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The economics of first-order linear differential equations

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The following theorem is a standard statement of the solution to a first order linear differential equation. I took it from Apostol [1, Theorems 8.2 and 8.3, pp. 309–310].

1 Theorem (First order linear differential equation) *Assume P, Q are continuous on the open interval I. Let $a \in I$, $b \in \boldsymbol{R}$.*

Then there is one and only one function $y = f(x)$ that satisfies the initial value problem

$$y' + P(x)y = Q(x)$$

with $f(a) = b$. It is given by

$$f(x) = be^{-A(x)} + e^{-A(x)} \int_a^x Q(t)e^{A(t)}\,dt$$

where

$$A(x) = \int_a^x P(t)\,dt.$$

The theorem appears a bit mysterious in this form, but I can give it an economic interpretation that makes it obvious (at least to me). The first thing we will do is change the variable on which y depends from x to time, t. Interpret $y(t)$ as the value of a savings account at time t. At each point of time it earns an instantaneous rate of return $r(t)$. Moreover, we add a "flow" of additional savings to the account at the rate $s(t)$. (Negative values of s indicate a flow of withdrawals.) Thus the rate of change of the value of the account is

$$y'(t) = r(t)y(t) + s(t). \tag{1}$$

Moreover, let's rewrite the initial condition as $y(t_0) = y_0$. This yields the following translation.

2 Theorem (First order linear differential equation) *Assume r, s are continuous on the open interval I. Let $t_0 \in I$, $y_0 \in \boldsymbol{R}$.*

Then there is one and only one function $y = f(t)$ that satisfies the initial value problem

$$y' = r(t)y + s(t)$$

with $f(t_0) = y_0$. It is given by

$$f(t) = [y_0 + S(t)]\,e^{\bar{r}(t)(t-t_0)}$$

where

$$\bar{r}(t) = \frac{1}{t-t_0}\int_{t_0}^t r(\tau)\,d\tau$$

and

$$S(t) = \int_{t_0}^t s(\tau)e^{-\bar{r}(\tau)(\tau-t_0)}\,d\tau.$$

But this version is obviously true!

Proof: We rely on the following well-known (easily proved) result: $\lim_{n\to\infty}\big(1+(r/n)\big)^{nt} = e^{rt}$, which is true for all r. That is, the result of compounding interest on a dollar continuously over t periods is e^{rt} dollars.

Case 1: $s = 0$. I claim that if $s(t) = 0$ for all t, then the value of the account at time t is given by

$$y(t) = y_0 e^{\bar r(t)(t-t_0)}, \tag{2}$$

where[1]

$\bar r(t) = \dfrac{1}{t-t_0}\displaystyle\int_{t_0}^{t} r(\tau)\,d\tau$ is the average interest rate over the interval $[t_0, t]$.

Case 2: General s. But in general, the additional savings $s(t)$ is not zero. In order to deal with the general case, we use the incredibly useful notion of present value.

The **present value** at time t_0 of \$1 at time t is $e^{-\bar r(t)(t-t_0)}$.

For if you invest $e^{-\bar r(t)(t-t_0)}$ at t_0, you will have $e^{-\bar r(t)(t-t_0)}e^{\bar r(t)(t-t_0)} = 1$ dollar at time t.

The present value $S(t)$ of all the additional savings up to time t is thus

$$S(t) = \int_{t_0}^{t} s(\tau)e^{-\bar r(\tau)(\tau-t_0)}\,d\tau.$$

But at time t the future value is

$$S(t)e^{\bar r(t)(t-t_0)}.$$

Thus the total value of the savings account at time t is given by

$$y(t) = \left(y_0 + S(t)\right)e^{\bar r(t)(t-t_0)}.$$

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References

[1] T. M. Apostol. 1967. *Calculus*, 2d. ed., volume 1. Waltham, Massachusetts: Blaisdell.

[1] We can verify this by showing that y given by (2) solves (1).

$$\begin{aligned}\frac{dy}{dt} &= \frac{d}{dt}y_0e^{\bar r(t)(t-t_0)} = y_0e^{\bar r(t)(t-t_0)}\frac{d}{dt}\left(\bar r(t)(t-t_0)\right)\\ &= y_0e^{\bar r(t)(t-t_0)}\frac{d}{dt}\int_{t_0}^{t} r(\tau)\,d\tau = y_0e^{\bar r(t)(t-t_0)}r(t)\\ &= r(t)y(t),\end{aligned}$$

which is (1) with $s = 0$. (Or it can be proven directly.)