Caltech Division of the Humanities and Social Sciences

Traditional Derivation of Demand Properties

KCB

v. 2016.01.14::23.58

Implicit assumptions

In the traditional analysis of demand functions, there are a number of implicit assumptions that if not made render the analysis vacuous. Among these are:

U.1 The utility function $u\colon \boldsymbol{R}^{\mathrm{n}}_{+} \to \boldsymbol{R}$ is continuous, and twice continuously differentiable on $\boldsymbol{R}^{\mathrm{n}}_{++}$.

U.2 At each point $x \gg 0$, we have $u'(x) \gg 0$, which is a strong monotonicity condition.

U.3 The utility satisfies the following strong quasiconcavity condition. At each $x \gg 0$, the Hessian is negative definite on tangent planes to indifference curves. That is, for all $v \in \boldsymbol{R}^{\mathrm{n}}$,

$$\sum_{i=1}^{n}\sum_{j=1}^{n} u_{ij}(x)v_iv_j < 0 \qquad v \neq 0 \text{ and } u'(x)\cdot v = 0.$$

(Here and throughout these notes we adopt the notational convention that subscripts can be used to denote partial differentiation, so that u_i denotes $D_i u = \frac{\partial u}{\partial x_i}$, and u_{ij} denotes $D_{ij}u = \frac{\partial^2 u}{\partial x_i \partial x_j}$.)

This is equivalent to

$$(-1)^p \begin{vmatrix} u_{11} & \dots & u_{1p} & u_1 \\ \vdots & & \vdots & \vdots \\ u_{p1} & \dots & u_{pp} & u_p \\ u_1 & \dots & u_p & 0 \end{vmatrix} > 0 \qquad p = 2,\dots,n.$$

In particular,

$$\begin{vmatrix} u_{11} & \dots & u_{1n} & u_1 \\ \vdots & & \vdots & \vdots \\ u_{n1} & \dots & u_{nn} & u_n \\ u_1 & \dots & u_n & 0 \end{vmatrix} \neq 0.$$

U.4 If $x \gg 0$, its indifference curve never approaches a point on the boundary of $\boldsymbol{R}^{\mathrm{n}}_{+}$.

Attention is restricted to strictly positive price vectors p and strictly positive income m. Under these conditions, the utility maximizing consumption is always unique, satisfies the budget with equality, and is strictly positive. Furthermore, utility maximization and expenditure minimization are equivalent. That is, if x^* maximizes $u(x)$ subject to $m - p \cdot x \geqslant 0$ and if $\hat{x}$ minimizes $p \cdot x$ subject to $u(x) \geqslant u^*$, where $u^* = u(x^*)$, then $x^* = \hat{x}$ and $p \cdot x^* = m$. Throughout these notes, $p \in \boldsymbol{R}^{\mathrm{n}}_{++}$ is a vector of strictly positive prices, and $m > 0$ is a strictly positive

Utility maximization

The constrained maximization problem is

$$\text{maximize } u(x) \quad \text{subject to } m - p \cdot x = 0.$$

We know there is a unique interior maximizer x^*. The gradient of the constraint is $-p \neq 0$, so the Lagrange Multiplier Theorem applies. Thus there is a Lagrange multiplier λ^* so that the first-order conditions

$$u_i(x^*) - \lambda^* p_i = 0 \quad i = 1, \ldots, n$$

are satisfied. Since $p \gg 0$ and each $u_i > 0$ by assumption U.2, we have

$$\lambda^* > 0.$$

The second-order conditions are that the Hessian matrix $[u_{ij}]$ be negative semidefinite under constraint, more specifically

$$\sum_i \sum_j u_{ij}(x^*) v_i v_j \leqslant 0 \quad \text{for all } v \text{ such that } \sum_i (-p_i) v_i = 0.$$

Since the first-order condition implies $u_i(x^*) = \lambda^* p_i$, we see that assumption U.3 guarantees that the strong second-order conditions are satisfied.

