Caltech Division of the Humanities and Social Sciences

Fixed point theory

KC Border

1 Sperner's Lemma

There are two commonly used definitions of a simplex. The one we use here follows Kuratowski [14] and makes simplexes open sets. The other definition corresponds to what we call closed simplexes.

1.1 Definition

A set $\{x^0, \dots, x^n\} \subset \boldsymbol{R}^{\mathrm{m}}$ is *affinely independent* if $\sum_{i=0}^n \lambda_i x^i = 0$ and $\sum_{i=0}^n \lambda_i = 0$ imply that $\lambda_0 = \cdots = \lambda_n = 0$.

An n*-simplex* is the set of all strictly positive convex combinations of an $n+1$ element affinely independent set. A *closed* n*-simplex* is the convex hull of an affinely independent set of $n+1$ vectors. The simplex $x^0 \cdots x^n$ (written without commas) is the set of strictly positive convex combinations of the x^i vectors, i.e.,

$$x^0 \cdots x^n = \left\{ \sum_{i=0}^n \lambda_i x^i : \lambda_i > 0,\ i = 0, \dots, n;\ \sum_{i=0}^n \lambda_i = 1 \right\}.$$

Each x^i is a *vertex* of $x^0 \cdots x^n$ and each k-simplex $x^{i_0} \cdots x^{i_k}$ is a *face* of $x^0 \cdots x^n$. By this definition each vertex is a face, and $x^0 \cdots x^n$ is a face of itself. It is easy to see that the closure of $x^0 \cdots x^n = \mathrm{co}\{x^0, \dots, x^n\}$. Given $y = \sum_{i=0}^n \lambda_i x^i \in \mathrm{co}\{x^0, \dots, x^n\}$, let $\chi(y) = \{i : \lambda_i > 0\}$. If $\chi(y) = \{i_0, \dots, i_k\}$, then $y \in x^{i_0} \cdots x^{i_k}$. This face is called the *carrier* of y. It follows that the union of the faces of $x^0 \cdots x^n$ is its closure.

If y belongs to the convex hull of the affinely independent set $\{x^0, \dots, x^n\}$, there is a unique set of numbers $\lambda_0, \dots, \lambda_n$ such that $y = \sum_{i=0}^n \lambda_i x^i$. Consequently y belongs to exactly one face of the simplex $x^0 \cdots x^n$. This means that the carrier as described above is well-defined. The numbers $\lambda_0, \dots, \lambda_n$ are called the *barycentric coordinates* of y.

The *standard* n*-simplex* is $\{y \in \boldsymbol{R}^{\mathrm{n+1}} : y_i > 0,\ i = 0, \dots, n;\ \sum_{i=0}^n y_i = 1\} = e^0 \cdots e^n$. Let Δ_n denote the closure of the standard n-simplex, which we call the *standard closed* n*-simplex*. (We may simply write Δ when n is apparent from the context.)

1.2 Definition

Let $T = x^0 \cdots x^n$ be an n-simplex. A *simplicial subdivision* of $\overline{T}$ is a finite collection of simplexes $\{T_i : i \in I\}$ satisfying $\bigcup_{i \in I} T_i = \overline{T}$ and such that for any $i, j \in I$, $\overline{T}_j \bigcap \overline{T}_i$ is either empty or equal to the closure of a common face. The *mesh* of a subdivision is the diameter of the largest subsimplex.

For course use only. This material is excerpted from Border [3].

1.3 Example

Refer to Figure 1. The collection

$$\{x^0x^2x^4, x^1x^2x^3, x^1x^3x^4, x^0x^2, x^0x^4,\\ x^1x^2, x^1x^3, x^1x^4, x^2x^3, x^3x^4, x^0, x^1, x^2, x^3, x^4\}$$

indicated by the solid lines is *not* a simplicial subdivision of $\overline{x^0x^1x^2}$. This is because $\overline{x^0x^2x^4} \bigcap \overline{x^1x^2x^3} = \overline{x^2x^3}$, which is not the closure of a face of $x^0x^2x^4$. By replacing $x^0x^2x^4$ by $x^0x^2x^3$, $x^0x^3x^4$ and x^0x^3 as indicated by the faint line, the result is a valid simplicial subdivision.

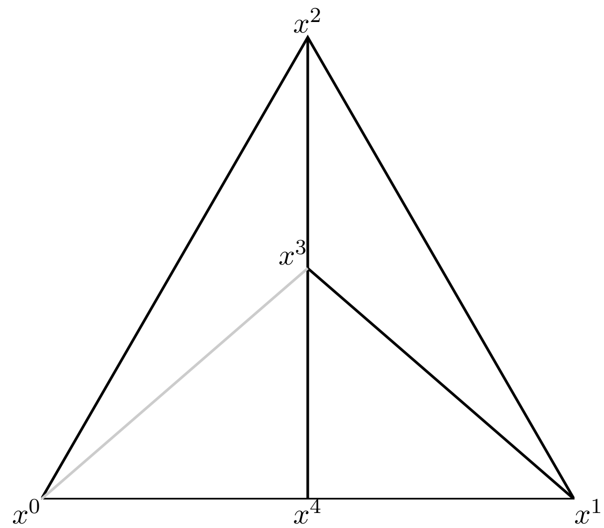


Figure 1. Not a simplicial subdivision.

