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The "Integrability Problem"

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The simplest formulation of the consumer's problem in economics is to choose a "market basket" (a point in $\boldsymbol{R}^{\mathrm{n}}_{+}$) of n goods subject to a budget constraint. We usually assume that the consumer has a **utility** function $u\colon \boldsymbol{R}^{\mathrm{n}}_{+} \to \boldsymbol{R}$ that indexes desirability and chooses a market basket $x^*(p,m)$ to maximize $u(x)$ subject to the budget constraint $m - p\cdot x \geqslant 0$, where $p \in \boldsymbol{R}^{\mathrm{n}}_{++}$ is a vector of prices and $m \geqslant 0$ is the consumer's money income. A problem that occupied economic theorists for several decades was to figure out what restrictions the assumption that utility maximization placed on demand functions.

Another way to phrase this issue is, given a **demand function** $\xi\colon \boldsymbol{R}^{\mathrm{n}}_{++} \times \boldsymbol{R}_{+} \to \boldsymbol{R}^{\mathrm{n}}_{+}$, is it the case that there is a utility function that generates it, and if so, how can it be recovered? Applied mathematicians tend to call this the inverse optimization problem. What was discovered is that under certain conditions, it is possible to solve differential equations to recover a utility function from a demand function. The following reasonably general result is taken from Hurwicz and Uzawa [11]. It involves the following function related to the demand function function. Given a function $\xi\colon \boldsymbol{R}^{\mathrm{n}}_{++} \times \boldsymbol{R}_{+} \to \boldsymbol{R}^{\mathrm{n}}_{+}$ define

$$\sigma_{i,j}(p,m) = \frac{\partial \xi_i(p,m)}{\partial p_j} + \xi_j(p,m)\frac{\partial \xi_i(p,m)}{\partial m}.$$

This is called the **Slutsky substitution function**.

1 Hurwicz–Uzawa Integrability Theorem *Let $\xi\colon \boldsymbol{R}^{\mathrm{n}}_{++} \times \boldsymbol{R}_{+} \to \boldsymbol{R}^{\mathrm{n}}_{+}$. Assume*

(B) *The budget exhaustion condition*

$$p \cdot \xi(p,m) = m$$

is satisfied for every $(p,m) \in \boldsymbol{R}^{\mathrm{n}}_{++} \times \boldsymbol{R}_{+}$.

(D) *Each component function ξ_i is differentiable everywhere on $\boldsymbol{R}^{\mathrm{n}}_{++} \times \boldsymbol{R}_{+}$.*

(S) *The Slutsky matrix is symmetric, that is, for every $(p,m) \in \boldsymbol{R}^{\mathrm{n}}_{++} \times \boldsymbol{R}_{+}$,*

$$\sigma_{i,j}(p,m) = \sigma_{j,i}(p,m) \qquad i,j = 1,\dots,n.$$

(NSD) *The Slutsky matrix is negative semidefinite, that is, for every $(p,m) \in \boldsymbol{R}^{\mathrm{n}}_{++} \times \boldsymbol{R}_{+}$, and every $v \in \boldsymbol{R}^{\mathrm{n}}$,*

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{i,j}(p,m) v_i v_j \leqslant 0.$$

(IB) *The function ξ satisfies the following boundedness condition on the partial derivative with respect to income. For every $0 \ll \underline{a} \ll \bar{a} \in \boldsymbol{R}^{\mathrm{n}}_{++}$, there exists a (finite) real number $M_{\underline{a},\bar{a}}$ such that for all $m \geqslant 0$*

$$\underline{a} \leqq p \leqq \bar{a} \quad \implies \quad \left| \frac{\partial \xi_i(p,m)}{\partial m} \right| \leqslant M_{\underline{a},\bar{a}} \quad i = 1, \ldots, n.$$

Let X denote the range of ξ,

$$X = \{\xi(p,m) \in \boldsymbol{R}^{\mathrm{n}}_{+} : (p,m) \in \boldsymbol{R}^{\mathrm{n}}_{++} \times \boldsymbol{R}_{+}\}.$$

Then there exists a utility function $u \colon X \to \boldsymbol{R}$ on the range X such that for each $(p,m) \in \boldsymbol{R}^{\mathrm{n}}_{++} \times \boldsymbol{R}_{+}$,

$$\xi(p,m) \text{ is the unique maximizer of } u \text{ over the budget set } \{x \in X : p \cdot x \leqslant m\}.$$

The proof is rather roundabout and requires a little motivation.

1 A little motivation

Consider a **demand function**

$$x^* \colon \boldsymbol{R}^{\mathrm{n}}_{++} \times \boldsymbol{R}_{++} \to \boldsymbol{R}^{\mathrm{n}}_{+}$$

derived by maximizing a locally nonsatiated utility function u. Let v be the **indirect utility**, that is, the optimal value function

$$v(p,m) = u\big(x^*(p,m)\big).$$

Since u is locally nonsatiated, the indirect utility function v is strictly increasing in m. The Hicksian **expenditure function** e is defined by

$$e(p,v) = \min\{p \cdot x : u(x) \geqslant v\}.$$

We know from the support function theorem or the envelope theorem that

$$\frac{\partial e(p,v)}{\partial p_i} = \hat{x}_i(p,v) = x_i^*\big(p, e(p,v)\big).$$

Ignoring v for the moment, we have the **total differential equation**

$$e'(p) = x^*\big(p, e(p)\big). \tag{1}$$

What does it mean to solve such an equation? And what happened to v?

