Caltech Division of the Humanities and Social Sciences

Euler's Theorem for Homogeneous Functions

KC Border

October 2000

v. 2017.10.27::16.34

1 Definition *Let X be a subset of $\boldsymbol{R}^{\mathrm{n}}$. A function $f \colon X \to \boldsymbol{R}$ is* ***homogeneous of degree k*** *if for all $x \in X$ and all $\lambda > 0$ with $\lambda x \in X$,*

$$f(\lambda x) = \lambda^k f(x).$$

- Note that if $0 \in X$ and f is homogeneous of degree $k \neq 0$, then $f(0) = f(\lambda 0) = \lambda^k f(0)$, so setting $\lambda = 2$, we see $f(0) = 2^k f(0)$, which implies $f(0) = 0$.
- A constant function is homogeneous of degree 0.
- If a function is homogeneous of degree 0, then it is constant on rays from the the origin.
- Linear functions are homogenous of degree one.
- The economists' favorite homogeneous function is the weighted geometric mean with domain $\boldsymbol{R}^{\mathrm{n}}_{+}$, which they know as the Cobb–Douglas function [3],
$$f(x_1, \dots, x_n) = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n},$$
where each $\alpha_i > 0$. It is homogeneous of degree $\alpha = \alpha_1 + \cdots + \alpha_n$.
- If $f_1, \dots, f_m$ are homogeneous of degree k and g is homogeneous of degree ℓ, then $h(x) = g(f_1(x), \dots, f_m(x))$ is homogeneous of degree $k + \ell$.

2 Remark Usually the domain of f will be taken to be a **cone**, that is, a set in $\boldsymbol{R}^{\mathrm{n}}$ closed under multiplication by positive scalars. Indeed, we often take the domain to be the set $\boldsymbol{R}^{\mathrm{n}}_{+}$ of vectors with nonnegative components, or the set $\boldsymbol{R}^{\mathrm{n}}_{++}$ of vectors with strictly positive components. Apostol [2, Exercises 8, 9, p. 287] proves the following theorem for any open domain. The proof is the same as the one used here.

3 Euler's theorem *Let $f\colon \boldsymbol{R}^n_+ \to \boldsymbol{R}$ be continuous and also differentiable on $\boldsymbol{R}^n_{++}$. Then f is homogeneous of degree k if and only if for all $x \in \boldsymbol{R}^n_{++}$,*

$$\sum_{i=1}^n D_i f(x) x_i = k f(x). \tag{$*$}$$

Proof: ($\Longrightarrow$) Assume f is homogeneous of degree k. Let $x \in \boldsymbol{R}^n_{++}$. Define the function $g\colon [0,\infty) \to \boldsymbol{R}$ (depending on x) by

$$g(\lambda) = f(\lambda x) - \lambda^k f(x),$$

and note that for all $\lambda > 0$,

$$g(\lambda) = 0.$$

Therefore

$$g'(\lambda) = 0$$

for all $\lambda > 0$. But by the chain rule,

$$g'(\lambda) = \sum_{i=1}^n D_i f(\lambda x) x_i - k\lambda^{k-1} f(x).$$

Evaluate this at $\lambda = 1$ to obtain ($*$).

($\Longleftarrow$) Suppose

$$kf(x) = \sum_{i=1}^n D_i f(x) x_i$$

for all $x \in \boldsymbol{R}^n_{++}$. Fix any such x and again define $g\colon I \to \boldsymbol{R}$ (depending on x) by

$$g(\lambda) = f(\lambda x) - \lambda^k f(x)$$

and note that $g(1) = 0$. Then for $\lambda > 0$,

$$\begin{aligned} g'(\lambda) &= \sum_{i=1}^n D_i f(\lambda x) x_i - k\lambda^{k-1} f(x) \\ &= \lambda^{-1} \left(\sum_{i=1}^n D_i f(\lambda x) \lambda x_i \right) - k\lambda^{k-1} f(x) \\ &= \lambda^{-1} k f(\lambda x) - k\lambda^{k-1} f(x), \end{aligned}$$

so

$$\begin{aligned} \lambda g'(\lambda) &= k\big(f(\lambda x) - \lambda^k f(x)\big) \\ &= kg(\lambda). \end{aligned}$$