1.4 Example: Barycentric Subdivision

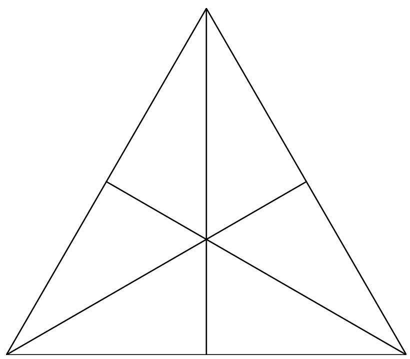
For any simplex $T = x^0 \cdots x^n$, the *barycenter* of T, denoted $b(T)$, is the point $\frac{1}{n+1}\sum_{i=0}^n x^i$. For simplexes T_1, T_2 define $T_1 > T_2$ to mean T_2 is a face of T_1 and $T_1 \neq T_2$. Given a simplex T, the family of all simplexes $b(T_0) \cdots b(T_k)$ such that $T \geqslant T_0 > T_1 > \cdots > T_k$ is a simplicial subdivision of $\overline{T}$ called the *first barycentric subdivision* of T. See Figure 2. Further barycentric subdivisions are defined recursively. It can be shown that there are barycentric subdivisions of arbitrarily small mesh.

1.5 Definition

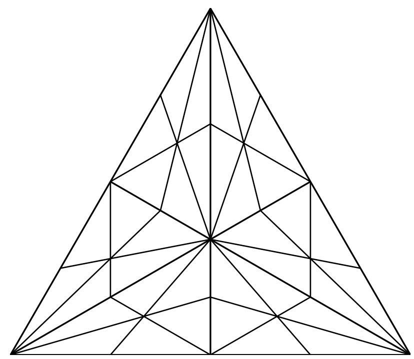
Let $\overline{T} = \overline{x^0 \cdots x^n}$ be simplicially subdivided. Let V denote the collection of all the vertexes of all the subsimplexes. (Note that each $x^i \in V$.) A function $\lambda\colon V \to \{0, \ldots, n\}$ satisfying

$$\lambda(v) \in \chi(v)$$

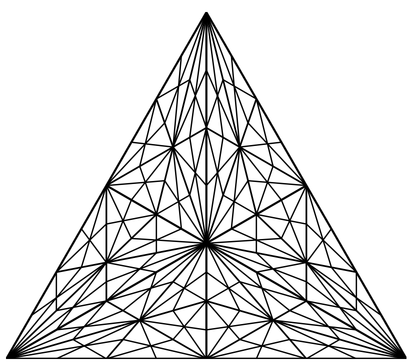
is called a *proper labeling* of the subdivision. (Recall the definition of the carrier χ from 1.1.) Call a subsimplex *completely labeled* if λ assumes all the values $0, \ldots, n$ on its set of vertexes.



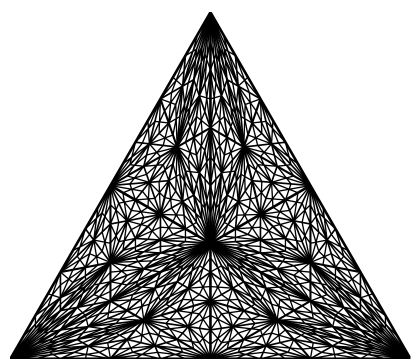
First barycentric subdivision.



Second barycentric subdivision.



Third barycentric subdivision.



Fourth barycentric subdivision.

Figure 2. Successive barycentric subdivisions of Δ_2.

1.6 Theorem (Sperner [19])

Let $\overline{T} = \overline{x^0 \cdots x^n}$ be simplicially subdivided and properly labeled by the function λ. Then there is an odd number of completely labeled subsimplexes in the subdivision.

1.7 Proof (Kuhn [13])

The proof is by induction on n. The case $n = 0$ is trivial. The simplex consists of a single point x^0, which must bear the label 0, and so there is one completely labeled subsimplex, x^0 itself.

We now assume the statement to be true for $n-1$ and prove it for n. Let

C denote the set of completely labeled n-simplexes;

A denote the set of almost completely labeled n-simplexes, i.e., those such that the range of λ is exactly $\{0, \ldots, n-1\}$;

B denote the set of $(n-1)$-simplexes on the boundary that bear all the labels $\{0, \ldots, n-1\}$; and

E denote the set of $(n-1)$-simplexes that bear all the labels $\{0, \ldots, n-1\}$.

An $n-1$ simplex either lies on the boundary and is the face of a single n-simplex in the subdivision or it is a common face of two n-simplexes. We can view this situation as a graph, i.e., a collection of nodes and edges joining them. Let $D = C \bigcup A \bigcup B$ be the set of nodes and E the set of edges. Define edge $e \in E$ and node $d \in D$ to be *incident* if either

(i) $d \in A \bigcup C$ and e is a face of d or

(ii) $e = d \in B$.

The *degree* of a node d, $\delta(d)$, is the number of edges incident at d. If $d \in A$, then one label is repeated and exactly two faces of d belong to E, so its degree is 2. The degree of $d \in B \bigcup C$ is 1. On the other hand, each edge is incident at exactly two nodes: If an $(n-1)$-simplex lies on the boundary and bears labels $\{0, \ldots, n-1\}$, then it is incident at itself (as a node in B) and at an n-simplex (which must be a node in either A or C). If an $(n-1)$-simplex is a common face of two n-simplexes, then each n-simplex belongs to either A or C.

Thus

$$\delta(d) = \begin{cases} 1 & d \in B \bigcup C \\ 2 & d \in A \end{cases}$$

A standard graph theoretic argument yields $\sum_{d \in D} \delta(d) = 2|E|$. That is, since each edge joins exactly two nodes, counting the number of edges incident at each node and adding them up counts each edge twice. By the definition of δ, $\sum_{d \in D} \delta(d) = 2|A| + |B| + |C|$. Thus $2|E| = 2|A| + |B| + |C|$ so that $|B| + |C|$ is even. Since $|B|$ is odd by the induction hypothesis, we must have that $|C|$ is odd.

2 The Knaster–Kuratowski–Mazurkiewicz lemma

2.1 Theorem (Knaster–Kuratowski–Mazurkiewicz [12])

Let $\Delta = \operatorname{co}\{e^0, \dots, e^m\} \subset \boldsymbol{R}^{\mathrm{m}}+1$ and let $\{F_0, \dots, F_m\}$ be a family of closed subsets of Δ such that for every $A \subset \{0, \dots, m\}$ we have

$$\operatorname{co}\{e^i : i \in A\} \subset \bigcup_{i \in A} F_i. \tag{1}$$

Then $\bigcap_{i=0}^m F_i$ is compact and nonempty.