An aside on solutions of differential equations

You may recall from your calculus classes that, in general, differential equations have many solutions, often indexed by “constants of integration.” For instance, take the simplest differential equation,

$$y' = a$$

for some constant a. The general form of the solution is

$$y(x) = ax + C,$$

where C is an arbitrary constant of integration. What this means is that the differential equation $y' = a$ has infinitely many solutions, one for each value of C. The parameter υ in our problem can be likened to a constant of integration.

You should also recall that we rarely specify C directly as a condition of the problem, since we don’t know the function y in advance. Instead we usually us an **initial condition** (x^0, y^0). That is, we specify that

$$y(x^0) = y^0.$$

In this simple case, the way to translate an initial condition into a constant of integration is to solve the equation

$$y^0 = ax^0 + C \implies C = y^0 - ax^0,$$

and rewrite the solution as

$$y(x) = ax + (y^0 - ax^0) = y^0 + a(x - x^0).$$

In order to make it really explicit that the solution depends on the initial conditions, differential equations texts may go so far as to write the solution as

$$y(x; x^0, y^0) = y^0 + a(x - x^0).$$

In our differential equation (1), an initial condition corresponding to the “constant of integration” υ is a pair (p^0, m^0) satisfying

$$e(p^0, \upsilon) = m^0.$$

From the equivalence of expenditure minimization and utility maximization under a budget constraint, this gives us the relation

$$\upsilon = v(p^0, m^0) = u\big(x^*(p^0, m^0)\big).$$

Following Hurwicz and Uzawa [11], define the **income compensation function** in terms of the Hicksian expenditure function e via[1]

$$\boxed{\mu(p; p^0, m^0) = e\big(p, v(p^0, m^0)\big).}$$

[1] In terms of preferences,

$$\mu(p; p^0, m^0) = \inf\{p \cdot x : x \succcurlyeq x^*(p^0, m^0)\}.$$

Lionel McKenzie [13] employs a similar construction to replace the expenditure function in a framework where only preferences were used, not a utility index. He defines a slightly different function $\mu(p; x^0) = \inf\{p \cdot x : x \succcurlyeq x^0\}$.

Observe that

$$\mu(p^0; p^0, m^0) = m^0$$

and

$$\frac{\partial \mu(p; p^0, m^0)}{\partial p_i} = \frac{\partial e(p, v^0)}{\partial p_i} = \hat{x}_i(p, v^0) = x_i^*(p, e(p, v^0)) = x_i^*(p, \mu(p; p^0, m^0)).$$

In other words,

> The function $e\colon \boldsymbol{R}^{\mathrm{n}}_{++} \to \boldsymbol{R}$ defined by
>
> $$e(p) = \mu(p; p^0, m^0)$$
>
> is the solution to differential equation (1),
>
> $$e'(p) = x^*(p, e(p)),$$
>
> that satisfies the initial condition
>
> $$e(p^0) = m^0.$$

We are now going to turn the income compensation function around and treat (p^0, m^0) as the variable of interest. Fix a price vector $p^* \in \boldsymbol{R}^{\mathrm{n}}_{++}$ and define the function $w\colon \boldsymbol{R}^{\mathrm{n}}_{++} \times \boldsymbol{R}_{++} \to \boldsymbol{R}$ by

$$\boxed{w(p, m) = \mu(p^*; p, m) = e(p^*, v(p, m)).}$$

The function w is another indirect utility. That is,

$$w(p, m) \geqslant w(p', m') \iff v(p, m) \geqslant v(p', m').$$

To see this, observe that since e is strictly increasing in v,

$$w(p, m) = e(p^*, v(p, m)) \geqslant e(p^*, v(p', m')) = w(p', m') \qquad \iff \qquad v(p, m) \geqslant v(p', m').$$

We can use w to find a utility U, at least on the range of x^* by

$$\boxed{U(x) = \mu(p^*; p, m) \quad \text{where } x = x^*(p, m).}$$

2 Recovering utility from demand: The plan

The discussion above leads us to the following approach. Given a demand function x^*:

1. Somehow solve the differential equation

$$\frac{\partial \mu(p)}{\partial p_i} = x_i^*(p, \mu(p)).$$

Write the solution explicitly in terms of the initial condition $\mu(p^0) = m^0$ as $\mu(p; p^0, m^0)$.

2. Fix a price vector p^* and define an indirect utility function w by

$$w(p, m) = \mu(p^*; p, m).$$

3. Invert the demand function to give (p, m) as a function of x^*.

4. Define the utility on the range of x^* by

$$U(x) = \mu(p^*; p, m) \quad \text{where } x = x^*(p, m).$$

This is easier said than done, and there remain a few questions. For instance, how do we know that the differential equation has a solution? If a solution exists, how do we know that the "utility" U so derived generates the demand function x^*? We shall address these questions presently, but I find it helps to look at some examples first.

3 Examples

In order to draw pictures, I will consider two goods x and y. By homogeneity of x^*, I may take good y as numéraire and fix $p_y = 1$, so the price of x will simply be denoted p.

