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Rough Notes on Growth with Endogenous Savings

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These notes are far from completed.

1 Household problem

At time t the typical household receives real wage income $w(t)$ and owns assets $k(t)$, which produces income at the interest rate $r(t)$ and depreciates at the rate n (due to population growth), and chooses consumption level $c(t)$.

The continuous time budget can be written as

$$\dot{k}(t) = w(t) + r(t)k(t) - c(t) - nk(t). \tag{1}$$

The objective is to maximize

$$\int_0^\infty u(c(t))e^{-(\rho-n)t}\,dt$$

where $\rho > n$. The present value Hamiltonian is

$$H(k, c, t, p) = u(c)e^{-(\rho-n)t} + p\Big(w(t) + r(t)k - c - nk\Big)$$

The first-order necessary conditions are

$$\dot{k} = D_p H = w + rk - c - nk \tag{2}$$

$$\dot{p} = -D_k H = -(r-n)p \tag{3}$$

$$D_c H = u'(c(t))e^{-(\rho-n)t} - p(t) = 0 \tag{4}$$

$$\lim_{t\to\infty} p(t)k(t) = 0 \tag{5}$$

Equation (4) implies

$$\begin{aligned}
p(t) &= u'\big(c(t)\big)e^{-(\rho-n)t} \\
\dot{p}(t) &= u''\big(c(t)\big)\dot{c}(t)e^{-(\rho-n)t} - (\rho-n)\underbrace{u'\big(c(t)\big)e^{-(\rho-n)t}}_{p(t)} \\
\frac{\dot{p}}{p} &= \frac{u''\big(c(t)\big)\dot{c}(t)e^{-(\rho-n)t}}{u'\big(c(t)\big)e^{-(\rho-n)t}} - (\rho-n) \\
&= \left(\frac{u''}{u'}c\right)\frac{\dot{c}}{c} - (\rho-n) \qquad (6)
\end{aligned}$$

The quantity $\varepsilon(c) = -\frac{u''(c)}{u'(c)}c$ is the *elasticity of marginal utility with respect to consumption*. It is constant for the utilities

$$u(c) = \begin{cases} c^{1-\varepsilon} & 0 < \varepsilon < 1 \\ \ln c & \varepsilon = 1. \end{cases}$$

So for these utilities, (6) implies

$$\frac{\dot{p}}{p} = -\varepsilon\frac{\dot{c}}{c} - (\rho-n).$$

On the other hand (3) implies

$$\frac{\dot{p}}{p} = -(r-n)$$

Combining these yields

$$\frac{\dot{c}}{c} = \frac{r-\rho}{\varepsilon} \qquad (7)$$

2 Firm's problem

A firm wishes to maximize

$$F(K,L) - rK - wL.$$

The first-order conditions are

$$D_K F - r = 0, \qquad D_L F - w = 0.$$

If F exhibits constant returns to scale we have

$$r = f'(k), \qquad w = f(k) - f'(k)k,$$

where $k = K/L$ and $f(k) = F(k,1)$.

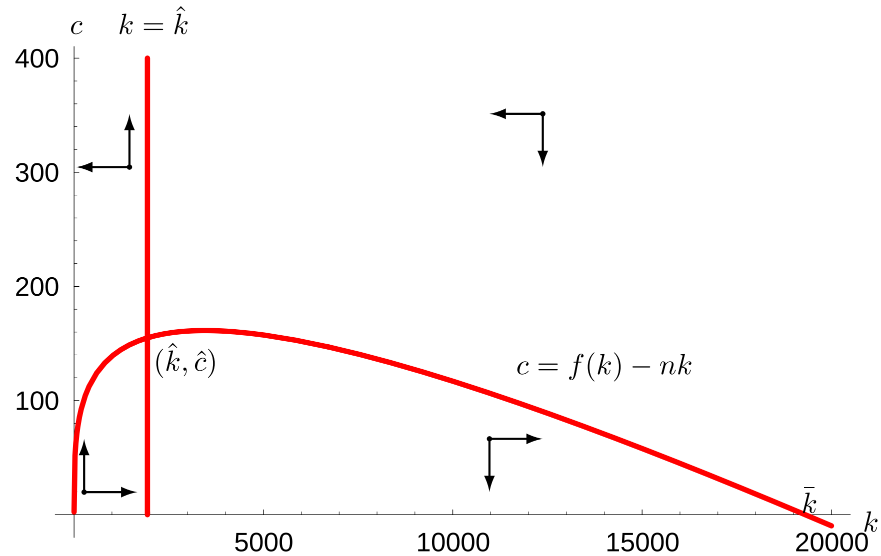

Figure 1. Stationary point for the system (8)–(9).