Consider the function $g \colon \boldsymbol{R}^{n+1} \times \boldsymbol{R}^{n+1} \to \boldsymbol{R}^{n+1}$ defined by the first-order conditions,

$$g^i(x, \lambda; p, m) = u_i(x) - \lambda p_i$$

for $i = 1, \ldots, n$, and the constraint,

$$g^{n+1}(x, \lambda; p, m) = m - p \cdot x.$$

The Jacobian of this function with respect to (x, λ) is

$$\begin{vmatrix} u_{11} & \ldots & u_{1n} & -p_1 \\ \vdots & & \vdots & \vdots \\ u_{n1} & \ldots & u_{nn} & -p_n \\ -p_1 & \ldots & -p_n & 0 \end{vmatrix} = \begin{vmatrix} u_{11} & \ldots & u_{1n} & -\frac{1}{\lambda} u_1 \\ \vdots & & \vdots & \vdots \\ u_{n1} & \ldots & u_{nn} & -\frac{1}{\lambda} u_n \\ -\frac{1}{\lambda} u_1 & \ldots & -\frac{1}{\lambda} u_n & 0 \end{vmatrix} = \frac{1}{\lambda^2} \begin{vmatrix} u_{11} & \ldots & u_{1n} & u_1 \\ \vdots & & \vdots & \vdots \\ u_{n1} & \ldots & u_{nn} & u_n \\ u_1 & \ldots & u_n & 0 \end{vmatrix}$$

where the first equality follows from the first-order conditions, and the second by multiplying the last row and last column by $-\lambda$. It follows from $\lambda^* > 0$ and U.3 this determinant is nonzero at (x^*, λ^*), so by the Implicit Function Theorem, since u is C^2, then x^* and λ^* are C^1 functions of (p, m), at least locally. Thus the first-order conditions imply that

$$u_i(x^*(p,m)) - \lambda^*(p,m)p_i = 0 \qquad i = 1, \dots, n.$$

and

$$m - \sum_i p_i x_i^*(p,m) = 0 \tag{FOC}$$

for all (p, m).

The left-hand side of each of these first-order conditions can be viewed as a constant function of (p, m), namely the zero function. So for each commodity i and each price p_j, differentiate the left-hand side of the first order condition for x_i with respect to p_j to get

$$\sum_k u_{ik} \frac{\partial x_k^*}{\partial p_j} - \frac{\partial \lambda^*}{\partial p_j} p_i - \lambda^* \delta_{ij} = 0 \qquad \begin{array}{l} i = 1, \dots, n, \\ j = 1, \dots, n. \end{array} \tag{1''}$$

and the constraint to get

$$-\sum_k p_k \frac{\partial x_k^*}{\partial p_j} - x_j^* = 0 \qquad j = 1, \dots, n. \tag{2''}$$

Differentiate each left-hand side with respect to m to get

$$\sum_k u_{ik} \frac{\partial x_k^*}{\partial m} - \frac{\partial \lambda^*}{\partial m} p_i = 0 \qquad i = 1, \dots, n. \tag{3''}$$

and

$$1 - \sum_k p_k \frac{\partial x_k^*}{\partial m} = 0. \tag{4''}$$

For aesthetic reasons that will become clear in a moment, I want to use the first-order conditions $u_i = \lambda^* p_i$ and do a little regrouping:

$$\sum_k u_{ik} \frac{\partial x_k^*}{\partial p_j} + u_i \frac{-1}{\lambda^*} \frac{\partial \lambda^*}{\partial p_j} = \lambda^* \delta_{ij} \qquad \begin{array}{l} i = 1, \dots, n, \\ j = 1, \dots, n. \end{array} \tag{1'}$$

$$\sum_k u_k \frac{\partial x_k^*}{\partial p_j} = -\lambda^* x_j^* \qquad j = 1, \dots, n. \tag{2'}$$

$$\sum_k u_{ik} \frac{\partial x_k^*}{\partial m} + u_i \frac{-1}{\lambda^*} \frac{\partial \lambda^*}{\partial m} = 0 \qquad i = 1, \dots, n. \tag{3'}$$

$$\sum_k u_k \frac{\partial x_k^*}{\partial m} = \lambda^*. \tag{4'}$$

We can view these in terms of n-vectors and rewrite them as

$$\begin{bmatrix} u_{i1},\dots,u_{in},u_i \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial x_1^*}{\partial p_j},\dots \frac{\partial x_n^*}{\partial p_j}, \frac{-1}{\lambda^*}\frac{\partial \lambda^*}{\partial p_j} \end{bmatrix} = \lambda^* \delta_{ij} \qquad \begin{matrix} i=1,\dots,n, \\ j=1,\dots,n. \end{matrix} \tag{1}$$

$$\begin{bmatrix} u_1,\dots,u_n,0 \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial x_1^*}{\partial p_j},\dots \frac{\partial x_n^*}{\partial p_j}, \frac{-1}{\lambda^*}\frac{\partial \lambda^*}{\partial p_j} \end{bmatrix} = -\lambda^* x_j^* \qquad j=1,\dots,n. \tag{2}$$