3.1 Deriving the income compensation function from a utility

For the Cobb–Douglas utility function

$$u(x, y) = x^\alpha y^\beta$$

where $\alpha + \beta = 1$, the demand functions are

$$x^*(p, m) = \frac{\alpha m}{p}, \quad y^*(p, m) = \beta m.$$

The indirect utility is thus

$$v(p, m) = m\beta^\beta \left(\frac{\alpha}{p}\right)^\alpha.$$

The expenditure function is

$$e(p, v) = v\beta^{-\beta} \left(\frac{p}{\alpha}\right)^\alpha.$$

Now pick (p^0, m^0) and define

$$\begin{aligned}
\mu(p; p^0, m^0) &= e(p; v(p^0, m^0)) \\
&= \left(m^0 \beta^\beta \left(\frac{\alpha}{p^0}\right)^\alpha\right) \beta^{-\beta} \left(\frac{p}{\alpha}\right)^\alpha \\
&= m^0 \left(\frac{p}{p^0}\right)^\alpha.
\end{aligned}$$

Evaluating this at $p = p^0$ we have

$$\mu(p^0; p^0, m^0) = m^0.$$

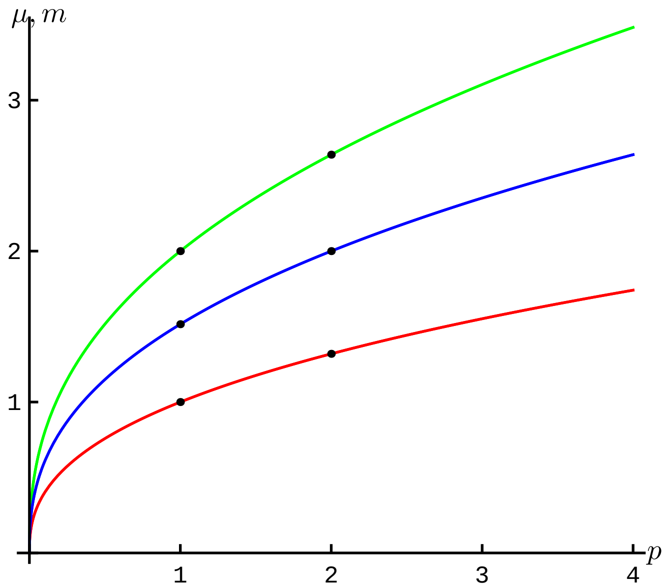


Figure 1. Graph of $\mu(p;p^0;m^0)$ for Cobb–Douglas $\alpha = 2/5$ utility and various values of (p^0, m^0).

That is, the point (p^0, m^0) lies on the graph of $\mu(\,\cdot\,;p^0,m^0)$. Figure 1 shows the graph of this function for different values of (p^0, m^0). For each fixed (p^0, m^0), the function $\mu(p) = \mu(p;p^0,m^0)$ satisfies the (ordinary) differential equation

$$\frac{d\mu}{dp} = \alpha\left[m^0(p^0)^{-\alpha}\right]p^{\alpha-1} = \frac{\alpha\mu(p)}{p} = x^*(p, \mu(p)).$$

Note that homogeneity and budget exhaustion have allowed us to reduce the dimensionality by 1. We have $n-1$ prices, as we have chosen a numéraire, and the demand for the n^{th} good is gotten from $x_n^* = m - \sum_{i=1}^{n-1} p_i x_i^*$.

3.2 Examples of recovering utility from demand

Let $n = 2$, and set $p_2 = 1$, so that there is effectively only one price p, and only one differential equation (for x_1)

$$\mu'(p) = x(p, \mu(p)).$$

2 Example In this example

$$x(p, m) = \frac{\alpha m}{p}.$$

(This x is the demand for x_1. From the budget constraint we can infer $x_2 = (1-\alpha)m$.)

The corresponding differential equation is

$$\mu' = \frac{\alpha\mu}{p} \quad \text{or} \quad \frac{\mu'}{\mu} = \frac{\alpha}{p}.$$

(For those of you more comfortable with y-x notation, this is $y' = \alpha y/x$.) Integrate both sides of the second form to get

$$\ln \mu = \alpha \ln p + C$$

so exponentiating each side gives

$$\mu(p) = Kp^\alpha$$

where $K = \exp(C)$ is a constant of integration. Given the initial condition (p^0, m^0), we must have

$$m^0 = K(p^0)^\alpha, \qquad \text{so } K = \frac{m^0}{(p^0)^\alpha},$$

or

$$\mu(p; p^0, m^0) = \frac{m^0}{(p^0)^\alpha} p^\alpha.$$

For convenience set $p^* = 1$, to get

$$w(p, m) = \mu(p^*; p, m) = \frac{m}{p^\alpha}.$$

To recover the utility u, we need to invert the demand function, that is, we need to know for what budget (p, m) is (x_1, x_2) chosen. The demand function is

$$x_1 = \frac{\alpha m}{p}, \quad x_2 = (1 - \alpha)m,$$

so solving for m and p, we have

$$m = \frac{x_2}{1 - \alpha}$$

$$x_1 = \frac{\alpha \frac{x_2}{1-\alpha}}{p} \qquad \Longrightarrow \qquad p = \frac{\alpha}{1 - \alpha} \frac{x_2}{x_1}.$$

Thus

$$\begin{aligned}
u(x_1, x_2) &= w(p, m) \\
&= w\left(\frac{\alpha}{1 - \alpha} \frac{x_2}{x_1}, \frac{x_2}{1 - \alpha}\right) \\
&= \frac{\frac{x_2}{1-\alpha}}{\left(\frac{\alpha}{1-\alpha} \frac{x_2}{x_1}\right)^\alpha} \\
&= \left(\frac{x_2}{1 - \alpha}\right)^{1-\alpha} \left(\frac{x_1}{\alpha}\right)^\alpha \\
&= c x_1^\alpha x_2^{1-\alpha},
\end{aligned}$$

where $c = (1 - \alpha)^{1-\alpha} \alpha^\alpha$, which is a Cobb–Douglas utility. □

