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Growth Accounting

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Revised Fall 2017
v. 2017.10.10::10.11

This note shows how to derive a commonly used expression for relating per capita growth rates that are used throughout the literature on economic growth. Greg Clark [1] goes so far as to call it the "fundamental equation of economic growth." The derivation is not difficult, but relies on a frequently unstated assumption that is standard in the macroeconomic/growth literature, as well as an obscure theorem that economists use all the time.

1 Production

We start with a **production function for output (GDP)**,

$$Y = A\,F(X_0, \dots, X_n),$$

where Y is total output, A is the **total factor productivity (TFP)**, X_0 is the total labor, and each X_i for $i > 0$ is some other factor of production, e.g., capital, land, human capital, R&D, etc.[1]

The reason I use X_0 instead of say N or L to denote the quantity of labor is so I can refer to all factors by writing something like $i = 0, \dots, n$. The reason for separating TFP from the production function F is that later we shall let A vary over time, while holding F fixed. Finally the use of upper case letters indicates totals, and later on we shall use lower case letters to indicate per capita quantities.

The following assumption is sometimes unstated.

1 Assumption *The production function F is differentiable and exhibits **constant returns to scale (CRS)**. That is, for every $X = (X_0, \dots, X_n)$ and every $t > 0$, we have*

$$F(tX_0, \dots, tX_n) = t\,F(X_0, \dots, X_n).$$

This assumption, which also known as *homogeneity* or *positive homogeneity of degree one*, is not as drastic as it may seem, since it is well-known that by adding one additional (possibly fictional) factor of production, we can convert an arbitrary production function to one with constant returns to scale.

[1]Since the early work of Cobb and Douglas [2], applied macro-economists have tended to work mostly with two factors of production, capital (K) and labor (L). This resulted in what was known as the "Cambridge controversy," [10] in which economists such as Nicholas Kaldor [3], Joan Robinson [5, 6], and Luigi Pasinetti [4] in Cambridge, England argued that the notion of aggregate capital was a misleading fiction that had no operational definition; while others, such as and Robert Solow, Paul Samuelson [7, 8], and Franco Modigliani [9] in Cambridge, Massachusetts argued that it was legitimate to construct such aggregates. The Cambridge, Massachusetts side prevailed in convincing the profession to use aggregate capital, but it may be the result of convenience more than theory.

1.1 On the Assumption of Constant Returns to Scale

Given an arbitrary function $G(X_0,\dots,X_n)$, define a new function $H(Z,X_0,\dots,X_n)$ with an extra argument Z by the formula

$$H(Z,X_0,\dots,X_n)=\begin{cases} ZG\Big(\frac{X_0}{Z},\dots,\frac{X_n}{Z}\Big) & Z>0\\ 0 & Z=0.\end{cases}$$

Then we can recover G from H:

$$G(X_0,\dots,X_n)=H(1,X_0,\dots,X_n),$$

but H exhibits constant returns to scale.

To see this, let $t,Z>0$,

$$\begin{aligned} H(tZ,tX_0,\dots,tX_n) &= t\,ZG\Big(\frac{tX_0}{tZ},\dots,\frac{tX_n}{tZ}\Big)\\ &= t\,\underbrace{ZG\Big(\frac{X_0}{Z},\dots,\frac{X_n}{Z}\Big)}\\ &= t\,H(Z,X_0,\dots,X_n). \end{aligned}$$

As long as we fix Z, we cannot tell which is the "true" production function.

2 Euler's Theorem

My on-line notes on Euler's Theorem prove the following well-known (to economists) and useful results about constant returns to scale.

From here on out we use subscripts on functions to indicate partial derivatives. For example, F_i denotes the partial derivative $D_iF(X)$, but X_i is still the quantity of the i^{th} input.

2 Euler's Theorem *Let $F\colon \boldsymbol{R}^{n+1}_{+}\to\boldsymbol{R}$ be continuous, and also differentiable on $\boldsymbol{R}^{n+1}_{++}$. Then F exhibits CRS if and only if for all $X\in\boldsymbol{R}^{n+1}_{++}$.*

$$F(X)=\sum_{i=0}^{n}F_i(X)X_i, \tag{1}$$

3 Corollary *Let $F\colon \boldsymbol{R}^{n+1}_{+}\to\boldsymbol{R}$ be continuous, and also differentiable on $\boldsymbol{R}^{n+1}_{++}$. If F exhibits CRS, then for all $X\in\boldsymbol{R}^{n+1}_{++}$ and $t>0$,*

$$F_i(tX)=F_i(X),\quad i=0,\dots,n \tag{2}$$

3 Marginal product, real wages, and factor shares

We take output as our numéraire good. That means that its price is set to unity, and wages of factors can be interpreted as being given in units of output. Now assume that output is produced by maximizing profits, taking factor prices as fixed, we have that

$$Y \text{ maximizes } A\,F(X_0,\dots,X_n)-r_0X_0-\cdots-r_nX_n,$$

where r_i is the real wage of factor i. The first order conditions for an interior maximum are

$$A\,F_i(X)-r_i=0\qquad i=0,\dots,n. \tag{3}$$

That is, every factor's wage is its marginal product, AF_i. More precisely, the quantity of each factor is chosen so that its marginal product is equal to its wage.