$$\begin{bmatrix} u_{i1},\dots,u_{in},u_i \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial x_1^*}{\partial m},\dots \frac{\partial x_n^*}{\partial m}, \frac{-1}{\lambda^*}\frac{\partial \lambda^*}{\partial m} \end{bmatrix} = 0 \qquad i=1,\dots,n. \tag{3}$$

$$\begin{bmatrix} u_1,\dots,u_n,0 \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial x_1^*}{\partial m},\dots \frac{\partial x_n^*}{\partial m}, \frac{-1}{\lambda^*}\frac{\partial \lambda^*}{\partial m} \end{bmatrix} = \lambda^* \qquad i=1,\dots,n. \tag{4}$$

Arranging all this in matrix terms gives

$$\begin{bmatrix} u_{11} & \dots & u_{1n} & u_1 \\ \vdots & & \vdots & \vdots \\ u_{n1} & \dots & u_{nn} & u_n \\ u_1 & \dots & u_n & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial x_1^*}{\partial p_1} & \dots & \frac{\partial x_1^*}{\partial p_n} & \frac{\partial x_1^*}{\partial m} \\ \vdots & & \vdots & \vdots \\ \frac{\partial x_n^*}{\partial p_1} & \dots & \frac{\partial x_n^*}{\partial p_n} & \frac{\partial x_n^*}{\partial m} \\ \frac{-1}{\lambda^*}\frac{\partial \lambda^*}{\partial p_1} & \dots & \frac{-1}{\lambda^*}\frac{\partial \lambda^*}{\partial p_n} & \frac{-1}{\lambda^*}\frac{\partial \lambda^*}{\partial m} \end{bmatrix} = \left[\begin{array}{ccc|c} \lambda^* & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & \lambda^* & 0 \\ \hline -\lambda^* x_1^* & \dots & -\lambda^* x_n^* & \lambda^* \end{array}\right]. \tag{5}$$

The entries in the $(n+1)\times(n+1)$ right-hand side matrix correspond to the equations (1)–(4) according to this scheme:

$$\left[\begin{array}{c|c} (1)_{n\times n} & (3)_{n\times 1} \\ \hline (2)_{1\times n} & (4)_{1\times 1} \end{array}\right]$$

Solving this gives

$$\begin{bmatrix} \frac{\partial x_1^*}{\partial p_1} & \dots & \frac{\partial x_1^*}{\partial p_n} & \frac{\partial x_1^*}{\partial m} \\ \vdots & & \vdots & \vdots \\ \frac{\partial x_n^*}{\partial p_1} & \dots & \frac{\partial x_n^*}{\partial p_n} & \frac{\partial x_n^*}{\partial m} \\ \frac{-1}{\lambda^*}\frac{\partial \lambda^*}{\partial p_1} & \dots & \frac{-1}{\lambda^*}\frac{\partial \lambda^*}{\partial p_n} & \frac{-1}{\lambda^*}\frac{\partial \lambda^*}{\partial m} \end{bmatrix} = \begin{bmatrix} u_{11} & \dots & u_{1n} & u_1 \\ \vdots & & \vdots & \vdots \\ u_{n1} & \dots & u_{nn} & u_n \\ u_1 & \dots & u_n & 0 \end{bmatrix}^{-1} \left[\begin{array}{ccc|c} \lambda^* & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & \lambda^* & 0 \\ \hline -\lambda^* x_1^* & \dots & -\lambda^* x_n^* & \lambda^* \end{array}\right]. \tag{6}$$

Set

$$A = \begin{bmatrix} u_{11} & \dots & u_{1n} & u_1 \\ \vdots & & \vdots & \vdots \\ u_{n1} & \dots & u_{nn} & u_n \\ u_1 & \dots & u_n & 0 \end{bmatrix}^{-1},$$

which exists by assumption U.3. Then we have

$$\frac{\partial x_i^*}{\partial m} = \lambda^* a_{i,n+1}$$

and

$$\frac{\partial x_i^*}{\partial p_j} = \lambda^* a_{ij} - \lambda^* x_j^* a_{i,n+1} = \lambda^* a_{ij} - x_j^* \frac{\partial x_i^*}{\partial m}.$$

In particular,

$$\boxed{\frac{\partial x_i^*}{\partial p_i} = \lambda^* a_{ii} - x_i^* \frac{\partial x_i^*}{\partial m}.} \qquad (7)$$

The natural question is, what is the economic interpretation of $\lambda^* a_{ij}$? The answer lies in the expenditure minimization problem.