3 Example In this example we find a utility that generates a linear demand for x. That is,

$$x(p, m) = \beta - \alpha p.$$

(Note the lack of m.) The differential equation is

$$\mu' = \beta - \alpha p.$$

This differential equation is easy to solve:

$$\mu(p) = \beta p - \frac{\alpha}{2} p^2 + C$$

For initial condition (p^0, m^0) we must choose $C = m^0 - \beta p^0 + \frac{\alpha}{2} {p^0}^2$, so the solution becomes

$$\mu(p; p^0, m^0) = \beta p - \frac{\alpha}{2} p^2 + m^0 - \beta p^0 + \frac{\alpha}{2} {p^0}^2.$$

So choosing $p^* = 0$ (not really allowed, but it works in this case), we have

$$w(p, m) = \mu(p^*; p, m) = m - \beta p + \frac{\alpha}{2} p^2.$$

Given (x, y) (let's use this rather than (x_1, x_2)), we need to find the (p, m) at which it is chosen. We know

$$x = \beta - \alpha p, \quad y = m - px = m - \beta p + \alpha p^2,$$

so

$$p = \frac{\beta - x}{\alpha}, \quad m = y + \beta p - \alpha p^2 = y + \beta \frac{\beta - x}{\alpha} - \alpha \left(\frac{\beta - x}{\alpha} \right)^2.$$

Therefore

$$\begin{aligned} u(x, y) = w(p, m) &= w \left(\frac{\beta - x}{\alpha}, y + \beta \frac{\beta - x}{\alpha} - \alpha \left(\frac{\beta - x}{\alpha} \right)^2 \right) \\ &= \underbrace{y + \beta \frac{\beta - x}{\alpha} - \alpha \left(\frac{\beta - x}{\alpha} \right)^2}_{m} - \beta \underbrace{\frac{\beta - x}{\alpha}}_{p} + \frac{\alpha}{2} \underbrace{\left(\frac{\beta - x}{\alpha} \right)^2}_{p^2} \\ &= y - \frac{(\beta - x)^2}{2\alpha}. \end{aligned}$$

Note that the utility is decreasing in x for $x > \beta$. Representative indifference curves are shown in Figure 2. The demand curve specified implies that x and y will be negative for some values of p and m, so we can't expect that this is a complete specification. I'll leave it to you to figure out when this makes sense. □

4 Existence of Solutions for Total Differential Equations

Given an open set $A \times B \subset \mathbf{R}^{\mathrm{n}} \times \mathbf{R}$ with typical element (p, m), and a function

$$\xi \colon A \times B \to \mathbf{R}^{\mathrm{n}},$$

a function $\mu \colon A \to B$ is a **local solution to the total differential equation**

$$M' = \xi(p, M) \tag{$*$}$$

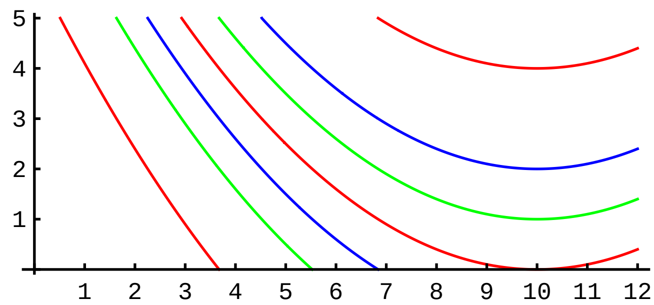


Figure 2. Indifference curves for Example 3 (linear demand) with $\beta = 10$, $\alpha = 5$.

on $U \subset A$ if

$$\mu'(p) = \xi(p, \mu(p)) \qquad \text{for all } p \in U. \tag{$**$}$$

The total differential equation is often written as a system of partial differential equations:

$$\frac{\partial M}{\partial p_i} = \xi_i(p, M) \qquad i = 1, \dots, n.$$

This equation is said to be **completely integrable** (in the sense of Clebsch) if for every $(p^0, m^0) \in A \times B$, there is a neighborhood U of p^0 and a unique continuously differentiable function $\mu \colon U \to B$ (depending on (p^0, m^0)) satisfying $(**)$ and the initial condition

$$\mu(p^0) = m^0.$$

The classical Frobenius Theorem tell us when such a system is completely integrable. The following theorem is a special case translated from Dieudonné [6, Theorems 10.9.4, pp. 308–310].

4 Frobenius's Theorem *Assume ξ is continuously differentiable. The total differential equation $(*)$ is completely integrable if and only if for every $(p, m) \in A \times B$, the function ξ satisfies the symmetry condition*

$$\sigma_{i,j}(p, m) = \sigma_{j,i}(p, m) \qquad i, j = 1, \dots, n.$$

above

This next theorem, also taken from Dieudonné [6, Theorems 10.9.4, pp. 310–311], extends the Frobenius theorem by asserting that the solution is also a continuously differentiable function of the initial conditions.

5 Theorem (Further properties of the solution) *If ξ is continuously differentiable, then each point $(\bar{p}, \bar{m}) \in A \times B$ has a neighborhood $U \times V$ satisfying*

1. *for any point (p^*, m^*) in $U \times V$, there is a unique solution $p \mapsto \mu(p; p^*, m^*)$ satisfying $(*)$ and $\mu(p^*; p^*, m^*) = m^*$, and*
2. *if ξ is C^k, then the mapping $(p, p^*, m^*) \mapsto \mu(p; p^*, m^*)$ is C^k on $U \times U \times V$.*

The classical results are not adequate for our purposes, as the solution is only asserted to exist locally. We want a theorem that asserts the existence of a global solution. Such an extension was provided by Leonid Hurwicz and Hirofumi Uzawa [11, Existence Theorem III], who built on the work of Hartman [8, 9], Nikliborc [14], Thomas [17], and Tsuji [18].

