sum_{i=1}^{n} p_i v_i = 0,$$

or

$$(-1)^k \begin{vmatrix} \dfrac{\partial^2 u}{\partial x_1^2} & \cdots & \dfrac{\partial^2 u}{\partial x_1 \partial x_k} & \dfrac{\partial u}{\partial x_1} \\ \vdots & & \vdots & \vdots \\ \dfrac{\partial^2 u}{\partial x_k \partial x_1} & \cdots & \dfrac{\partial^2 u}{\partial x_k^2} & \dfrac{\partial u}{\partial x_k} \\ \dfrac{\partial u}{\partial x_1} & \cdots & \dfrac{\partial u}{\partial x_k} & 0 \end{vmatrix} > 0, \qquad k = 2, \ldots, n.$$

References

[1] T. M. Apostol. 1957. *Mathematical analysis: A modern approach to advanced calculus*. Addison-Wesley series in mathematics. Reading, Massachusetts: Addison Wesley.

[2] ——— . 1969. *Calculus*, 2d. ed., volume 2. Waltham, Massachusetts: Blaisdell.

[3] K. C. Border. Maximization with more than one variable. http://www.its.caltech.edu/~kcborder/Notes/Max2.pdf

[4] ——— . More than you wanted to know about quadratic forms. http://www.its.caltech.edu/~kcborder/Notes/QuadraticForms.pdf

[5] ——— . Notes on the implicit function theorem. http://www.its.caltech.edu/~kcborder/Notes/IFT.pdf

[6] W. Rudin. 1976. *Principles of mathematical analysis*, 3d. ed. International Series in Pure and Applied Mathematics. New York: McGraw Hill.

[7] P. A. Samuelson. 1965. *Foundations of economic analysis*. New York: Athenaeum. Reprint of the 1947 edition published by Harvard University Press.