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Crib Sheet on Demand Theory

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Assume $p \gg 0$, $w > 0$, u is continuous and locally nonsatiated on $\boldsymbol{R}^n_+$, and u is C^2, $u' \gg 0$ and strongly quasiconcave (its Hessian is negative definite on the subspace orthogonal to the gradient) on $\boldsymbol{R}^n_{++}$.

Utility Maximization	**Expenditure Minimization**
$\underset{x}{\text{maximize}}\, u(x)$ subject to $w - p \cdot x = 0$	$\underset{x}{\text{minimize}}\, p \cdot x$ subject to $u(x) - v = 0$

Solution

Ordinary (Walrasian) Demand	Hicksian Compensated Demand
$x^*(p, w)$	$\hat{x}(p, v)$
x^* is homogeneous of degree zero in (p, w).	$\hat{x}$ is homogeneous of degree zero in p.

Value function

Indirect Utility	Expenditure Function
$v(p, w) = u(x^*(p, w))$	$e(p, v) = p \cdot \hat{x}(p, v)$
v is quasiconvex in p, decreasing in p, strictly increasing in w, homogeneous of degree zero in (p, w).	e is concave in p, increasing in p, strictly increasing in v, homogeneous of degree 1 in p.

Statement of Equivalence

$x^*(p, w) = \hat{x}(p, v(p, w))$	$\hat{x}(p, v) = x^*(p, e(p, v))$
$w = e(p, v(p, w))$	$v = v(p, e(p, v))$

Utility Maximization	**Expenditure Minimization**
Lagrangean	
$\mathcal{L}(x,\lambda;p,w) = u(x) + \lambda(w - p\cdot x)$	$\mathcal{L}(x,\mu;p,\upsilon) = p\cdot x - \mu\big(u(x) - \upsilon\big)$
Lagrangean partials with respect to parameters	
$\dfrac{\partial\mathcal{L}(x,\lambda;p,w)}{\partial p_j} = -\lambda x_j$	$\dfrac{\partial\mathcal{L}(x,\mu;p,\upsilon)}{\partial p_j} = x_j$
$\dfrac{\partial\mathcal{L}(x,\lambda;p,w)}{\partial w} = \lambda$	$\dfrac{\partial\mathcal{L}(x,\mu;p,\upsilon)}{\partial \upsilon} = \mu$
Envelope Theorem	
$\dfrac{\partial v(p,w)}{\partial p_j} = -\lambda^*(p,w)x_j^*(p,w)$	$\dfrac{\partial e(p,\upsilon)}{\partial p_j} = \hat{x}_j(p,\upsilon)$
$\dfrac{\partial v(p,w)}{\partial w} = \lambda^*(p,w)$	$\dfrac{\partial e(p,\upsilon)}{\partial \upsilon} = \hat{\mu}(p,\upsilon)$
Roy's Law	Hotelling/Shephard's Lemma
$x_j^*(p,w) = -\dfrac{\dfrac{\partial v(p,w)}{\partial p_j}}{\dfrac{\partial v(p,w)}{\partial w}}$	$\hat{x}_j(p,\upsilon) = \dfrac{\partial e(p,\upsilon)}{\partial p_j}$

The Slutsky equation

From the equivalence

$$\hat{x}(p,\upsilon) = x^*\big(p, e(p,\upsilon)\big)$$

differentiating with respect to p_j yields

$$\frac{\partial \hat{x}_i(p,\upsilon)}{\partial p_j} = \frac{\partial x_i^*\big(p,e(p,\upsilon)\big)}{\partial p_j} + \frac{\partial x_i^*\big(p,e(p,\upsilon)\big)}{\partial w}\frac{\partial e(p,\upsilon)}{\partial p_j}.$$