Let θ_i denote factor i's **share of total output**. The share of output is just the real wage paid to factor i times the quantity of factor i employed divided by the total output. That is,

$$\theta_i = \frac{r_i X_i}{A\,F(X)} = \frac{F_i(X) X_i}{F(X)}, \tag{4}$$

where the last equality follows from (3). Note that CRS and Euler's Theorem guarantee via equation (1) that $\sum_{i=0}^n \theta_i = 1$.

Alternatively we could assume that all the output is paid to some factor of production, and that cost is minimized. The Lagrangean for the cost minimization problem is

$$\sum_{i=0}^n r_i X_i - \mu\big(AF(X_0, \dots, X_n) - Y\big),$$

and the first order conditions for an interior minimizer are

$$r_i - \mu A F_i = 0, \quad i = 0, \dots, n.$$

Multiplying by X_i and summing over i gives

$$Y = \sum_{i=0}^n r_i X_i = \mu A \sum_{i=0}^n F_i X_i = \mu Y,$$

where the first equality is simply that all the output is paid out to the factors and the last one is equation (1). Thus $\mu = 1$, so the wage of each factor is just its marginal product, just as in (3).

4 Per capita analysis

One of the nice features of CRS is that it makes per capita analysis possible. By per capita, we mean per unit of labor used, not necessarily per unit of population, although labor and population tend to be proportional within a country. Given $X \in \boldsymbol{R}^{\mathrm{n}+1}$ define $x \in \boldsymbol{R}^{\mathrm{n}}$ by

$$x_i = X_i / X_0, \quad i = 1, \dots, n,$$

so that x_i is the quantity of factor i per capita, and set

$$y = Y / X_0,$$

so that y is **output per capita**. Then we may write

$$y = A\, f(x_1, \dots, x_n), \tag{5}$$

where

$$f(x_1, \dots, x_n) = F(1, x_1, \dots, x_n). \tag{6}$$

This implies (in light of Corollary 3) that

$$f_i(x) = F_i(X), \quad i = 1, \dots, n. \tag{7}$$

It also implies that (4) can be rewritten as

$$\theta_i = \frac{f_i(x) x_i}{f(x)}, \quad i = 1, \dots, n. \tag{8}$$

To see this, use equations (4) and (7) to write

$$\theta_i = \frac{F_i(X) X_i}{F(X)} = \frac{f_i(x) X_i / X_0}{F(X)/X_0} = \frac{f_i(x) x_i}{f(x)}.$$

5 Growth accounting

Now let's make everything dynamic. Given any time-varying quantity $z(t)$, its **instantaneous rate of growth**, which Clark denotes by g_z, is given by

$$g_z(t) = \frac{\dot{z}(t)}{z(t)} = \frac{d}{dt}\ln\bigl(z(t)\bigr). \tag{9}$$

So if

$$Y(t) = A(t)\,F\bigl(X_0(t),\dots,X_n(t)\bigr),$$

in per capita terms we get

$$y(t) = A(t)\,f\bigl(x(t)\bigr).$$

We can now write the rate of growth of output per capita as follows.

$$\begin{aligned}
g_y(t) &= \frac{d}{dt}\ln\Bigl(A(t)\,f\bigl(x(t)\bigr)\Bigr) && \text{(by (9))}\\
&= \frac{d}{dt}\Bigl[\ln\bigl(A(t)\bigr) + \ln\Bigl(f\bigl(x(t)\bigr)\Bigr)\Bigr] && \text{(log of a product is sum of logs)}\\
&= \frac{d}{dt}\ln\bigl(A(t)\bigr) + \frac{d}{dt}\ln\Bigl(f\bigl(x(t)\bigr)\Bigr) && \text{(derivative of a sum is sum of derivatives)}\\
&= \frac{\dot{A}(t)}{A(t)} + \frac{\dot{f}\bigl(x(t)\bigr)}{f\bigl(x(t)\bigr)} && \text{(by (9))}\\
&= g_A(t) + \frac{\sum_{i=1}^n f_i\bigl(x(t)\bigr)\dot{x}_i(t)}{f\bigl(x(t)\bigr)} && \text{(Chain Rule for } \dot{f}\text{)}\\
&= g_A(t) + \sum_{i=1}^n \frac{f_i\bigl(x(t)\bigr)\dot{x}_i(t)}{f\bigl(x(t)\bigr)} && \text{(distributive law)}\\
&= g_A(t) + \sum_{i=1}^n \frac{f_i\bigl(x(t)\bigr)x_i(t)}{f\bigl(x(t)\bigr)}\,\frac{\dot{x}_i(t)}{x_i(t)} && \text{(multiply and divide by } x_i(t)\text{)}\\
&= g_A(t) + \sum_{i=1}^n \theta_i(t) g_{x_i}(t) && \text{(by (8)).}
\end{aligned}$$

Suppressing the dependence on t we may write

$$\boxed{g_y = g_A + \theta_1 g_{x_1} + \cdots + \theta_n g_{x_n}.}$$

This is what Clark calls the fundamental equation of economic growth.

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