6 Hurwicz–Uzawa Global Existence Theorem *Let* $\xi \colon \boldsymbol{R}^{\mathrm{n}}_{++} \times \boldsymbol{R}_{+} \to \boldsymbol{R}^{\mathrm{n}}_{+}$. *Assume*

(D) *Each component function* ξ_i *is differentiable everywhere on* $\boldsymbol{R}^{\mathrm{n}}_{++} \times \boldsymbol{R}_{+}$.[2]

(S) *The Slutsky matrix is symmetric, that is, for every* $(p, m) \in \boldsymbol{R}^{\mathrm{n}}_{++} \times \boldsymbol{R}_{+}$,

$$\sigma_{i,j}(p,m) = \sigma_{j,i}(p,m) \qquad i,j = 1,\dots,n.$$

(IB) *The function* ξ *satisfies the following boundedness condition on the partial derivative with respect to income. For every* $0 \ll \underline{a} \ll \bar{a} \in \boldsymbol{R}^{\mathrm{n}}_{++}$, *there exists a (finite) real number* $M_{\underline{a},\bar{a}}$ *such that for all* $m \geqslant 0$

$$\underline{a} \leqq p \leqq \bar{a} \quad \Longrightarrow \quad \left| \frac{\partial \xi_i(p,m)}{\partial m} \right| \leqslant M_{\underline{a},\bar{a}} \quad i = 1,\dots,n.$$

(0) *For each* $i = 1, \dots, n$, *and each* $p \in \boldsymbol{R}^{\mathrm{n}}_{++}$, *we have*

$$\xi_i(p, 0) = 0.$$

Then for every initial condition $(p^0, m^0) \in \boldsymbol{R}^{\mathrm{n}}_{++} \times \boldsymbol{R}^{\mathrm{n}}_{+}$, *there exists a unique continuous function* $\mu(\,\cdot\,; p^0, m^0) \colon \boldsymbol{R}^{\mathrm{n}}_{++} \to \boldsymbol{R}_{+}$ *such that for every* $p \in \boldsymbol{R}^{\mathrm{n}}_{++}$, *the partial differential equation*

$$\frac{\partial \mu(p; p^0, m^0)}{\partial p_i} = \xi_i\big(p, \mu(p; p^0, m^0)\big) \qquad i = 1,\dots,n.$$

is satisfied, and

$$m^0 = \mu(p^0; p^0, m^0).$$

Moreover $\mu(p; \cdot\,, \cdot\,)$ *is also continuous with respect to the initial condition* (p^0, m^0) *for each* p.

[2] Note that this domain is not open. Hurwicz and Uzawa use Graves's [7] definition of differential, which defines the differential at accumulation points of the domain, not just interior points. Let $f \colon X \subset \boldsymbol{R}^{\mathrm{n}} \to \boldsymbol{R}$, and let x be an accumulation point of X. A linear function $Df(x) \colon v \mapsto Df(x)(v)$ from $\boldsymbol{R}^{\mathrm{n}}$ to $\boldsymbol{R}$ is the differential of f at x if for every $\varepsilon > 0$ there exists some $\delta > 0$ such that for every v with $0 < \|v\| < \delta$ and $x + v \in X$, we have

$$\frac{|f(x+v) - f(x) - Df(x)(v)|}{\|v\|} < \varepsilon.$$

The difference between this definition and the one in say Apostol [1, 2] or Dieudonné [6] is that they only consider interior points, so for small enough $\|v\|$, the condition that $x + v \in X$ is automatic. In particular, at boundary points $(p, 0)$, we need only have one-sided directional derivatives with respect to m.

5 "Integrability" of demand

We are now in a position to prove the main theorem.

Proof of Theorem 1: Given such a function ξ, the budget exhaustion condition **(B)** implies condition **(0)** of the Hurwicz–Uzawa Existence Theorem. Thus all the hypotheses of that theorem are satisfied, there is a unique function

$$\mu\colon \boldsymbol{R}^{\mathrm{n}}_{++} \times \boldsymbol{R}^{\mathrm{n}}_{++} \times \boldsymbol{R}_{+} \to \boldsymbol{R}$$

such that

$$\text{for each initial condition } (p^0, m^0), \quad \mu(\,\cdot\,; p^0, m^0) \text{ satisfies } (**) \text{ and } \mu(p^0; p^0, m^0) = m^0$$

and such that $\mu(p; \cdot\,, \cdot\,)$ is continuous for each p.

We start by observing that while different initial conditions (p^0, m^0) may or may not define different functions $\mu(\,\cdot\,; p^0, m^0)$, the graphs of these functions do not intersect each other, but are vertically ordered. Formally we have the following lemma.

