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Monotonicity and local nonsatiation

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While there are clearly preference relations that are locally nonsatiated but not monotonic, from the point of view of competitive demand theory, local nonsatiation is no more general than monotonicity. By this I mean that if $\succcurlyeq$ is a locally nonsatiated upper semicontinuous preference, then there is a monotonic preference $\overset{*}{\succcurlyeq}$ that generates the same demand.

Let there be n commodities, so the neoclassical consumption set is $\boldsymbol{R}^{\mathrm{n}}_{+}$, the set of nonnegative vectors in $\boldsymbol{R}^{\mathrm{n}}$. A **preference** is a reflexive, transitive, total binary relation $\succcurlyeq$ on $\boldsymbol{R}^{\mathrm{n}}_{+}$. The corresponding strict preference is denoted $\succ$. The preference $\succcurlyeq$ is **locally nonsatiated** if for each $x \in \boldsymbol{R}^{\mathrm{n}}_{+}$ and $\varepsilon > 0$, there is some $y \in \boldsymbol{R}^{\mathrm{n}}_{+}$ such that $d(x, y) < \varepsilon$ [1] and $y \succ x$. The preference $\succcurlyeq$ is **monotonic** if $x \gg y$ implies $x \succ y$.[2] Clearly every monotonic preference relation is locally nonsatiated. A preference is **upper semicontinuous** if for each x, the upper contour set $\{y \in \boldsymbol{R}^{\mathrm{n}}_{+} : y \succcurlyeq x\}$ is closed, or equivalently the strict lower contour set $\{y \in \boldsymbol{R}^{\mathrm{n}}_{+} : x \succ y\}$ is relatively open in $\boldsymbol{R}^{\mathrm{n}}_{+}$. The preference $\succcurlyeq$ is **convex** if for each y, the upper contour set $\{x \in \boldsymbol{R}^{\mathrm{n}}_{+} : x \succcurlyeq y\}$ is convex.

Given a price vector $p \in \boldsymbol{R}^{\mathrm{n}}_{++}$ and income level $w > 0$, the **budget set** $\beta(p, w)$ is the compact convex set $\{x \in \boldsymbol{R}^{\mathrm{n}}_{+} : p \cdot x \leqslant w\}$. It is a fact that if $\succcurlyeq$ is upper semicontinuous, then every compact set has a $\succcurlyeq$-greatest element; and if $\succcurlyeq$ is locally nonsatiated, then every $\succcurlyeq$-greatest element x^* in $\beta(p, w)$ satisfies $p \cdot x^* = w$.

To simplify the explicit description of the desired monotonic relation, let us introduce the following notation. Given a set A,

$$x \succcurlyeq A \quad \text{means} \quad (\forall z \in A)\ [\, x \succcurlyeq z \,]$$

and let

$$D(x) = \{y \in \boldsymbol{R}^{\mathrm{n}}_{+} : 0 \leqq y \leqq x\}.$$

Note that $D(x)$ is nonempty for $x \in \boldsymbol{R}^{\mathrm{n}}_{+}$ as it always contains x.

1 Proposition *Let $\succcurlyeq$ be a locally nonsatiated and upper semicontinuous regular preference on $\boldsymbol{R}^{\mathrm{n}}_{+}$. Then the binary relation $\overset{*}{\succcurlyeq}$ on $\boldsymbol{R}^{\mathrm{n}}_{+}$ defined by*

$$x \overset{*}{\succcurlyeq} y \quad \textit{if there exists } v \in \boldsymbol{R}^{\mathrm{n}}_{+} \textit{ such that } x \geqq v \succcurlyeq D(y)$$

is a monotonic upper semicontinuous preference relation that generates the same demand as $\succcurlyeq$ for all $(p, w) \in \boldsymbol{R}^{\mathrm{n}}_{++} \times \boldsymbol{R}_{+}$. Moreover, if $\succcurlyeq$ is convex, then $\overset{*}{\succcurlyeq}$ is convex.

[1] Here d denotes the usual Euclidean metric.

[2] The partial order $\geqq$ on $\boldsymbol{R}^{\mathrm{n}}_{+}$ is defined by $x \geqq y$ if $x_i \geqslant y_i$ for $i = 1, \ldots, n$. The partial order $\gg$ is defined by $x \gg y$ if $x_i > y_i$ for $i = 1, \ldots, n$. The partial order $>$ is defined by $x > y$ if $x \geqq y$ and $x \neq y$. The nonnegative orthant is $\boldsymbol{R}^{\mathrm{n}}_{+} = \{x \in \boldsymbol{R}^{\mathrm{n}} : x \geqq 0\}$, and the strictly positive orthant is $\boldsymbol{R}^{\mathrm{n}}_{++} = \{x \in \boldsymbol{R}^{\mathrm{n}} : x \gg 0\}$.