Utility maximization and expenditure minimization

Fix (p, m) and let $v = u(x^*(p, m)) = v(p, m)$. Consider the problem

$$\underset{x}{\text{minimize}}\, p \cdot x \quad \text{subject to } u(x) \geqslant v.$$

When is this problem equivalent to the utility maximization problem?

To answer that let me introduce a new definition. We say that the utility function $u \colon X \to \boldsymbol{R}$ is **locally nonsatiated at $\boldsymbol{x}$** if for every $\varepsilon > 0$, there is some $z \in X$ satisfying $\|x - z\| < \varepsilon$ and $u(z) > u(x)$. (Note that this is a joint assumption on X and u.)

Lemma 1 (Budget exhaustion) *Let $u \colon X \to \boldsymbol{R}$ and suppose $\bar{x} \in X$ maximizes $u(x)$ over the budget set $\beta(p, m) = \{x \in X : p \cdot x \leqslant m\}$. If u is locally nonsatiated at $\bar{x}$, then $\bar{x}$ exhausts the budget, that is,*

$$p \cdot \bar{x} = m.$$

Proof: If $p \cdot \bar{x} < m$, then there is an $\varepsilon > 0$ such that $y \in X$ and $\|y - x\| < \varepsilon$ implies $p \cdot y < m$, and thus $y \in \beta(p, m)$. Thus $u(\bar{x}) \geqslant u(y)$ for all such y, so u is not locally nonsatiated at $\bar{x}$. The lemma now follows by contraposition. ∎

Proposition 1 *Let $u \colon \boldsymbol{R}^{\mathrm{n}}_+ \to \boldsymbol{R}$ be locally nonsatiated everywhere.*

1. *If $\bar{x}$ maximizes $u(x)$ subject to $p \cdot x \leqslant m$, then $\bar{x}$ minimizes $p \cdot x$ subject to $u(x) \geqslant u(\bar{x})$.*

2. *If u is also continuous, and $\bar{x}$ minimizes $p \cdot x$ subject to $u(x) \geqslant u(\bar{x})$, and $p \cdot \bar{x} > 0$, then $\bar{x}$ maximizes $u(x)$ subject to $p \cdot x \leqslant p \cdot \bar{x}$.*

Proof: (1) Assume $\bar{x}$ maximizes $u(x)$ subject to $p \cdot x \leqslant m$. Then clearly $u(y) > u(\bar{x})$ implies $p \cdot y > p \cdot \bar{x}$. Now suppose by way of contradiction that $u(y) = u(\bar{x})$, but $p \cdot y < p \cdot \bar{x}$. Since u is locally nonsatiated at y there is some $z \in X$ close to y with $p \cdot z < p \cdot \bar{x}$ and $u(z) > u(y) = u(x)$, which contradicts the maximality of $\bar{x}$ over he budget set.

(2) Assume $\bar{x}$ minimizes $p \cdot x$ subject to $u(x) \geqslant u(\bar{x})$, and $p \cdot \bar{x} > 0$. Then clearly $p \cdot y < p \cdot \bar{x}$ implies $u(y) < u(\bar{x})$. Now consider the case $p \cdot y = p \cdot \bar{x} > 0$. The for $0 \leqslant \lambda < 1$ we have $p \cdot \lambda y < p \cdot \bar{x}$, so $u(\lambda y) < u(\bar{x})$. Since u is continuous $u(y) \leqslant u(\bar{x})$. Thus $p \cdot y \leqslant p \cdot \bar{x}$ implies $u(y) \leqslant u(\bar{x})$. ∎

The assumption that $p \cdot \bar{x} > 0$ in part (2) above is needed, if we wish to allow nonnegative price vectors that are not strictly positive. For instance, let $u(x, y) = x + \sqrt{y}$, and $p = (0, 1)$. Then $(1, 0)$ minimizes $p \cdot (x, y)$ over $\boldsymbol{R}^2_+$ subject to $u(x, y) \geqslant 1$ as $p \cdot (1, 0) = 0$, but it does not maximize u subject to $p \cdot (x, y) \leqslant 0$, since $u(x, 0) = x$ and $p \cdot (x, 0) = 0$ for all x.