7 Lemma *Consider two different initial conditions* (p^0, m^0) *and* (p^1, m^1). *Either they define the same function* μ *or one always lies above the other. That is, either*

$$\mu(\,\cdot\,; p^0, m^0) = \mu(\,\cdot\,; p^1, m^1),$$

or

$$\mu(\,\cdot\,; p^0, m^0) > \mu(\,\cdot\,; p^1, m^1),$$

or

$$\mu(\,\cdot\,; p^0, m^0) < \mu(\,\cdot\,; p^1, m^1).$$

Proof: First suppose that for some $\bar{p}$ we have $\mu(\bar{p}; p^0, m^0) = \mu(\bar{p}; p^1, m^1) = \bar{m}$. Consider the initial condition $(\bar{p}, \bar{m})$. By the Global Existence Theorem, there is a unique function satisfying the differential equation $(**)$ through the point $(\bar{p}, \bar{m})$. But $\mu(\,\cdot\,; p^0, m^0)$ and $\mu(\,\cdot\,; p^1, m^1)$ satisfy $(**)$ and pass through $(\bar{p}, \bar{m})$, so we must have

$$\mu(\,\cdot\,; p^0, m^0) = \mu(\,\cdot\,; p^1, m^1) = \mu(\,\cdot\,; \bar{p}, \bar{m}).$$

Now suppose that for some $\bar{p}$ we have $\mu(\bar{p}; p^0, m^0) > \mu(\bar{p}; p^1, m^1)$, but that for some $\tilde{p}$ we have $\mu(\tilde{p}; p^0, m^0) < \mu(\tilde{p}; p^1, m^1)$. But μ is continuous in its first variable, so by the Intermediate Value Theorem for some $0 < \lambda < 1$ we must have

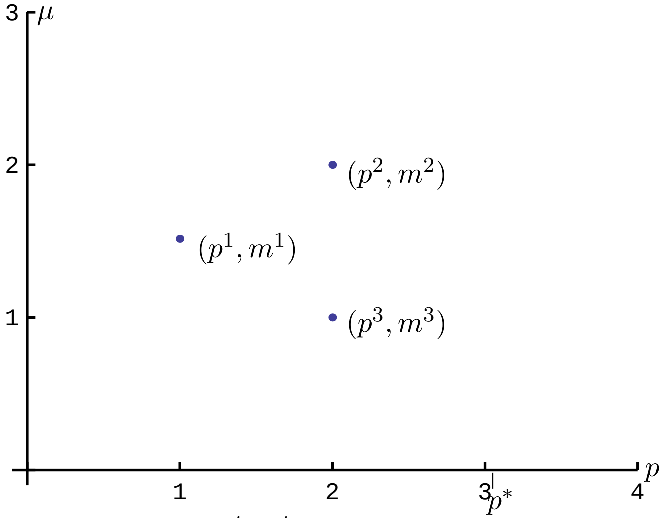
$$\mu\big((1-\lambda)\bar{p} + \lambda\tilde{p}; p^0, m^0\big) = \mu\big((1-\lambda)\bar{p} + \lambda\tilde{p}; p^1, m^1\big),$$

which by the first paragraph implies that

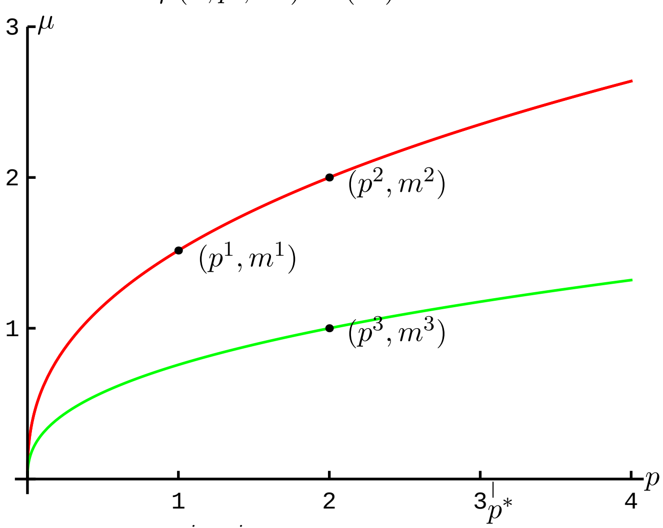
$$\mu(\,\cdot\,; p^0, m^0) = \mu(\,\cdot\,; p^1, m^1).$$

Of course this contradicts $\mu(\tilde{p}; p^0, m^0) < \mu(\tilde{p}; p^1, m^1)$. This completes the proof. ∎

Fix p^*:



For each (p^i, m^i), find the solution $\mu(\,\cdot\,; p^i, m^i)$ of $(**)$ with initial condition (p^i, m^i):



Evaluate $\mu(\,\cdot\,; p^i, m^i)$ at p^* to get $w(p^i, m^i)$:

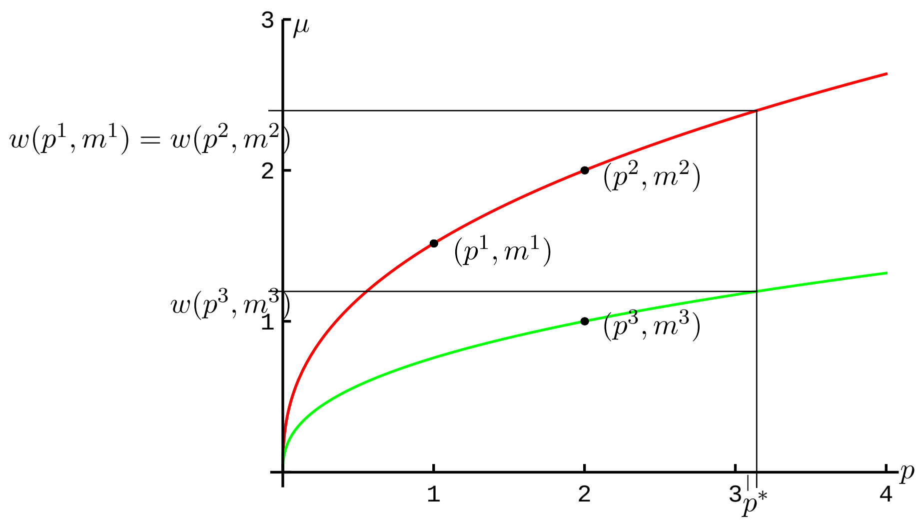


Figure 3. Construction of the “indirect utility” w.

