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Technical Results on Regular Preferences and Demand

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Preferences

For the purposes of this note, a **preference relation** (or simply a **preference**) $\succcurlyeq$ on a set X is a reflexive, total, transitive binary relation on X. Following Richter [3], I usually call this a **regular preference**. Mas-Colell, Whinston, and Green [2] call this a **rational preference**. There are times we wish to consider a weaker notion of preference that may be incomplete or non-transitive, but for now a preference is always reflexive, total, and transitive.[1] The binary relations $\succ$ and $\sim$ are the **asymmetric** and **symmetric parts** of $\succcurlyeq$, defined by

$$x \succ y \quad \text{if } x \succcurlyeq y \text{ and not } y \succcurlyeq x$$

and

$$x \sim y \quad \text{if } x \succcurlyeq y \ \& y \succcurlyeq x.$$

The symmetric part $\sim$ of a preference relation is called the induced **indifference** relation, and is reflexive, transitive, and symmetric. That is, $\sim$ is an **equivalence** relation. The equivalence classes of $\sim$ are traditionally called **indifference curves**, even though in general they may not be "curves."[2] The asymmetric part $\succ$ of $\succcurlyeq$ is called the induced **strict preference** relation. It is transitive, irreflexive, and asymmetric.

Recall that a function $u\colon X \to \boldsymbol{R}$ is a **utility for** $\succcurlyeq$ if

$$x \succcurlyeq y \qquad \Longleftrightarrow \qquad u(x) \geqslant u(y).$$

[1] Recall that a binary relation $\succcurlyeq$ on X is (i) **reflexive** if $(\forall x \in X)\ [x \succcurlyeq x]$; (ii) **total** if $(\forall x, y \in X)\ [x \neq y \implies (x \succcurlyeq y \text{ or } y \succcurlyeq x)]$; and (iii) **transitive** if $(\forall x, y, z \in X)\ [(x \succcurlyeq y \ \&\ y \succcurlyeq z) \implies x \succcurlyeq z]$. Some authors use the term **complete** to mean reflexive and total, that is, $(\forall x, y \in X)\ [x \succcurlyeq y \text{ or } y \succcurlyeq x]$. Debreu [1] calls a reflexive and transitive relation a **preorder** or **quasiorder**. Getting ahead of ourselves for a moment, a binary relation $\sim$ on X is (iv) **symmetric** if $(\forall x, y \in X)\ [x \sim y \implies y \sim x]$. The relation $\succ$ is (v) **irreflexive** if $(\forall x \in X)\ [\neg x \succ x]$; and (vi) **asymmetric** if $(\forall x, y \in X)\ [x \succ y \implies \neg y \succ x]$.

[2] The **equivalence class** $[x]$ of an element x of X is simply $\{y \in X : y \sim x\}$. The equivalence classes partition X. That is, each x belongs to some equivalence class (namely $[x]$); and for two equivalence classes $[x]$ and $[y]$ either $[x] = [y]$ (which occurs when $x \sim y$) or $[x] \cap [y] = \varnothing$. Economists often phrase this latter fact as "indifference curves do not cross."

Nonsatiation

A preference relation $\succcurlyeq$ on a set X has a **satiation point** x if x is a greatest element, that is, if $x \succcurlyeq y$ for all $y \in X$. A preference relation is **nonsatiated** if it has no satiation point. That is, for every x there is some $y \in X$ with $y \succ x$.

If (X, d) is a metric space, the preference relation $\succcurlyeq$ is **locally nonsatiated** if for every $x \in X$ and every $\varepsilon > 0$, there exists a point $y \in X$ with $d(y, x) < \varepsilon$ and $y \succ x$. Note that this is a joint condition on X and $\succcurlyeq$. In particular, if X is nonempty, it must be that for each point $x \in X$ and every $\varepsilon > 0$ there *exists* a point $y \neq x$ belonging to X with $d(y, x) < \varepsilon$. That is, X cannot have isolated points.

Monotonicity

Now let X be a subset of $\boldsymbol{R}^{\mathrm{n}}$. We use the following notation for partial orders on $\boldsymbol{R}^{\mathrm{n}}$:

$$\begin{aligned} x \geqq y \quad &\text{if} \quad x_i \geqslant y_i, \ i = 1, \dots, n; \\ x > y \quad &\text{if} \quad x \geqq y \ \& \ x \neq y; \text{ and} \\ x \gg y \quad &\text{if} \quad x_i > y_i, \ i = 1, \dots, n. \end{aligned}$$

A preference $\succcurlyeq$ on X is

$$\begin{aligned} \textbf{monotonic} \text{ if } x \gg y &\implies x \succ y, \text{ and} \\ \textbf{strictly monotonic} \text{ if } x > y &\implies x \succ y. \end{aligned}$$

The next lemma demonstrates that monotonicity implies local nonsatiation. The converse is not true, but we shall see in Proposition 23 below that the apparent increase in generality is illusory.

