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Duality Approach to Demand Properties

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The constrained maximization problem is

$$\text{maximize } u(x) \quad \text{subject to } m - p \cdot x = 0.$$

We assume there is a unique interior maximizer. The Lagrangean for this problem is

$$L(x, \lambda; p, m) = u(x) + \lambda(m - p \cdot x).$$

The gradient of the constraint is $-p \neq 0$, so the Lagrange Multiplier Theorem applies. Let $x^*(p, m)$ be the solution with Lagrange multiplier $\lambda^*(p, m)$. The first-order conditions are

$$u_i(x^*) - \lambda^* p_i = 0 \quad i = 1, \ldots, n.$$

Since $p \gg 0$ and each $u_i > 0$ by assumption U.2, we have

$$\lambda^* > 0.$$

Define the **indirect utility function** v to be the value function for this problem, that is,

$$v(p, m) = u\big(x^*(p, m)\big).$$

Then by the Envelope Theorem,

$$\frac{\partial v(p,m)}{\partial m} = \frac{\partial L}{\partial m} = \lambda^*(p, m) \quad \text{and} \quad \frac{\partial v(p,m)}{\partial p_j} = \frac{\partial L}{\partial p_j} = -\lambda^*(p, m) x_j^*(p, m).$$

Together these imply **Roy's Identity**, namely:

$$x_j^*(p, m) = -\frac{\dfrac{\partial v(p,m)}{\partial p_j}}{\dfrac{\partial v(p,m)}{\partial m}}.$$

Compare this with the expenditure minimization problem:

$$\underset{x}{\text{minimize}} \ p \cdot x \quad \text{subject to } u(x) - v \geqslant 0.$$

The Lagrangean for this problem is:

$$p \cdot x - \mu\big(u(x) - v\big).$$

Let $\hat{x}(p,\upsilon)$ solve the problem and let $\hat{\mu}(p,\upsilon)$ be the Lagrange multiplier. The function $\hat{x}$ is also known as the **Hicksian compensated demand function**. Define the **expenditure function** e to be the value function for this problem, that is,

$$e(p,\upsilon) = p \cdot \hat{x}(p,\upsilon).$$

Then by the Envelope Theorem,

$$\frac{\partial e(p,\upsilon)}{\partial \upsilon} = \hat{\mu}(p,\upsilon) \qquad \text{and} \qquad \frac{\partial e(p,\upsilon)}{\partial p_j} = \hat{x}^j(p,\upsilon).$$

Moreover, by the Support Function Theorem, e is concave in p. Thus e is twice differentiable in p almost everywhere, and where it is differentiable, then

$$\begin{bmatrix} \dfrac{\partial^2 e(p,\upsilon)}{\partial^2 p_1} & \cdots & \dfrac{\partial^2 e(p,\upsilon)}{\partial p_n \partial p_1} \\ \vdots & & \vdots \\ \dfrac{\partial^2 e(p,\upsilon)}{\partial p_1 \partial p_n} & \cdots & \dfrac{\partial e(p,\upsilon)}{\partial^2 p_n} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \hat{x}^1}{\partial p_1} & \cdots & \dfrac{\partial \hat{x}^1}{\partial p_n} \\ \vdots & & \vdots \\ \dfrac{\partial \hat{x}^n}{\partial p_1} & \cdots & \dfrac{\partial \hat{x}^n}{\partial p_n} \end{bmatrix}$$

is symmetric and negative semidefinite. In particular then,

$$\frac{\partial \hat{x}^j}{\partial p_j} \leqslant 0$$

and

$$\frac{\partial \hat{x}^j}{\partial p_i} = \frac{\partial \hat{x}^i}{\partial p_j}$$

for $i, j = 1, \dots, n$,

What is the relationship between this and ordinary demand? From the equivalence of expenditure minimization and utility maximization we have

$$x^{*i}(p, e(p,\upsilon)) = \hat{x}^i(p,\upsilon),$$

which implies

$$\frac{\partial x^{*i}}{\partial p_j} + \frac{\partial x^{*i}}{\partial m}\frac{\partial e}{\partial p_j} = \frac{\partial \hat{x}^i}{\partial p_j}.$$

Rearranging,

$$\frac{\partial x^{*i}}{\partial p_j} = \frac{\partial \hat{x}^i}{\partial p_j} - \frac{\partial x^{*i}}{m}\frac{\partial e}{\partial p_j}.$$

Now use $\frac{\partial e}{\partial p_j} = \hat{x}^j$ and $\hat{x}^j(p,\upsilon) = x^{*j}(p,m)$ where $m = e(p,\upsilon)$ to conclude

$$\frac{\partial x^{*i}(p,m)}{\partial p_j} = \frac{\partial \hat{x}^i(p,\upsilon(p,m))}{\partial p_j} - x^{*j}(p,m)\frac{\partial x^{*i}(p,m)}{\partial m}.$$

This is the famous **Slutsky equation**. Or rearranging another way we find

$$\begin{bmatrix} \dfrac{\partial^2 e}{\partial^2 p_1} & \cdots & \dfrac{\partial^2 e}{\partial p_n \partial p_1} \\ \vdots & & \vdots \\ \dfrac{\partial^2 e}{\partial p_1 \partial p_n} & \cdots & \dfrac{\partial e}{\partial^2 p_n} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \hat{x}^1}{\partial p_1} & \cdots & \dfrac{\partial \hat{x}^1}{\partial p_n} \\ \vdots & & \vdots \\ \dfrac{\partial \hat{x}^n}{\partial p_1} & \cdots & \dfrac{\partial \hat{x}^n}{\partial p_n} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{\partial x^{*1}}{\partial p_1} + x^{*1}\dfrac{\partial x^{*1}}{\partial m} & \cdots & \dfrac{\partial x^{*1}}{\partial p_n} + x^{*n}\dfrac{\partial x^{*1}}{\partial m} \\ \vdots & & \vdots \\ \dfrac{\partial x^{*n}}{\partial p_1} + x^{*1}\dfrac{\partial x^{*n}}{\partial m} & \cdots & \dfrac{\partial x^{*n}}{\partial p_j} + x^{*n}\dfrac{\partial x^{*n}}{\partial m} \end{bmatrix}$$

is symmetric and negative semidefinite.

The problem with Hicksian compensated demand is that the utility level v is in principle not observable. This problem can be solved by considering the **Slutsky compensated demand**, which solves the following problem.

$$\underset{x}{\text{maximize}} \;\; u(x) \qquad \text{subject to } p \cdot x \leqslant p \cdot \omega.$$

These demands are also called **offer curves**. Let $\tilde{x}(p, \omega)$ denote the solution to this problem. These demands are in prinicple observable. Note that

$$\tilde{x}(p, \omega) = x^*(p, p \cdot \omega).$$

Thus

$$\frac{\partial \tilde{x}^i(p, \omega)}{\partial p_j} = \frac{\partial x^{*i}(p, p \cdot \omega)}{\partial p_j} + \frac{\partial x^{*i}(p, p \cdot \omega)}{\partial m}\omega_j.$$

Thus, fixing $(\bar{p}, \bar{m})$ and setting $\bar{x} = x^*(\bar{p}, \bar{m})$, and $\bar{v} = u(\bar{x})$, we have

$$\frac{\partial \tilde{x}^i(\bar{p}, \bar{x})}{\partial p_j} = \frac{\partial x^{*i}(\bar{p}, \bar{m})}{\partial p_j} + \frac{\partial x^{*i}(\bar{p}, \bar{m})}{\partial m}\bar{x}_j = \frac{\partial \hat{x}^i(\bar{p}, \bar{v})}{\partial p_j}.$$