Now fix some $p^* \in \boldsymbol{R}^{\mathrm{n}}_{+}+$, define $w\colon \boldsymbol{R}^{\mathrm{n}}_{++} \times \boldsymbol{R}_{+} \to \boldsymbol{R}$ by

$$w(p,m) = \mu(p^*;p,m),$$

and define $u\colon X \to \boldsymbol{R}$ by

$$u(\xi(p,m)) = w(p,m) = \mu(p^*;p,m).$$

See Figure 3. It follows from Lemma 7 that the ordering of budgets (p,m) induced by w is independent of the choice of p^*.

If indeed ξ is derived from maximizing a utility, then we know its income compensation function satisfies the differential equation $(**)$. From the uniqueness part of the existence theorem we will have found it, and that w is an indirect utility, and the u we have constructed is a utility. (Different choices of p^* give different utilities.) But we don't know that ξ is a bona fide demand function, so we have to prove somehow that this u satisfies the conclusion of the theorem. There are many steps on this road. The first step makes use of the relationship between the Weak Axiom of Revealed Preference and **(NSD)**, cf. [16, equation (70), p. 109] and Kihlstrom, Mas-Colell, and Sonnenschein [12].

8 Lemma (Hurwicz–Uzawa [11, Lemma 4, p. 126]) *Let $x^0 = \xi(p^0,m^0)$ and $x^1 = \xi(p^1,m^1)$ and assume $x^0 \neq x^1$. If*

$$m^1 \geqslant \mu(p^1;p^0,m^0).$$

then

$$p^0 \cdot x^1 > p^0 \cdot x^0.$$

We shall also have occasion to use this result in a transmogrified form. Take the contrapositive to get: If

$$p^0 \cdot x^0 \geqslant p^0 \cdot x^1,$$

then

$$m^1 < \mu(p^1;p^0,m^0).$$

Proof of Lemma 8: Define

$$p(t) = (1-t)p^0 + tp^1, \quad m(t) = \mu(p(t);p^0,m^0), \quad \text{and} \quad x(t) = \xi(p(t),m(t)).$$

Note that $p(0) = p^0$ and $p(1) = p^1$, so

$$x(0) = \xi(p(0),\mu(p(0);p^0,m^0)) = \xi(p^0,\mu(p^0;p^0,m^0)) = \xi(p^0,m^0) = x^0,$$

and

$$x(1) = \xi(p^1,\mu(p^1;p^0,m^0)).$$

Let

$$\psi(t) = p^0 \cdot x(t) = \sum_{i=1}^{n} p^0_i \xi_i(p(t),\mu(p(t);p^0,m^0)).$$

Then by the chain rule,

$$\begin{aligned}
\psi'(t) &= \sum_{i=1}^n p_i^0 \frac{d}{dt}\xi_i(p(t),\mu(p(t);p^0,m^0)) \\
&= \sum_{i=1}^n p_i^0 \sum_{j=1}^n \left(\frac{\partial \xi_i}{\partial p_j} + \frac{\partial \xi_i}{\partial m}\frac{\partial \mu}{\partial p_j}\right)(p_j^1 - p_j^0) \\
&= \sum_{i=1}^n \sum_{j=1}^n \left(\frac{\partial \xi_i}{\partial p_j} + \frac{\partial \xi_i}{\partial m}\xi_j\right)(p_j^1 - p_j^0)p_i^0. \qquad (2)
\end{aligned}$$

By the budget exhaustion condition **(B)** we have

$$m(t) = p(t)\cdot x(t) = \sum_{i=1}^n p_i(t)\xi_i(p(t),\mu(p(t);p^0,m^0))$$

for all t. Differentiating the left hand side with respect to t yields

$$\frac{d}{dt}m(t) = \sum_{i=1}^n \frac{\partial \mu}{\partial p_i}\frac{d}{dt}p(t) = \sum_{i=1}^n \xi_i(p(t),m(t))(p_i^1 - p_i^0).$$

Differentiating the right hand side with respect to t yields

$$\sum_{i=1}^n \left\{ (p_i^1 - p_i^0)\xi_i(p(t),\mu(p(t);p^0,m^0)) + p_i(t)\sum_{j=1}^n \left(\frac{\partial \xi_i}{\partial p_j} + \frac{\partial \xi_i}{\partial m}\xi_j\right)(p_j^1 - p_j^0) \right\}.$$

Equating the two implies after some cancellation that

$$\sum_{i=1}^n \sum_{j=1}^n \left(\frac{\partial \xi_i}{\partial p_j} + \frac{\partial \xi_i}{\partial m}\xi_j\right)(p_j^1 - p_j^0)p_i(t) = 0.$$

Subtracting this from (2) gives

$$\begin{aligned}
\psi'(t) &= \sum_{i=1}^n \sum_{j=1}^n \left(\frac{\partial \xi_i}{\partial p_j} + \frac{\partial \xi_i}{\partial m}\xi_j\right)(p_j^1 - p_j^0)(p_i^0 - p_i(t)) \\
&= -t\sum_{i=1}^n \sum_{j=1}^n (- - -)(p_i^1 - p_i^0)(p_j^1 - p_j^0) \\
&= -t\sum_{i=1}^n \sum_{j=1}^n \sigma_{i,j}(p(t),m(t))(p_i^1 - p_i^0)(p_j^1 - p_j^0).
\end{aligned}$$