Proof: For $y \in \boldsymbol{R}^{n}_{+}$, define $\mu(y)$ to be the set of $\succcurlyeq$-greatest elements of $D(y)$. That is,

$$\mu(y) = \{x \in D(y) : x \succcurlyeq D(y)\}.$$

Note that for each y the set $D(y)$ is compact, and since $\succcurlyeq$ is upper semicontinuous, the set $\mu(y)$ is nonempty. Note that $x \succcurlyeq^* y$ if and only if there exist $v \leqq x$ and $u \in \mu(y)$ with $v \succcurlyeq u$.

First we show that $\succcurlyeq^*$ is

i. reflexive: Let $u \in \mu(x)$. Then $x \geqq u \succcurlyeq D(x)$, so $x \succcurlyeq^* x$.

ii. transitive: Assume $x \succcurlyeq^* y$ and $y \succcurlyeq^* z$. Then there exist v_y, v_z, u_y, and u_z such that $x \geqq v_y \succcurlyeq u_y \in \mu(y)$ and $y \geqq v_z \succcurlyeq u_z \in \mu(z)$.

   Since $y \geqq v_z$ and $u_y \succcurlyeq D(y)$ we have $u_y \succcurlyeq v_z$. Thus $x \geqq v_y \succcurlyeq u_y \succcurlyeq v_z \succcurlyeq D(z)$, so $x \succcurlyeq^* z$.

iii. total: If $x \not\succcurlyeq^* y$, then by definition, for every $0 \leqq u \leqq x$, there is some $0 \leqq v \leqq y$ with $v \succ u$.

   Let $u^* \in \mu(x)$, and let $v^* \leqq y$ satisfy $v^* \succ u^*$. Thus $y \geqq v^* \succ u^* \succcurlyeq D(x)$. Thus $y \succcurlyeq^* x$.

This also proves that $\succ^*$, the asymmetric part of $\succcurlyeq^*$, satisfies

$$y \succ^* x \text{ if and only if } (\,\exists v\,)\ \big[\, y \geqq v \succ D(x) \,\big].$$

Next we show that $\succcurlyeq^*$ is monotonic, that is, if $x \gg y$, then $x \succ^* y$. So assume $x \gg y$ and let $z \in \mu(y) \subset D(y)$, so $x \gg z$. Then for $\varepsilon > 0$ small enough, $d(v, z) < \varepsilon$ implies $x \gg v$. By local nonsatiation, at least one such v satisfies $v \succ z \succcurlyeq D(y)$. Thus $x \gg z \succ D(y)$, so $x \succ^* y$.

To see that $\succcurlyeq^*$ is upper semicontinuous, I shall prove that if $y \succ^* x$, then there is an $\varepsilon > 0$ such that $d(x, x') < \varepsilon$ implies $y \succ^* x'$ too. So assume $y \succ^* x$. Then there exists $v \leqq y$ such that $v \succ D(x)$. Since $D(x)$ is compact, I claim there is some $\varepsilon > 0$ such that $v \succ N_\varepsilon\big(D(x)\big)$.[3] Then if $d(x, x') < \varepsilon$, we have $D(x') \subset N_\varepsilon\big(D(x)\big)$ too, so $v \succ D(x')$. But this implies $y \succ^* x'$.

Finally we show that for $p \gg 0$, a point x^* is $\succcurlyeq$-greatest in $\beta(p, w)$ if and only if x^* is also $\succcurlyeq^*$-greatest.

Assume first that x^* is $\succcurlyeq$-greatest in $\beta(p, w)$. Let $y \in \beta(p, w)$. Then $D(y) \subset \beta(p, w)$, so $x^* \geqq x^* \succcurlyeq D(y)$. Thus $x^* \succcurlyeq^* y$. Therefore x^* is $\succcurlyeq^*$-greatest in $\beta(p, w)$.