2.2 Proof (Knaster–Kuratowski–Mazurkiewicz [12])

The intersection is clearly compact, being a closed subset of a compact set. Let $\eta > 0$ be given and subdivide Δ into subsimplexes of diameter $\leqslant \eta$. For a vertex v of the subdivision belonging to the face $e^{i_0} \cdots e^{i_k}$, by 1 there is some index i in $\{i_0, \dots, i_k\}$ with $v \in F_i$. If we label all the vertexes this way, then the labeling satisfies the hypotheses of Sperner's lemma, so there is a completely labeled subsimplex ${}^\eta p^0 \cdots {}^\eta p^m$, with ${}^\eta p^i \in F_i$ for each i. As $\eta \downarrow 0$, choose a convergent subsequence ${}^\eta p^i \to z$. Since F_i is closed and ${}^\eta p^i \in F_i$ for each i, we have $z \in \bigcap_{i=0}^m F_i$.

2.3 Corollary

Let $K = \operatorname{co}\{a^0, \dots, a^m\} \subset \boldsymbol{R}^k$ and let $\{F_0, \dots, F_m\}$ be a family of closed sets such that for every $A \subset \{0, \dots, m\}$ we have

$$\operatorname{co}\{a^i : i \in A\} \subset \bigcup_{i \in A} F_i. \tag{2}$$

Then $K \cap \bigcap_{i=0}^m F_i$ is compact and nonempty.

2.4 Proof

Again compactness is immediate. Define the mapping $\sigma \colon \Delta \to K$ by $\sigma(z) = \sum_{i=0}^m z_i a^i$. If $\{a^0, \dots, a^m\}$ is not an affinely independent set, then σ is not injective, but it is nevertheless continuous. Put $E_i = \sigma^{-1}[F_i \bigcap K]$ for each i. Since σ is continuous, each E_i is a closed subset of Δ. It is straightforward to verify that 1 is satisfied by $\{E_0, \dots, E_m\}$, so let $z \in \bigcap_{i=0}^m E_i \neq \varnothing$. Then $\sigma(z) \in \bigcap_{i=0}^m F_i \neq \varnothing$.

2.5 Corollary (Fan [7])

Let $X \subset \boldsymbol{R}^{\mathrm{m}}$, and for each $x \in X$ let $F(x) \subset \boldsymbol{R}^{\mathrm{m}}$ be closed. Suppose:

(i) For any finite subset $\{x^1, \dots, x^k\} \subset X$, $\operatorname{co}\{x^1, \dots, x^k\} \subset \bigcup_{i=1}^k F(x^i)$.

(ii) $F(x)$ is compact for some $x \in X$.

Then $\bigcap_{x \in X} F(x)$ is compact and nonempty.

2.6 Proof

The conclusion follows from Corollary 2.3 and the fact that in a compact set, a family of closed sets with the finite intersection property has a nonempty intersection. (Rudin [17, 2.36].)

3 Brouwer's fixed point theorem

3.1 Remark

The basic fixed point theorem that we will use is due to Brouwer [5]. For our purposes the most useful form of Brouwer's fixed point theorem is Corollary 3.7 below, but the simplest version to prove is Theorem 3.2.

3.2 Theorem

Let $f\colon \Delta_m \to \Delta_m$ be continuous. Then f has a fixed point.

3.3 Proof

Let $\eta > 0$ be given and subdivide Δ simplicially into subsimplexes of diameter $\leqslant \eta$. Let V be the set of vertexes of the subdivision and define a labeling function $\lambda\colon V \to \{0, \dots, m\}$ as follows. For $v \in e^{i_0} \cdots e^{i_k}$ choose

$$\lambda(v) \in \{i_0, \dots, i_k\} \bigcap \{i : f_i(v) \leqslant v_i\}.$$

(This intersection is nonempty, for if $f_i(v) > v_i$ for all $i \in \{i_0, \dots, i_k\}$, we would have

$$1 = \sum_{i=0}^{m} f_i(v) > \sum_{j=0}^{k} v_{i_j} = \sum_{i=0}^{m} v_i = 1,$$

a contradiction, where the second equality follows from $v \in x^{i_0} \cdots x^{i_k}$.) Since λ so defined satisfies the hypotheses of Sperner's lemma (1.6), there exists a completely labeled subsimplex. That is, there is a simplex ${}^{\eta}p^0 \cdots {}^{\eta}p^m$ such that $f_i({}^{\eta}p^i) \leqslant^{\eta} p^i_i$ for each i. Letting $\eta \downarrow 0$ we can extract a convergent subsequence (as Δ is compact) of simplexes such that ${}^{\eta}p^i \to z$ as $\eta \to 0$ for all $i = 0, \dots, m$. Since f is continuous we must have $f_i(z) \leqslant z_i$, $i = 0, \dots, m$, so $f(z) = z$.

3.4 Definition

A set A is *homeomorphic* to the set B if there is a bijective continuous function $h\colon A \to B$ such that h^{-1} is also continuous. Such a function h is called a *homeomorphism*.

3.5 Corollary

Let K be homeomorphic to Δ and let $f\colon K \to K$ be continuous. Then f has a fixed point.

3.6 Proof

Let $h\colon \Delta \to K$ be a homeomorphism. Then $h^{-1} \circ f \circ h\colon \Delta \to \Delta$ is continuous, so there exists z' with $h^{-1} \circ f \circ h(z') = z'$. Set $z = h(z')$. Then $h^{-1}(f(z)) = h^{-1}(z)$, so $f(z) = z$ as h is injective.

3.7 Corollary

Let $K \subset \boldsymbol{R}^{\mathrm{m}}$ be convex and compact and let $f\colon K \to K$ be continuous. Then f has a fixed point.