Thus by **(NSD)** we have $\psi'(t) \geqslant 0$, so $\psi(1) \geqslant \psi(0)$, or

$$p^0 \cdot x(1) \geqslant p^0 \cdot x^0 = m^0.$$

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9 Lemma (Weak Axiom of Revealed Preference) *Letting* $x^0 = \xi(p^0, m^0)$ *and* $x^1 = \xi(p^1, m^1)$,

$$x^0 \neq x^1 \text{ and } p^0 \cdot x^0 \geqslant p^0 \cdot x^1 \quad \Longrightarrow \quad p^1 \cdot x^0 > p^1 \cdot x^1.$$

Proof: Assume $x^0 \neq x^1$ and x^0 is revealed preferred to x^1, that is, $p^0 \cdot x^0 \geqslant p^0 \cdot x^1$. By the contrapositive form of Lemma 8,

$$\mu(p^1; p^1, m^1) = m^1 < \mu(p^1; p^0, m^0).$$

Therefore by Lemma 7

$$\mu(p^0; p^1, m^1) < \mu(p^0; p^0, m^0) = m^0,$$

so a fortiori, $m^0 \geqslant \mu(p^0; p^1, m^1)$. Applying Lemma 8 to this inequality (and interchanging the roles of 0 and 1), we conclude $p^1 \cdot x^0 > p^1 \cdot x^1$. ∎

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We first need to show that u is well defined. That is,

10 Proposition *Assume* ξ *satisfies* **(NSD)**. *If* $\xi(p, m) = \xi(p', m')$, *then* $\mu(p^*; p, m) = \mu(p^*; p', m')$.

Proof: (Hurwicz and Uzawa [11, Lemma 7, p. 129].) ∎

We are finally in a position to prove that u is a utility that generates the demand ξ. That is,

11 Lemma *Under the hypotheses of the theorem, for any* $(p, m) \in \boldsymbol{R}^{\mathrm{n}}_{++} \times \boldsymbol{R}_+$,

$$u(\xi(p, m)) > u(x) \quad \text{for all } x \in X \text{ such that } p \cdot x \leqslant m, x \neq \xi(p, m).$$

This completes the proof of the integrability theorem. ∎

References

[1] T. M. Apostol. 1957. *Mathematical analysis: A modern approach to advanced calculus*. Addison-Wesley series in mathematics. Reading, Massachusetts: Addison Wesley.

[2] ——— . 1969. *Calculus*, 2d. ed., volume 2. Waltham, Massachusetts: Blaisdell.

[3] M. S. Berger and N. G. Meyers. 1966. On a system of nonlinear partial differential equations arising in mathematical economics. *Bulletin of the American Mathematical Society* 72:954–958. DOI: 10.1090/S0002-9904-1966-11600-2

[4] ——— . 1971. Utility functions for finite-dimensional commodity spaces. In Chipman et al. [5], chapter 7, pages 149–162.

[5] J. S. Chipman, L. Hurwicz, M. K. Richter, and H. F. Sonnenschein, eds. 1971. *Preferences, utility, and demand: A Minnesota symposium*. New York: Harcourt, Brace, Jovanovich.

[6] J. Dieudonné. 1969. *Foundations of modern analysis*. Number 10-I in Pure and Applied Mathematics. New York: Academic Press. Volume 1 of Treatise on Analysis.

[7] L. M. Graves. 1956. *Theory of functions of real variables*. New York: McGraw–Hill.

[8] P. Hartman. 1964. *Ordinary differential equations*. New York: Wiley.

[9] ——— . 1970. Frobenius theorems under Carathéodory type conditions. *Journal of Differential Equations* 7(2):307–333. DOI: 10.1016/0022-0396(70)90113-0

[10] L. Hurwicz. 1971. On the problem of integrability of demand functions. In Chipman et al. [5], chapter 9, pages 174–214.

[11] L. Hurwicz and H. Uzawa. 1971. On the integrability of demand functions. In Chipman et al. [5], chapter 6, pages 114–148.

[12] R. Kihlstrom, A. Mas-Colell, and H. F. Sonnenschein. 1976. The demand theory of the weak axiom of revealed preference. *Econometrica* 44(5):971–978. http://www.jstor.org/stable/1911539

[13] L. W. McKenzie. 1957. Demand theory without a utility index. *Review of Economic Studies* 24(3):185–189. http://www.jstor.org/stable/2296067

[14] W. Nikliborc. 1929. Sur les équations linéaires aux différentielles totales. *Studia Mathematica* 1:41–49. http://matwbn.icm.edu.pl/ksiazki/sm/sm1/sm112.pdf

[15] P. A. Samuelson. 1950. The problem of integrability in utility theory. *Economica N.S.* 17(68):355–385. http://www.jstor.org/stable/2549499

[16] ——— . 1965. *Foundations of economic analysis*. New York: Athenaeum. Reprint of the 1947 edition published by Harvard University Press.

[17] T. Y. Thomas. 1934. Systems of total differential equations defined over simply connected domains. *Annals of Mathematics* 35(4):730–734. http://www.jstor.org/stable/1968488

[18] M. Tsuji. 1948. On a system of total differential equations. *Japanese Journal of Mathematics* 19:383–393.