Expenditure minimization

Let $\hat{x}(p, \upsilon)$ minimize $p \cdot x$ subject to $u(x) - \upsilon = 0$, so that $\hat{x}$ minimizes the cost of achieving utility level υ. The Lagrangean for this is

$$p \cdot x - \mu(u(x) - \upsilon)$$

and by the Lagrange Multiplier Theorem first-order conditions are (multiplying by -1)

$$p_i - \hat{\mu} u_i(\hat{x}) = 0 \quad i = 1, \ldots, n,$$

and the second-order conditions for a minimum are

$$-\sum_i \sum_j u_{ij}(\hat{x}) v_i v_j \geqslant 0 \quad \text{for all } v \text{ such that } \sum_i u_i(\hat{x}) v_i = 0.$$

Again assumption U.3 guarantees the second-order conditions are satisfied, and that the Jacobian of the system is nonsingular.

Differentiate each of the first-order conditions with respect to p_j to get

$$\delta_{ij} - \hat{\mu} \sum_k u_{ik} \frac{\partial \hat{x}_k}{\partial p_j} - \frac{\partial \hat{\mu}}{\partial p_j} u_i = 0 \qquad \begin{array}{l} i = 1, \ldots, n, \\ j = 1, \ldots, n \end{array}$$

or, dividing by $\hat{\mu}$,

$$\sum_k u_{ik} \frac{\partial \hat{x}_k}{\partial p_j} + \frac{\partial \hat{\mu}}{\partial p_j} \frac{u_i}{\hat{\mu}} = \frac{\delta_{ij}}{\hat{\mu}} \qquad \begin{array}{l} i = 1, \ldots, n, \\ j = 1, \ldots, n. \end{array}$$

Differentiating with respect to υ to get

$$-\hat{\mu} \sum_k u_{ik} \frac{\partial \hat{x}_k}{\partial \upsilon} - \frac{\partial \hat{\mu}}{\partial \upsilon} u_i = 0 \qquad i = 1, \ldots, n.$$

or, dividing by $-\hat{\mu}$,

$$\sum_k u_{ik} \frac{\partial \hat{x}_k}{\partial v} + \frac{\partial \hat{\mu}}{\partial v} \frac{u_i}{\hat{\mu}} = 0 \qquad i = 1, \dots, n.$$

Now take the constraint $u(\hat{x}) - v = 0$, and differentiate with respect to p_j to get

$$\sum_k u_k \frac{\partial \hat{x}_k}{\partial p_j} = 0 \qquad j = 1, \dots, n,$$

so $\sum_k \frac{\partial \hat{x}_k}{\partial p_j} \frac{u_i}{\hat{\mu}} = 0$, and differentiate with respect to v to get

$$\sum_i u_i \frac{\partial \hat{x}_i}{\partial v} - 1 = 0.$$

Arranging in matrix terms gives

$$\begin{bmatrix} u_{11} & \dots & u_{1n} & \frac{1}{\hat{\mu}} u_1 \\ \vdots & & \vdots & \vdots \\ u_{n1} & \dots & u_{nn} & \frac{1}{\hat{\mu}} u_n \\ \frac{1}{\hat{\mu}} u_1 & \dots & \frac{1}{\hat{\mu}} u_n & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial \hat{x}_1}{\partial p_1} & \dots & \frac{\partial \hat{x}_1}{\partial p_n} & \frac{\partial \hat{x}_1}{\partial v} \\ \vdots & & \vdots & \vdots \\ \frac{\partial \hat{x}_n}{\partial p_1} & \dots & \frac{\partial \hat{x}_n}{\partial p_n} & \frac{\partial \hat{x}_n}{\partial v} \\ \frac{\partial \hat{\mu}}{\partial p_1} & \dots & \frac{\partial \hat{\mu}}{\partial p_n} & \frac{\partial \hat{\mu}}{\partial v} \end{bmatrix} = \left[\begin{array}{ccc|c} \frac{1}{\hat{\mu}} & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & \frac{1}{\hat{\mu}} & 0 \\ \hline 0 & \dots & 0 & \frac{1}{\hat{\mu}} \end{array}\right]$$