3.8 Proof

Since K is compact, it is contained in some sufficiently large simplex T. Define $h\colon \overline{T} \to K$ by setting $h(x)$ equal to the point in K closest to x. Then h is continuous and is equal to the identity on K. So $f \circ h\colon \overline{T} \to K \subset \overline{T}$ has a fixed point z. Such a fixed point cannot belong to $\overline{T} \setminus K$, as $f \circ h$ maps into K. Thus $z \in K$ and $f \circ h(z) = z$; but $h(z) = z$, so $f(z) = z$.

3.9 Note

The above method of proof provides a somewhat more general theorem. Following Borsuk [4], we say that E is an *r-image* of F if there are continuous functions $h\colon F \to E$ and $g\colon E \to F$ such that $h \circ g$ is the identity on E. Such a function h is called an *r-map* of F onto E. In particular, if h is a homeomorphism, then it is an r-map. In the special case where $E \subset F$ and g is the inclusion map, i.e., the identity map on E, we say that E is a *retract* of F and that h is a *retraction*.

3.10 Theorem

Let E be an r-image of a compact convex set $K \subset \boldsymbol{R}^{\mathrm{m}}$, and let $f\colon E \to E$ be continuous. Then f has a fixed point.

3.11 Proof

The map $g \circ f \circ h\colon K \to K$ has a fixed point z, $(g \circ f)(h(z)) = z$. Set $x = h(z) \in E$. Then $(g \circ f)(x) = z$, so $h \circ g \circ f(x) = h(z) = x$, but $h \circ g$ is the identity on E, so $f(x) = x$.

3.12 Remark

Let B_m be the unit ball in $\boldsymbol{R}^{\mathrm{m}}$, i.e., $B_m = \{x \in \boldsymbol{R}^{\mathrm{m}} : |x| \leqslant 1\}$, and let $\partial B_m = \{x \in \boldsymbol{R}^{\mathrm{m}} : |x| = 1\}$. The following theorem is equivalent to the fixed point theorem.

3.13 Theorem

∂B_m is not an r-image of B_m.

3.14 Proof

Suppose ∂B is an r-image of B. Then there are continuous functions $g\colon \partial B \to B$ and $h\colon B \to \partial B$ such that $h \circ g$ is the identity. Define $f(x) = g(-h(x))$. Then f is continuous and maps B into itself and so by 3.7 has a fixed point z. That is, $z = g(-h(z))$ and so $h(z) = (h \circ g)(-h(z)) = -h(z)$. Thus $h(z) = 0 \notin \partial B$, a contradiction.

3.15 Exercise: Theorem 3.13 implies the fixed point theorem for balls

Hint: Let $f\colon B \to B$ be continuous and suppose that f has no fixed point. For each x let $\lambda(x) = \max\{\lambda : |x + \lambda(f(x) - x)| = 1\}$ and put $h(x) = x + \lambda(x)(f(x) - x)$. Then h is an r-map of B onto ∂B.

3.16 Note

For any continuous function $f\colon E \to \boldsymbol{R}^{\mathrm{m}}$, the set of fixed points $\{x : x = f(x)\}$ is a closed (but possibly empty) subset of E. If E is compact, then the set of fixed points is also compact.

4 Maximization of binary relations

4.1 Definition

A *binary relation* U on a set K associates to each $x \in K$ a set $U(x) \subset K$, which may be interpreted as the set of those objects in K that are "better" "larger" or "after" x. Define $U^{-1}(x) = \{y \in K : x \in U(y)\}$. An element $x \in K$ is *U-maximal* if $U(x) = \varnothing$. The *U-maximal set* is $\{x \in K : U(x) = \varnothing\}$. The *graph* of U is $\{(x, y) : y \in U(x)\}$.

4.2 Theorem (cf. Sonnenschein [18])

Let $K \subset \boldsymbol{R}^{\mathrm{m}}$ be compact and convex and let U be a relation on K satisfying the following:

(i) $x \notin \operatorname{co} U(x)$ for all $x \in K$.

(ii) if $y \in U^{-1}(x)$, then there exists some $x' \in K$ (possibly $x' = x$) such that $y \in \operatorname{int} U^{-1}(x')$.

Then K has a U-maximal element, and the U-maximal set is compact.

4.3 Proof (cf. Fan [7, Lemma 4]; Sonnenschein [18, Theorem 4])

Note that $\{x : U(x) = \varnothing\}$ is just $\bigcap_{x\in K} \big(K \setminus U^{-1}(x)\big)$. By hypothesis (ii),

$$\bigcap_{x\in K} (K \setminus U^{-1}(x)) = \bigcap_{x'\in K} (K \setminus \operatorname{int} U^{-1}(x')).$$

This latter intersection is clearly compact, being the intersection of compact sets.

For each x, put $F(x) = K \setminus \operatorname{int} U^{-1}(x)$. As noted above, each $F(x)$ is compact. If $y \in \operatorname{co}\{x^i : i = 1, \ldots, n\}$, then $y \in \bigcup_{i=1}^n F(x^i)$: Suppose that $y \notin \bigcup_{i=1}^n F(x^i)$. Then $y \in U^{-1}(x^i)$ for all i, so $x^i \in U(y)$ for all i. But then $y \in \operatorname{co}\{x^i\} \subset \operatorname{co} U(y)$, which violates (i). It then follows from the Knaster–Kuratowski–Mazurkiewicz lemma as extended by Fan (2.5) that $\bigcap_{x\in K} F(x) \neq \varnothing$.

4.4 Corollary (Fan's Lemma [7, Lemma 4])

Let $K \subset \boldsymbol{R}^{\mathrm{m}}$ be compact and convex. Let $E \subset K \times K$ be closed and suppose

(i) $(x, x) \in E$ for all $x \in K$.

(ii) for each $y \in K, \{x \in K : (x, y) \notin E\}$ is convex (possibly empty).

