Caltech Division of the Humanities and Social Sciences

# The First Welfare Theorem

KC Border

January 2000
v. 2016.01.19::11.26

**First Welfare Theorem** *Let*

$$\left((X_i, \underset{i}{\succcurlyeq}, \omega^i)_{i=1}^m, (Y_j)_{j=1}^n, (\theta_j^i)_{j=1,\dots,n}^{i=1,\dots,m}\right)$$

*be a private ownership economy (see the notes on the Arrow–Debreu–McKenzie model), and let*

$$(\bar{x}^1, \dots, \bar{x}^m, \bar{y}^1, \dots, \bar{y}^n, \bar{p})$$

*be a Walrasian equilibrium. Assume that every preference relation is locally nonsatiated. Then*

$$(\bar{x}^1, \dots, \bar{x}^m, \bar{y}^1, \dots, \bar{y}^n)$$

*is an efficient allocation.*

*Proof*: Suppose by way of contradiction that the allocation is inefficient. That is, that there exists another allocation

$$(\hat{x}^1, \dots, \hat{x}^m, \hat{y}^1, \dots, \hat{y}^n)$$

such that

$$\hat{x}^i \underset{i}{\succcurlyeq} \bar{x}^i \text{ for all } i \quad \text{and} \quad \hat{x}^i \underset{i}{\succ} \bar{x}^i \text{ for some } i.$$

Since every preference relation is locally nonsatiated (see below), and consumers are maximizing utility we have

$$\hat{x}^i \underset{i}{\succcurlyeq} \bar{x}^i \implies \bar{p}\cdot\hat{x}^i \geqslant \bar{p}\cdot\bar{x}^i \quad \text{and} \quad \hat{x}^i \underset{i}{\succ} \bar{x}^i \implies \bar{p}\cdot\hat{x}^i > \bar{p}\cdot\bar{x}^i.$$

Summing over $i$ gives

$$\bar{p}\cdot\sum_{i=1}^m \hat{x}^i > \bar{p}\cdot\sum_{i=1}^m \bar{x}^i.$$

Since firms are maximizing profits, for each $j$,

$$\bar{p} \cdot \bar{y}^j \geqslant \bar{p} \cdot \hat{y}^j,$$

so summing gives

$$\bar{p} \cdot \sum_{j=1}^{n} \bar{y}^j \geqslant \bar{p} \cdot \sum_{j=1}^{n} \hat{y}^j.$$

On the other hand, by definition of allocation we have

$$\sum_{i=1}^{m} \bar{x}^i = \sum_{i=1}^{m} \omega^i + \sum_{j=1}^{n} \bar{y}^j$$

and

$$\sum_{i=1}^{m} \hat{x}^i = \sum_{i=1}^{m} \omega^i + \sum_{j=1}^{n} \hat{y}^j.$$

Stringing these together gives

$$\begin{aligned} \bar{p} \cdot \left( \sum_{i=1}^{m} \omega^i + \sum_{j=1}^{n} \bar{y}^j \right) &\geqslant \bar{p} \cdot \left( \sum_{i=1}^{m} \omega^i + \sum_{j=1}^{n} \hat{y}^j \right) \\ &= p \cdot \sum_{i=1}^{m} \hat{x}^i \\ &> \bar{p} \cdot \sum_{i=1}^{m} \bar{x}^i \\ &= \bar{p} \cdot \left( \sum_{i=1}^{m} \omega^i + \sum_{j=1}^{n} \bar{y}^j \right), \end{aligned}$$

a contradiction. ∎

# 1 Local nonsatiation

A preference relation $\succcurlyeq$ on a set $X$ in $\boldsymbol{R}^\ell$ is **locally nonsatiated** if

$$(\forall \varepsilon > 0)\ (\forall x \in X)\ (\exists y \in X)\ [\, \|y - x\| < \varepsilon \text{ and } y \succ x \,].$$

Fix a price vector $p \in \boldsymbol{R}^\ell$ and income level $m \in \boldsymbol{R}$. Set

$$B = \{x \in X : p \cdot x \leqslant m\}.$$

Assume $x^*$ maximizes $\succcurlyeq$ over $B$, that is, $x^* \in B$ and for every $x \in B$, $x^* \succcurlyeq x$. Then it follows that $y \succ x^*$ implies $y \notin B$, that is, $p \cdot y > m \geqslant p \cdot x^*$.

**Lemma 1** *If $\succcurlyeq$ is locally nonsatiated, and $x^*$ maximizes $\succcurlyeq$ over $B$, then $y \succcurlyeq x^*$ implies $p \cdot y \geqslant m$. Consequently, setting $y = x^*$, we see $p \cdot x^* = m$.*

*Proof*: Suppose by way of contradiction that $p \cdot y < m$. Then there is some $\varepsilon > 0$ such that $\|z - y\| < \varepsilon$ implies $p \cdot z < m$ (so that $z \in B$). By local nonsatiation, one such $z$ satisfies $z \succ y \succcurlyeq x^*$, which contradicts the $\succcurlyeq$-maximality of $x^*$ in $B$. ∎

A consequence of this is Walras' Law. Let

$$\big((X_i, \underset{i}{\succcurlyeq}, \omega^i)_{i=1}^m, (Y_j)_{j=1}^n, (\theta_j^i)_{j=1,\dots,n}^{i=1,\dots,m}\big)$$

be a private ownership economy, and let $p \in \boldsymbol{R}^\ell$ be a price vector (not necessarily an equilibrium price vector).

Let

$$(\hat{x}^1, \dots, \hat{x}^m, \hat{y}^1, \dots, \hat{y}^n, \hat{p}),$$

satisfy the first two parts of the definition of Walrasian equilibrium. That is,

1. (Profit Maximization) For every firm $j$,

$$\hat{y}^j \in Y_j \quad \text{and} \quad p \cdot \hat{y}^j \geqslant p \cdot y^j.$$

2. (Preference Maximization) For every consumer $i$,

$$\hat{x}^i \in B_i = \{x^i \in X_i : p \cdot x^i \leqslant p \cdot \omega^i + \sum_{j=1}^n \theta_j^i p \cdot \hat{y}^j\} \quad \text{and} \quad \hat{x}^i \underset{i}{\succcurlyeq} x^i \text{ for all } x^i \in B_i.$$

However, we do not assume that it is an allocation (markets may not clear). We say that

$$z = \sum_{i=1}^m \hat{x}^i - \sum_{i=1}^m \omega^i - \sum_{j=1}^n \hat{y}^j$$

is the **excess demand** at price $p$.

**Walras' Law** *Assume every preference $\underset{i}{\succcurlyeq}$ is locally nonsatiated. Let $z$ be the excess demand at price $p$. Then*

$$p \cdot z = 0.$$

*Proof*: From the budget constraint and the lemma above we have

$$p \cdot x^i = p \cdot \omega^i + \sum_{j=1}^{n} \theta^i_j p \cdot \hat{y}^j$$

for all $i$, so summing gives

$$\begin{aligned} p \cdot \sum_{i=1}^{m} x^i &= p \cdot \sum_{i=1}^{m} \omega^i + \sum_{i=1}^{m} \sum_{j=1}^{n} \theta^i_j p \cdot \hat{y}^j \\ &= p \cdot \sum_{i=1}^{m} \omega^i + \sum_{j=1}^{n} \left( \sum_{i=1}^{m} \theta^i_j \right) p \cdot \hat{y}^j \\ &= p \cdot \sum_{i=1}^{m} \omega^i + p \cdot \sum_{j=1}^{n} \hat{y}^j, \end{aligned}$$

and rearranging gives the desired result. ∎