Now assume that x^* is $\succcurlyeq^*$-greatest in $\beta(p, w)$, and let $\bar{x}$ be $\succcurlyeq$-greatest. Since $x^* \succcurlyeq^* \bar{x}$ there is some $u \leqq x^*$ with $u \succcurlyeq D(\bar{x})$. In particular, $u \succcurlyeq \bar{x} \in D(\bar{x})$, so u too is $\succcurlyeq$-greatest. But by local nonsatiation $p \cdot u = w$, so $u \leqq x^* \in \beta(p, w)$ implies $u = x^*$, so x^* is also $\succcurlyeq$-greatest.

To see that $\succcurlyeq^*$ is convex if $\succcurlyeq$ is convex, let $x, x' \succcurlyeq^* y$, where $x \geqq v \succcurlyeq D(y)$ and let $x' \geqq v' \succcurlyeq D(y)$. Then $(1-\lambda)x + \lambda x' \geqq (1-\lambda)v + \lambda v'$ and assuming $\succcurlyeq$ is convex, $(1-\lambda)v + \lambda v' \succcurlyeq D(y)$. Thus $(1-\lambda)x + \lambda x' \succcurlyeq^* y$. ∎

[3]Here $N_\varepsilon(A)$ denotes the ε-neighborhood of A, that is, $\big\{x : \big(\,\exists y \in A\,\big)\ \big[\,d(x, y) < \varepsilon\,\big]\big\}$. It is an open set being the union of the open balls of radius ε centered on points of A. To see why this claim is true, let F denote the closed upper contour set $\{u : u \succcurlyeq v\}$. Then the distance function $d(z, F) = \inf\{d(z, u) : u \in F\}$ is (Lipschitz) continuous and so achieves its minimum over the compact set $D(x)$. Since F and $D(x)$ are disjoint closed sets this minimum is strictly greater than zero. Choose $\varepsilon > 0$ less than this minimum.

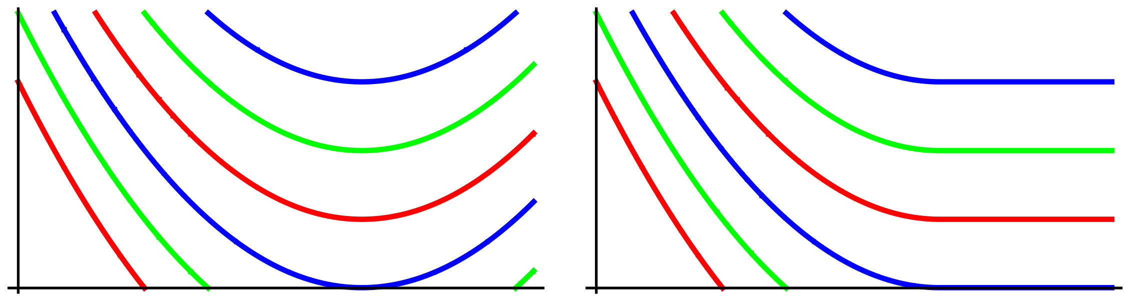

Figure 1. Indifference curves for locally nonsatiated utility $u(x, y) = y - (1 - x)^2$ and monotone utility with same demand.

2 Remark Finally, let u be an upper semicontinuous utility function representing the preference $\succcurlyeq$. (Such a utility exists, see Richter [2].) Its **monotonic hull** v is defined by $v(x) = \max\{u(y) : 0 \leqq y \leqq x\}$, and is the smallest monotonic function that dominates u. The Berge Maximum Theorem implies that v is upper semicontinuous.[4] It is easy to see that v represents $\overset{*}{\succcurlyeq}$.

3 Example Consider the quasilinear utility function for two goods x and y defined by

$$u(x, y) = y - (1 - x)^2$$

which gives a linear demand function for x. It is locally nonsatiated but not monotone. It has the property that the demand for x never exceeds 1 unit. It has the same demand behavior as the monotone utility

$$v(x, y) = \begin{cases} y - (1 - x)^2 & x \leqslant 1 \\ y & x \geqslant 1 \end{cases}$$

which is its monotonic hull. See Figure 1. □

References

[1] Aliprantis, C. D. and K. C. Border. 2006. *Infinite dimensional analysis: A hitchhiker's guide*, 3d. ed. Berlin: Springer–Verlag.

[2] Richter, M. K. 1980. Continuous and semi-continuous utility. *International Economic Review* 21:293–299.

[4] The correspondence $y \mapsto D(y)$ is compact-valued and upper hemicontinuous, so the upper hemicontinuous half of the Berge Maximum Theorem (see, e.g., [1, Lemma 17.30, p. 569] implies that v, which is the optimal value function for maximizing u over $D(y)$ is upper semicontinuous.