Then there exists $\bar{y} \in K$ such that $K \times \{\bar{y}\} \subset E$. The set of such $\bar{y}$ is compact.

4.5 Corollary (Fan's Lemma – Alternate Statement)

Let $K \subset \boldsymbol{R}^{\mathrm{m}}$ be compact and let U be a relation on K satisfying:

(i) $x \notin U(x)$ for all $x \in K$.

(ii) $U(x)$ is convex for all $x \in K$.

(iii) $\{(x, y) : y \in U(x)\}$ is open in $K \times K$.

Then the U-maximal set is compact and nonempty.

4.6 Exercise

Show that both statements of Fan's lemma are special cases of Theorem 4.2.

4.7 Definition

A set $C \subset \boldsymbol{R}^{\mathrm{m}}$ is called *σ-compact* if there is a sequence $\{C_n\}$ of compact subsets of C satisfying $\bigcup_n C_n = C$. The euclidean space $\boldsymbol{R}^{\mathrm{m}}$ is itself σ-compact as $\boldsymbol{R}^{\mathrm{m}} = \bigcup_n \{x : |x| \leqslant n\}$. So is any closed convex cone in $\boldsymbol{R}^{\mathrm{m}}$. Another example is the open unit ball, $\{x : |x| < 1\} = \bigcup_n \{x : |x| \leqslant 1 - \frac{1}{n}\}$.

Let $C = \bigcup_n C_n$, where $\{C_n\}$ is an increasing sequence of nonempty compact sets. A sequence $\{x^k\}$ is said to be *escaping from C* (relative to $\{C_n\}$) if for each n there is an M such that for all $k \geqslant M$, $x^k \notin C_n$. A *boundary condition* on a binary relation on C puts restrictions on escaping sequences. Boundary conditions can be used to guarantee the existence of maximal elements for sets that are not compact. Theorems 4.8 and 4.10 below are two examples.

4.8 Proposition

Let $C \subset \boldsymbol{R}^{\mathrm{m}}$ be convex and σ-compact and let U be a binary relation on C satisfying

(i) $x \notin \operatorname{co} U(x)$ for all $x \in C$.

(ii) $U^{-1}(x)$ is open (in C) for each $x \in C$.

Let $D \subset C$ be compact and satisfy

(iii) for each $x \in C \setminus D$, there exists $z \in D$ with $z \in U(x)$.

Then C has a U-maximal element. The set of U-maximal elements is a compact subset of D.

4.9 Proof

Since C is σ-compact, there is a sequence $\{C_n\}$ of compact subsets of C satisfying $\bigcup_n C_n = C$. Set $K_n = \operatorname{co} \left(\bigcup_{j=1}^n C_j \bigcup D \right)$. Then $\{K_n\}$ is an increasing sequence of compact convex sets each containing D with $\bigcup_n K_n = C$. By Theorem 4.2, it follows from (i) and (ii) that each K_n has a U-maximal element x^n, i.e., $U(x^n) \bigcap K_n = \varnothing$. Since $D \subset K_n$, (iii) implies that $x^n \in D$. Since D is compact, we can extract a convergent subsequence $x^n \to \bar{x} \in D$.

Suppose that $U(\bar{x}) \neq \varnothing$. Let $z \in U(\bar{x})$. By (ii) there is a neighborhood W of $\bar{x}$ contained in $U^{-1}(z)$. For large enough n, $x^n \in W$ and $z \in K_n$. Thus $z \in U(x^n) \bigcap K_n$, contradicting the maximality of x^n. Thus $U(\bar{x}) = \varnothing$.

Hypothesis (iii) implies that any U-maximal element must belong to D, and (ii) implies that the U-maximal set is closed. Thus the U-maximal set is a compact subset of D.

4.10 Theorem

Let $C = \bigcup_n C_n$, where $\{C_n\}$ is an increasing sequence of nonempty compact convex subsets of $\boldsymbol{R}^{\mathrm{m}}$. Let U be a binary relation on C satisfying the following:

(i) $x \notin \operatorname{co} U(x)$ for all $x \in C$.

(ii) $U^{-1}(x)$ is open (in C) for each $x \in C$.

(iii) For each escaping sequence $\{x^n\}$, there is a $z \in C$ such that $z \in U(x^n)$ for infinitely many n.

Then C has a U-maximal element and the U-maximal set is a closed subset of C.

4.11 Proof

By 4.2 each C_n has a U-maximal element x^n, i.e., $U(x^n) \bigcap C_n = \varnothing$. Suppose the sequence $\{x^n\}$ were escaping from C. Then by the boundary condition (iii), there is a $z \in C$ such that $z \in U(x^n)$ infinitely often. But since $\{C_n\}$ is increasing, $z \in C_k$ for all sufficiently large k. Thus for infinitely many n, $z \in U(x^n) \bigcap C_k$, which contradicts the U-maximality of x^k. Thus $\{x^n\}$ is not escaping from C. This means that some subsequence of $\{x^n\}$ must lie entirely in some C_k, which is compact. Thus there is a subsequence of $\{x^n\}$ converging to some $\bar{x} \in C$.

This $\bar{x}$ is U-maximal: Let $x^n \to \bar{x}$ be a convergent subsequence and suppose that there exists some $y \in U(\bar{x})$. Then for sufficiently large k, $y \in C_k$, and by (ii) there is a neighborhood of $\bar{x}$ contained in $U^{-1}(y)$. So for large enough k, $y \in C_k \bigcap U(x^k)$, again contradicting the maximality of x^k. Thus $U(\bar{x}) = \varnothing$. The closedness of the U-maximal set follows from (ii).

5 Variational inequalities, price equilibrium, and complementarity

5.1 Lemma (Hartman and Stampacchia [11, Lemma 1.1])

Let $K \subset \boldsymbol{R}^{\mathrm{m}}$ be compact and convex and let $f \colon K \to \boldsymbol{R}^{\mathrm{m}}$ be continuous. Then there exists $\bar{p} \in K$ such that for all $p \in K$,

$$\bar{p} \cdot f(\bar{p}) \geqslant p \cdot f(\bar{p}).$$

Furthermore, the set of such $\bar{p}$ is compact.