Since λ is arbitrary, g satisfies the following differential equation:

$$g'(\lambda) - \frac{k}{\lambda} g(\lambda) = 0$$

and the initial condition $g(1) = 0$. By Theorem 7 below,

$$g(\lambda) = 0 \cdot e^{A(\lambda)} + e^{-A(\lambda)} \int_1^\lambda 0 \cdot e^{A(t)}\,dt = 0$$

where, irrelevantly, $A(\lambda) = -\int_1^\lambda \frac{k}{t}\,dt = -k \ln \lambda$. This implies g is identically zero, so f is homogeneous on $\boldsymbol{R}^{\mathrm{n}}_{++}$. Continuity guarantees that f is homogeneous on X. ∎

4 Corollary *Let $f \colon \boldsymbol{R}^{\mathrm{n}}_{+} \to \boldsymbol{R}$ be continuous and differentiable on $\boldsymbol{R}^{\mathrm{n}}_{++}$. If f is homogeneous of degree k, then $D_j f(x)$ is homogeneous of degree $k-1$.*

Proof if f is twice differentiable: By the first half of Euler's theorem,

$$\sum_{i=1}^n D_i f(x) x_i = k f(x)$$

so differentiating both sides with respect to the j^{th} variable,

$$D_j \left(\sum_{i=1}^n D_i f(x) x_i \right) = k D_j f(x)$$

or

$$\sum_{i=1}^n D_{ij} f(x) x_i + D_j f(x) = k D_j f(x)$$

or

$$\sum_{i=1}^n D_{ij} f(x) x_i = (k-1) D_j f(x). \tag{1}$$

Thus $D_j f(x)$ is homogeneous of degree $(k-1)$ by second half of Euler's theorem. ∎

Proof without twice differentiability: The difference quotients satisfy

$$\frac{f(\lambda x + \lambda h) - f(\lambda x)}{\|\lambda h\|} = \frac{\lambda^k f(x+h) - \lambda^k f(x)}{\lambda \|h\|} = \lambda^{k-1} \frac{f(x+h) - f(x)}{\|h\|}$$

whenever $\lambda > 0$. Thus f is differentiable at λx if and only if it is differentiable at x and $D_i f(\lambda x) = \lambda^{k-1} D_i f(x)$ for all $i = 1, \ldots, n$.

∎

5 Corollary *If f is homogeneous of degree k, then*

$$\frac{D_i f(\lambda x)}{D_j f(\lambda x)} = \frac{D_i f(x)}{D_j f(x)}$$

for $\lambda > 0$ and $x \in \boldsymbol{R}^{\mathrm{n}}_{++}$.

Proof: By Corollary 4 each f_i satisfies $D_i f(\lambda x) = \lambda^{k-1} D_i f(x)$, so

$$\frac{D_i f(\lambda x)}{D_j f(\lambda x)} = \frac{\lambda^{k-1} D_i f(x)}{\lambda^{k-1} D_j f(x)} = \frac{D_i f(x)}{D_j f(x)}.$$

∎

6 Corollary *If f is homogeneous of degree 1 and twice differentiable, then the Hessian matrix $[D_{ij} f(x)]$ is singular for all $x \in \boldsymbol{R}^{\mathrm{n}}_{++}$.*

Proof: By (1),

$$\sum_{i=1}^{n} D_{ij} f(x) x_i = (k-1) D_j f(x).$$

When $k = 1$ this becomes $[D_{ij} f(x)]x = 0$ in matrix terms, so for $x \neq 0$ we conclude that $[D_{ij} f(x)]$ is singular. ∎

7 Theorem (Solution of first order linear differential equations) *Assume P, Q are continuous on the open interval I. Let $a \in I$, $b \in \boldsymbol{R}$.*

Then there is one and only one function $y = f(x)$ that satisfies the initial value problem

$$y' + P(x)y = Q(x)$$

with $f(a) = b$. It is given by

$$f(x) = be^{-A(x)} + e^{-A(x)} \int_a^x Q(t) e^{A(t)} \, dt$$

where

$$A(x) = \int_a^x P(t) \, dt.$$

For a proof see [1, Theorems 8.2 and 8.3, pp. 309–310].

References

[1] Apostol, T. M. 1967. *Calculus*, 2d. ed., volume 1. Waltham, Massachusetts: Blaisdell.

[2] Apostol, T. M. 1969. *Calculus*, 2d. ed., volume 2. Waltham, Massachusetts: Blaisdell.

[3] Cobb, C. W. and P. H. Douglas. 1928. A theory of production. *American Economic Review* 18(1):139–165. Supplement, Papers and Proceedings of the Fortieth Annual Meeting of the American Economic Association.