Once again, let's rearrange things to get

$$\begin{bmatrix} \frac{\partial x_1^*}{\partial p_1} & \dots & \frac{\partial x_1^*}{\partial p_n} & \frac{\partial x_1^*}{\partial v} \\ \vdots & & \vdots & \vdots \\ \frac{\partial x_n^*}{\partial p_1} & \dots & \frac{\partial x_n^*}{\partial p_n} & \frac{\partial x_n^*}{\partial v} \\ \frac{1}{\mu^*} \frac{\partial \mu^*}{\partial p_1} & \dots & \frac{1}{\mu^*} \frac{\partial \mu^*}{\partial p_n} & \frac{1}{\mu^*} \frac{\partial \mu^*}{\partial v} \end{bmatrix} = \frac{1}{\hat{\mu}} \begin{bmatrix} u_{11} & \dots & u_{1n} & u_1 \\ \vdots & & \vdots & \vdots \\ u_{n1} & \dots & u_{nn} & u_n \\ u_1 & \dots & u_n & 0 \end{bmatrix}^{-1} .$$

We know from our results on matrices negative definite under constraint that the matrix

$$\begin{bmatrix} \frac{\partial \hat{x}_1}{\partial p_1} & \dots & \frac{\partial \hat{x}_1}{\partial p_n} \\ \vdots & & \vdots \\ \frac{\partial \hat{x}_n}{\partial p_1} & \dots & \frac{\partial \hat{x}_n}{\partial p_n} \end{bmatrix}$$

is negative semidefinite of rank $n - 1$. Consequently, for each i,

$$\frac{\partial \hat{x}_i}{\partial p_i} \leqslant 0.$$

Moreover we also know that the null space of this matrix is the one-dimensional linear space spanned by $u'(\hat{x})$.

Combining the two

Look at equivalent expenditure minimization and utility maximization problems. That is, set $\upsilon = u(x^*(p,m))$. Then

$$x^*(p,m) = \hat{x}(p,\upsilon) \quad \text{and} \quad \lambda^* = \frac{1}{\hat{\mu}}.$$

Thus we have just shown in (7) that

$$\frac{\partial x_i^*}{\partial p_j} = \frac{\partial \hat{x}_i}{\partial p_j} - x_j^* \frac{\partial x_i^*}{\partial m},$$

a formula known as the **Slutsky decomposition**.

Now define the **expenditure function**

$$e(p,\upsilon) = p \cdot \hat{x}(p,\upsilon),$$

an observe that

$$\frac{\partial e}{\partial p_j} = \sum_{i=1}^n p_i \frac{\partial \hat{x}_i}{\partial p_j} + \hat{x}_j.$$

Now cleverly notice that

$$\begin{bmatrix} \frac{\partial \hat{x}_1}{\partial p_1} & \cdots & \frac{\partial \hat{x}_1}{\partial p_n} \\ \vdots & & \vdots \\ \frac{\partial \hat{x}_n}{\partial p_1} & \cdots & \frac{\partial \hat{x}_n}{\partial p_n} \end{bmatrix} \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix} = 0$$

since $p = \hat{\mu} u'(\hat{x})$ is in the null space. Thus,

$$\frac{\partial e(p,\upsilon)}{\partial p_j} = \hat{x}_j(p,\upsilon).$$

Also, defining the **indirect utility function**

$$v(p,m) = u(x^*(p,m)),$$

we have

$$\frac{\partial v}{\partial p_j} = \sum_{i=1}^n u_i \frac{\partial x_i^*}{\partial p_j} = \sum_{i=1}^n \lambda^* p_i \frac{\partial x_i^*}{\partial p_j},$$

and

$$\frac{\partial v}{\partial m} = \sum_{i=1}^n u_i \frac{\partial x_i^*}{\partial m} = \sum_{i=1}^n \lambda^* p_i \frac{\partial x_i^*}{\partial m} = \lambda^*.$$

Therefore

$$\begin{aligned}
\frac{\frac{\partial v}{\partial p_j}}{\frac{\partial v}{\partial m}} &= \sum_{i=1}^{n} \lambda^* p_i \frac{\partial x_i^*}{\partial p_j} \\
&= \sum_{i=1}^{n} \lambda^* p_i \left(\frac{\partial \hat{x}_i}{\partial p_j} - x_j^* \frac{\partial x_i^*}{\partial m} \right) \\
&= \sum_{i=1}^{n} \lambda^* p_i \frac{\partial \hat{x}_i}{\partial p_j} - x_j^* \sum_{i=1}^{n} \lambda^* p_i \frac{\partial x_i^*}{\partial m} \\
&= 0 - x_j^* \cdot (1).
\end{aligned}$$

Which gives us **Roy's Law**:

$$x_j^* = -\frac{\frac{\partial v}{\partial p_j}}{\frac{\partial v}{\partial m}}.$$