5.2 Proof

Define the relation U on K by $q \in U(p)$ if and only if

$$q \cdot f(p) > p \cdot f(p).$$

Since f is continuous, U has open graph. Also $U(p)$ is convex and $p \notin U(p)$ for each $p \in K$. Thus by Fan's lemma (4.5), there is a $\bar{p} \in K$ with $U(\bar{p}) = \varnothing$, i.e., for each $p \in K$ it is not true that $p \cdot f(\bar{p}) > \bar{p} \cdot f(\bar{p})$. Thus for all $p \in K$, $\bar{p} \cdot f(\bar{p}) \geqslant p \cdot f(\bar{p})$. Conversely, any such $\bar{p}$ is U-maximal, so the U-maximal set is compact by 4.5.

5.3 Equilibrium Theorem

Let $f \colon \Delta_m \to \boldsymbol{R}^{m+1}$ be continuous and satisfy

$$p \cdot f(p) \leqslant 0 \text{ for all } p.$$

Then the set $\{p \in \Delta : f(p) \leqq 0\}$ of free disposal equilibrium prices is compact and nonempty.

5.4 Proof

Compactness is immediate. From 5.1 and Walras' law, there is a $\bar{p} \in \Delta$ such that $p \cdot f(\bar{p}) \leqslant \bar{p} \cdot f(\bar{p}) \leqslant 0$ for all $p \in \Delta$. Therefore $f(\bar{p}) \leqq 0$.

5.5 Definition

Let $S_m = \{x \in \Delta_m : x_i > 0,\ i = 0, \dots, m\}$, the standard m-simplex. The function $f \colon S \to \boldsymbol{R}^{\mathrm{m}+1}$ satisfies the *boundary condition* (B1) if the following holds.

(B1) there is a $p^* \in S$ and a neighborhood V of $\Delta \setminus S$ in Δ such that for all $p \in V \bigcap S$, $p^* \cdot f(p) > 0$.

5.6 Theorem (Neuefeind [15, Lemma 1])

Let $f \colon S \to \boldsymbol{R}^{\mathrm{n}+1}$ be continuous and satisfy the strong form of Walras' law and the boundary condition (B1):

(SWL) $p \cdot f(p) = 0$.

(B1) there is a $p^* \in S$ and a neighborhood V of $\Delta \setminus S$ in Δ such that for all $p \in V \bigcap S$, $p^* \cdot f(p) > 0$.

Then the set $\{p : f(p) = 0\}$ of equilibrium prices for f is compact and nonempty.

5.7 Proof (cf. Aliprantis and Brown [1])

Define the binary relation U on Δ by

$$p \in U(q) \text{ if } \begin{cases} p \cdot f(q) > 0 \text{ and } p, q \in S \\ \qquad\qquad \text{or} \\ \quad p \in S, \; q \in \Delta \setminus S. \end{cases}$$

There are two steps in the proof. The first is to show that the U-maximal elements are precisely the equilibrium prices. The second step is to show that U satisfies the hypotheses of 4.2.

First suppose that $\bar{p}$ is U-maximal, i.e., $U(\bar{p}) = \varnothing$. Since $U(p) = S$ for all $p \in \Delta \setminus S$, we have that $\bar{p} \in S$. Since $\bar{p} \in S$ and $U(\bar{p}) = \varnothing$, we have

$$\text{for each } q \in S, q \cdot f(\bar{p}) \leqslant 0.$$

Therefore $f(\bar{p}) \leqslant 0$. But the strong form of Walras' law says that $\bar{p} \cdot f(\bar{p}) = 0$. Since $\bar{p} \in S$, we must have that $f(\bar{p}) = 0$.

Conversely, if $\bar{p}$ is an equilibrium price, then $0 = f(\bar{p})$ and since $p \cdot 0 = 0$ for all p, $U(\bar{p}) = \varnothing$.

Verify that U satisfies the hypotheses of 4.2:

(ia) $p \notin U(p)$: For $p \in S$ this follows from Walras' law. For $p \in \Delta \setminus S$, $p \notin S = U(p)$.

(ib) $U(p)$ is convex: For $p \in S$, this is immediate. For $p \in \Delta \setminus S$, $U(p) = S$, which is convex.

(ii) If $q \in U^{-1}(p)$, then there is a p' with $q \in \operatorname{int} U^{-1}(p')$: There are two cases: (a) $q \in S$ and (b) $q \in \Delta \setminus S$.

(iia) $q \in S \bigcap U^{-1}(p)$. Then $p \cdot f(q) > 0$. Let $H = \{z : p \cdot z > 0\}$. Then by continuity of f, $f^{-1}[H]$ is a neighborhood of q contained in $U^{-1}(p)$.

(iib) $q \in (\Delta \setminus S) \bigcap U^{-1}(p)$. By boundary condition (B1) $q \in \operatorname{int} U^{-1}(p^*)$.

6 Alternate arguments

6.1 Brouwer's Theorem (3.7) Implies the K–K–M Lemma (2.3)

Let $K = \operatorname{co}\{a^i : i = 0, \dots, m\}$. Then K is convex and compact. Suppose by way of contradiction that $\bigcap_{i=0}^m F_i = \varnothing$. Then $\{F_i^c\}$ is an open cover of K and so there is a partition of unity $f_0, \dots, f_m$ subordinate to it. Define $g \colon K \to K$ by $g(x) = \sum_{i=0}^m f_i(x) a^i$. This g is continuous and hence by 3.7 has a fixed point z. Let $A = \{i : f_i(z) > 0\}$. Then $z \in \operatorname{co}\{a^i : i \in A\}$ and $z \notin F_i$ for each $i \in A$, which contradicts $\operatorname{co}\{a^i : i \in A\} \subset \bigcup_{i \in A} F_i$.