But $\frac{\partial e(p,\upsilon)}{\partial p_j} = \hat{x}_j(p,\upsilon) = x_j^*\big(p,e(p,\upsilon)\big)$. Set $w = e(p,\upsilon)$, and write

$$\frac{\partial \hat{x}_i(p,\upsilon)}{\partial p_j} = \frac{\partial x_i^*(p,w)}{\partial p_j} + x_j^*(p,w)\frac{\partial x_i^*(p,w)}{\partial w}$$

which implies the **Slutsky equation**

$$\boxed{\frac{\partial x_i^*(p,w)}{\partial p_j} = \frac{\partial \hat{x}_i(p,\upsilon)}{\partial p_j} - x_j^*(p,w)\frac{\partial x_i^*(p,w)}{\partial w},}$$

where $\upsilon = v(p,w)$, which decomposes the effect of a price change into its **substitution effect** and **income effect**.

But

$$\frac{\partial \hat{x}_i(p,\upsilon)}{\partial p_j} = \frac{\partial^2 e(p,\upsilon)}{\partial p_i \partial p_j},$$

so since e is concave in p, its Hessian is negative semidefinite (and symmetric), so the matrix

$$\left[\frac{\partial x_i^*(p,w)}{\partial p_j} + x_j^*(p,w)\frac{\partial x_i^*(p,w)}{\partial w}\right] \text{ is negative semidefinite and symmetric.}$$

Consequently the diagonal terms satisfy

$$\frac{\partial \hat{x}_i(p,\upsilon)}{\partial p_i} = \frac{\partial x_i^*(p,w)}{\partial p_i} + x_i^*(p,w)\frac{\partial x_i^*(p,w)}{\partial w} \leqslant 0,$$

and we have the unusual **reciprocity** relation

$$\frac{\partial x_i^*(p,w)}{\partial p_j} + x_j^*(p,w)\frac{\partial x_i^*(p,w)}{\partial w} = \frac{\partial x_j^*(p,w)}{\partial p_i} + x_i^*(p,w)\frac{\partial x_j^*(p,w)}{\partial w}.$$

Quasiconvexity of indirect utility

Recall that a function f is **quasiconvex** if for every $\alpha \in \boldsymbol{R}$ the lower contour set

$$\{x : f(x) \leqslant \alpha\} \text{ is convex .} \tag{1}$$

This is equivalent to the following statement: For every x and y and every $0 < \lambda < 1$,

$$f\big((1-\lambda)z + \lambda y\big) \leqslant \max\{f(z), f(y)\}. \tag{2}$$

The proof of equivalence is easy. Let $\alpha = \max\{f(z), f(y)\}$ and define $L = \{x : f(x) \leqslant \alpha\}$. To see that (1) $\implies$ (2), let $x, y \in L$. Since L is convex, $(1-\lambda)z + \lambda y \in L$, that is, $f\big((1-\lambda)z + \lambda y\big) \leqslant \alpha = \max\{f(z), f(y)\}$. Conversely if (2) holds, then $(1-\lambda)z + \lambda y \in L$, so L is convex.

To see that the indirect utility v is quasiconvex in p, let $p^\lambda = (1-\lambda)p^0 + \lambda p^1$, with $0 < \lambda < 1$, and let x satisfy the budget constraint

$$p^\lambda \cdot x \leqslant w.$$

This implies at least one of $p^0 \cdot x \leqslant w$ or $p^1 \cdot x \leqslant w$ must hold.[1] In the first case $u(x) \leqslant v(p^0, w)$ and in the second case $u(x) \leqslant v(p^1, w)$. Therefore

$$v(p^\lambda, w) = \max_{x : p^\lambda \cdot x \leqslant w} u(x) \leqslant \max\{v(p^0, w), v(p^1, w)\}.$$

[1]Suppose not. Then $p^0 \cdot x > w$ and $p^1 \cdot x > w$. Thus $(1-\lambda)p^0 \cdot x > (1-\lambda)w$ and $\lambda p^1 \cdot x > \lambda w$. Adding the two gives $p^\lambda \cdot x > w$, contrary to our choice of x.