3 Equilibrium

Substituting for r in (7) yields

$$\frac{\dot{c}}{c} = \frac{f'(k) - \rho}{\varepsilon} \tag{8}$$

and from (1) we have

$$\dot{k} = f(k) - nk - c. \tag{9}$$

This pair of differential equations governs the evolution of the economy.

4 Phase diagram

This system of equations can be analyzed in terms of a *phase diagram*. Let $\hat{k}$ satisfy

$$f'(\hat{k}) = \rho.$$

From (8), when $k = \hat{k}$, then $\dot{c} = 0$. The vertical line at $\hat{k}$ separates the plane into two regions. To the right, $\dot{c} < 0$, and to the left $\dot{c} > 0$. From (9), along the curve $c = f(k) - nk$, we have $\dot{k} = 0$. Above this curve $\dot{k} < 0$, and below

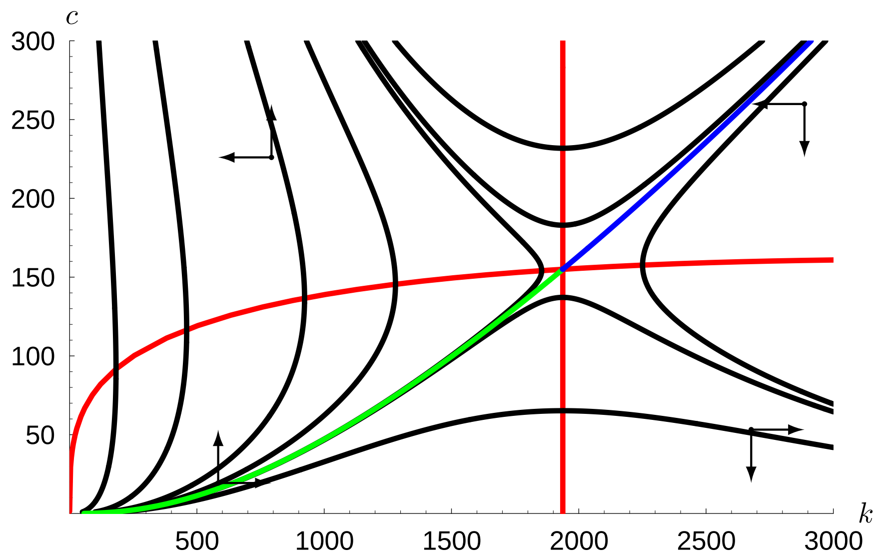

Figure 2. Trajectories for the system (8)–(9).

it, $\dot{k} > 0$. This curve intersects the vertical line at the point $(\hat{k}, \hat{c})$, which is a steady state. The arrows in Figure 1 indicate the direction of the time derivatives in these regions. The maximum sustainable level of k is $\bar{k}$, where $f(\bar{k}) = n\bar{k}$.

The curves in Figure 2 show the trajectories of the mathematical solutions to the system (8)–(9). Not all of these solutions have an economic interpretation. The reason is that the system (8)–(9) is derived from a subset of the first order conditions for the producer and consumer. But it takes more than that to make an equilibrium. Above the green and blue curves, the paths $\big(k(t), c(t)\big)$ have $k(t) \to 0$ and $c(t) \to \infty$, so these are clearly infeasible. Below the curves, $c(t) \to 0$ and $k(t) \to \bar{k}$. Thus $r(t) = f'\big(k(t)\big) \to f'(\bar{k}) < n < \rho$. Since $\dot{p}(t)/p(t) = -\big(r(t) - n\big)$, along these paths $p(t)$ grows, so the transversality condition $p(t)k(t) \to 0$ fails. Thus the equilibrium curves are the green and blue curves that converge monotonically to the steady state.

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