6.2 Another Proof of the K–K–M Lemma (2.1) Using Brouwer's Theorem (cf. Peleg [16])

Let $F_0, \dots, F_m$ satisfy the hypotheses of 2.1. Set $g_i(x) = \operatorname{dist}(x, F_i)$ and define $f \colon \Delta \to \Delta$ by

$$f_i(x) = \frac{x_i + g_i(x)}{1 + \sum_{j=0}^m g_j(x)}.$$

The function f is clearly continuous, so by Brouwer's theorem it has a fixed point $\bar{x}$. Now $\bar{x} \in \bigcup_{i=0}^m F_i$ by hypothesis, so some $g_i(\bar{x}) = 0$. For this particular i,

$$\bar{x}_i = \frac{\bar{x}_i}{1 + \sum_{j=0}^m g_j(\bar{x})},$$

which implies $g_j(\bar{x}) = 0$ for all j. That is, $\bigcap_{j=0}^m F_j \neq \varnothing$.

6.3 The K–K–M Lemma (2.1) Implies the Brouwer Theorem (3.2) (K–K–M [12])

Let $f\colon \Delta_m \to \Delta_m$ be continuous. Put $F_i = \{z \in \Delta : f_i(z) \leqslant z_i\}$. The collections $\{e^0, \dots, e^m\}$ and $\{F_0, \dots, F_m\}$ satisfy the hypotheses of the K–K–M lemma: For suppose $z \in e^{i_0} \dots e^{i_k}$, then $\sum_{i=0}^m f_i(z) = \sum_{j=0}^k z_{i_j}$ and therefore at least one $f_{i_j}(z) \leqslant z_{i_j}$, so $z \in F_{i_j}$. Also each F_i is closed as f is continuous. Thus $\bigcap_{i=0}^m F_i$ is compact and nonempty but $\bigcap_{i=0}^m F_i$ is $\{x \in \Delta : f(x) \leqq x\}$ which is just the set of fixed points of f.

6.4 The K–K–M Lemma (2.1) Implies the Equilibrium Theorem (5.3) (Gale [9])

Put $F_i = \{p \in \Delta : f_i(p) \leqslant 0\}, i = 0, \dots, m$. Then $\{e^0, \dots, e^m\}$ and $\{F_0, \dots, F_m\}$ satisfy the hypotheses of the K–K–M lemma: For if $p \in \operatorname{co}\{e^{i_0}, \dots, e^{i_k}\}$, we cannot have $f_{i_j}(p) > 0$ for all $j = 0, \dots, k$, since then $p \cdot f(p) = \sum_{j=0}^k p_{i_k} f_{i_k}(p) > 0$, a contradiction. Thus $\operatorname{co}\{e^i : i \in A\} \subset \bigcup_{i \in A} F_i$, for any $A \subset \{0, \dots, m\}$, and each F_i is closed as f is continuous. Thus $\{p : f(p) \leqq 0\} = \bigcap_{i=0}^m F_i$ is compact and nonempty.

6.5 The Equilibrium Theorem (5.3) Implies the Brouwer Theorem (3.2) (Uzawa [20])

Let $f\colon \Delta_m \to \Delta_m$ be continuous. Define $g\colon \Delta \to \boldsymbol{R}^{m+1}$ via

$$g(x) = f(x) - \frac{x \cdot f(x)}{x \cdot x} x.$$

Then g is continuous and satisfies

$$x \cdot g(x) = x \cdot f(x) - \frac{x \cdot f(x)}{x \cdot x} x \cdot x = 0 \text{ for all } x,$$

i.e., g projects $f(x)$ onto the hyperplane through zero to which x is normal. Thus by 5.3 there is a $p \in \Delta$ with $g(p) \leqq 0$, i.e.,

$$f_i(p) \leqslant \frac{p \cdot f(p)}{p \cdot p} p_i \quad i = 0, \dots, n. \tag{3}$$

If $p_i = 0$ then 3, implies $f_i(p) \leqslant 0$ but $f_i(p) \geqslant 0$ as $f(p) \in \Delta$; so $f_i(p) = 0$ and hence

$$f_i(p) = \frac{p \cdot f(p)}{p \cdot p} p_i.$$

If, on the other hand, $p_i > 0$, then $p \cdot g(p) = 0$ and $g(p) \leqslant 0$ imply $g_i(p) = 0$ or

$$f_i(p) = \frac{p \cdot f(p)}{p \cdot p} p_i.$$

Thus 3 must hold with equality for each i. Summing then over i yields $\frac{p \cdot f(p)}{p \cdot p} = 1$, so $p = f(p)$.

Thus $g(p) \leqslant 0$ implies $p = f(p)$, and the converse is clearly true. Hence $\{p : g(p) \leqq 0\} = \{p : p = f(p)\}$.

6.6 Fan's Lemma (4.5) Implies the Equilibrium Theorem (5.3) (Brown [6])

For each $p \in \Delta$ define $U(p) = \{q \in \Delta : q \cdot f(p) > 0\}$. Then $U(p)$ is convex for each p and Walras' law implies that $p \notin U(p)$. The continuity of f implies that U has open graph. If p is U-maximal, then $U(p) = \varnothing$, so for all $q \in \Delta$, $q \cdot f(p) \leqslant 0$. Thus $f(p) \leqq 0$. If $f(p) \leqq 0$, then $q \cdot f(p) \leqslant 0$ for all $q \in \Delta$; so by 4.5, $\{p : f(p) \leqq 0\}$ is compact and nonempty.

6.7 Fan's Lemma (4.5) Implies Brouwer's Theorem (3.7) (cf. Fan [8, Theorem 2])

Let $f\colon K \to K$ be continuous, and for each x set $U(x) = \{y : |y - f(x)| < |x - f(x)|\}$. Then for each x, U(x) is convex, $x \notin U(x)$, and U has open graph. If x is U-maximal, then for all $y \in K$, $|x - f(x)| \leqslant |y - f(x)|$. Picking $y = f(x)$ yields $|x - f(x)| = 0$, so $f(x) = x$. Conversely, if x is a fixed point, then $U(x) = \{y : |y - f(x)| < 0\} = \varnothing$. The conclusion is now immediate from 4.5.

6.8 Remark

The above argument implies the following generalization of Brouwer's fixed point theorem, which in turn yields another proof of Lemma 5.1.

6.9 Proposition (Fan [8, Theorem 2])

Let $K \subset \boldsymbol{R}^{\mathrm{m}}$ be nonempty compact and convex, and let $f\colon K \to \boldsymbol{R}^{\mathrm{m}}$ be continuous. Then there exists a point $\bar{x} \in K$ such that

$$|\bar{x} - f(\bar{x})| \leqslant |x - f(\bar{x})| \text{ for all } x \in K.$$

(Consequently, if $f(K) \subset K$, then $\bar{x}$ is a fixed point of f.)

6.10 The Brouwer Theorem Implies Theorem 4.2 (cf. Anderson [2, p. 66])

Suppose $U(x) \neq \varnothing$ for each x. Then for each x there is $y \in U(x)$ and so $x \in U^{-1}(y)$. Thus $\{U^{-1}(y) : y \in K\}$ covers K. By (ii), $\{\operatorname{int} U^{-1}(y) : y \in K\}$ is an open cover of K. Let $f^1, \dots, f^k$ be a partition of unity subordinate to the finite subcover $\{\operatorname{int} U^{-1}(y^1), \dots, \operatorname{int} U^{-1}(y^k)\}$. Define the continuous function $g\colon K \to K$ by $g(x) = \sum_{i=l}^{k} f^i(x) y^i$. It follows from the Brouwer fixed point theorem that g has a fixed point $\bar{x}$. Let $A = \{i : f^i(x) > 0\}$. Then $\bar{x} \in U^{-1}(y^i)$ or $y^i \in U(\bar{x})$ for all $i \in A$. Thus $\bar{x} \in \operatorname{co}\{y^i : i \in A\} \subset \operatorname{co} U(\bar{x})$, a contradiction. Thus $\{x : U(x) = \varnothing\}$ is nonempty. It is clearly closed, and hence compact, as K is compact.

6.11 The Brouwer Theorem (3.2) Implies the Equilibrium Theorem (5.3)

Define the price adjustment function $h\colon \Delta \to \Delta$ by

$$h(p) = \frac{p + f(p)^+}{1 + \sum_i f(p)_i^+}$$

where $f_i(p)^+ = \max\{f_i(p), 0\}$ and $f(p)^+ = (f_0(p)^+, \dots, f_n(p)^+)$. This is readily seen to satisfy the hypotheses of 3.2 and so has a fixed point $\bar{p}$, i.e.,

$$\bar{p} = \frac{\bar{p} + f(\bar{p})^+}{1 + \sum_i f_i(\bar{p})^+}.$$

By Walras' law $\bar{p} \cdot f(\bar{p}) \leqslant 0$; so for some i, we must have $\bar{p}_i > 0$ and $f_i(\bar{p}) \leqslant 0$. (Otherwise $\bar{p} \cdot f(\bar{p}) > 0$.) For this i, $f(\bar{p})^+ = 0$, and since

$$\bar{p} = \frac{\bar{p} + f(\bar{p})^+}{1 + \sum_i f_i(\bar{p})^+},$$

it follows that $\sum_i f_i(\bar{p})^+ = 0$. But this implies $f(\bar{p}) \leqq 0$.

7 What good is a completely labeled subsimplex?

7.1 Theorem

Let $\{F_0, \dots, F_m\}$ satisfy the hypotheses of the K–K–M lemma (2.1). Let Δ be simplicially subdivided and labeled as in 2.2. Set $F = \bigcap_{i=0}^m F_i$. Then for every $\eta > 0$ there is a $\delta > 0$, such that if the mesh of the subdivision is less than δ, then every completely labeled subsimplex lies in $N_\eta(F)$.

7.2 Proof

Put $g^i(x) = \operatorname{dist}(x, F_i)$ and $g = \max_i g^i$. Since $K \setminus (N_\eta(F))$ is compact, and g is continuous, it follows that g achieves a minimum value $\delta > 0$. Let $x^0 \cdots x^m$ be a completely labeled subsimplex of diameter $< \delta$ containing the point x. Since $x^0 \cdots x^m$ is completely labeled, $x^i \in F_i$ and so $\operatorname{dist}(x, F_i) \leqslant |x - x_i| < \delta$ for all i. Thus $g(x) < \delta$, so $x \in N_\eta(F)$.

7.3 Theorem

Let $f\colon \Delta \to \Delta$ and put $F = \{z : f(z) = z\}$. Let Δ be subdivided and labeled as in 3.3. Then for every $\eta > 0$ there is a $\delta > 0$, such that if the mesh of the subdivision is less than δ, then every completely labeled subsimplex lies in $N_\eta(F)$.

7.4 Proof

Put $F_i = \{z : f_i(z) \leqslant z_i\}$. Then each F_i is closed and $F = \bigcap_{i=0}^m F_i$. If the simplex $x^0 \cdots x^m$ is completely labeled, then $x^i \in F_i$ and the conclusion follows from 7.1.

7.5 Theorem

Let f satisfy the hypotheses of Brouwer's fixed point theorem (3.7) and let F be the set of fixed points of f. Then for every $\eta > 0$ there is a $\delta > 0$ such that $|f(z) - z| < \delta$ implies $z \in N_\eta(F)$.

7.6 Proof (Green [10])

Set $g(z) = |f(z) - z|$. Since $C = K \setminus N_\eta(F)$ is compact and g is continuous, $\delta = \min_{z \in C} g(z)$ satisfies the conclusion of the theorem.

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