1 Lemma (Monotonicity implies local nonsatiation for $\boldsymbol{R}^{\mathrm{n}}_{+}$) *Let $\succcurlyeq$ be a monotonic preference on $\boldsymbol{R}^{\mathrm{n}}_{+}$. Then $\succcurlyeq$ is locally nonsatiated.*

Proof: Let x belong to $\boldsymbol{R}^{\mathrm{n}}_{+}$, let $\mathbf{1} \in \boldsymbol{R}^{\mathrm{n}}$ be the vector whose components are all one, and let $\varepsilon > 0$ be given. Now for any $\lambda > 0$, we have that $x + \lambda\mathbf{1} \in \boldsymbol{R}^{\mathrm{n}}_{+}$ and $x + \lambda\mathbf{1} \gg x$, so by monotonicity, $x + \lambda\mathbf{1} \succ x$. Now $d(x + \lambda\mathbf{1}, x) = \|x + \lambda\mathbf{1} - x\| = \lambda\|\mathbf{1}\| = \lambda\sqrt{n}$, so for $\lambda < \varepsilon/\sqrt{n}$, the point $y = x + \lambda\mathbf{1}$ satisfies $d(y, x) < \varepsilon$ and $y \succ x$. Since x and ε were arbitrary, $\succcurlyeq$ is locally nonsatiated. ∎

Continuity

Given a preference relation $\succcurlyeq$ on a set X, define the **strict** and **weak upper contour sets**

$$P(x) = \{y \in X : y \succ x\} \qquad \text{and} \qquad U(x) = \{y \in X : y \succcurlyeq x\}.$$

We also define the **strict** and **weak lower contour sets**

$$P^{-1}(x) = \{y \in X : x \succ y\} \qquad \text{and} \qquad U^{-1}(x) = \{y \in X : x \succcurlyeq y\}.$$

When (X, d) is a metric space, we say that $\succcurlyeq$ is **upper semicontinuous** if for each x, the set $U(x)$ is closed, or equivalently (since $\succcurlyeq$ is total), $P^{-1}(x)$ is open in X. Similarly, $\succcurlyeq$ is **lower**

semicontinuous if for each x, the set $U^{-1}(x)$ is closed, or equivalently, $P(x)$ is open in X. A preference relation $\succcurlyeq$ is **continuous** if and only if it is both upper and lower semicontinuous.

The next lemma gives several equivalent characterization of continuity for preferences.

2 Lemma (Continuity of preferences) *For a total, transitive, reflexive preference relation $\succcurlyeq$ on a metric space X, the following are equivalent.*

1. *The graph of $\succcurlyeq$ is closed. That is, if $y_n \to y$, $x_n \to x$, and $y_n \succcurlyeq x_n$ for each n, then $y \succcurlyeq x$.*
2. *The graph of $\succ$ is open. That is, if $y \succ x$, there is an $\varepsilon > 0$ such that if $d(y', y) < \varepsilon$ and $d(x', x) < \varepsilon$, then $y' \succ x'$.*
3. *For each x, the weak contour sets $U(x) = \{y \in X : y \succcurlyeq x\}$ and $U^{-1}(x) = \{y \in X : x \succcurlyeq y\}$ are closed.*
4. *For each x, the strict contour sets $P(x) = \{y \in X : y \succ x\}$ and $P^{-1}(x) = \{y \in X : x \succ y\}$ are open.*

Proof: Since $\succcurlyeq$ is total, it is clear that (1) $\iff$ (2) and (3) $\iff$ (4). Moreover it is also immediate that (1) $\implies$ (3) and (2) $\implies$ (4). So it suffices to prove that (4) implies (1).

So assume by way of contradiction that $y_n \to y$, $x_n \to x$, and $y_n \succcurlyeq x_n$ for each n, but $x \succ y$. Since $P(y)$ is open by condition (4) and $x \in P(y)$ by hypothesis, there is some $\varepsilon > 0$ such that $d(z, x) < \varepsilon$ implies $z \in P(y)$, or $z \succ y$. Similarly, since $P^{-1}(x)$ is open and $y \in P^{-1}(x)$ there is some $\varepsilon' > 0$ such that $d(w, y) < \varepsilon'$ implies $x \succ w$. Since $x_n \to x$ and $y_n \to y$, for large enough n, we have $d(x_n, x) < \varepsilon$ and $d(y_n, y) < \varepsilon'$, so

$$x \succ y_n \succcurlyeq x_n \succ y$$

for these large n. Pick one such n, call it n_0, and observe that

$$x \succ x_{n_0} \succ y.$$

Now condition (4) implies $P(x_{n_0})$ is open and since $x \in P(x_{n_0})$, there is some $\eta > 0$ such that $d(z, x) < \eta$ implies $z \succ x_{n_0}$. Similarly, since $P^{-1}(x_{n_0})$ and $y \in P^{-1}(x_{n_0})$, there is $\eta' > 0$ such that $d(w, y) < \eta'$ implies $x_{n_0} \succ w$. Now for large enough n we have $d(x_n, x) < \eta$ and $d(y_n, y) < \eta'$, so

$$x_n \succ x_{n_0} \succ y_n,$$

which contradicts $y_n \succcurlyeq x_n$ for all n. ∎

The following proposition is almost trivial. The converse is considerably more involved, and requires some additional assumptions on X (such as second countability and connectedness).

3 Proposition *If $\succcurlyeq$ has a utility that is a continuous function, then $\succcurlyeq$ is a continuous preference.*

4 Lemma *If $\succcurlyeq$ is upper semicontinuous and locally nonsatiated, then $U(x)$ is the closure of $P(x)$.*

Proof: $\overline{P(x)} \subset U(x)$: Let y belong to $\overline{P(x)}$. That is, there is a sequences y_n in $P(x) \subset U(x)$ with $y_n \to y$. Now $U(x)$ is closed by upper semicontinuity, so $y \in U(x)$.

$U(x) \subset \overline{P(x)}$: Let y belong to $U(x)$. By local nonsatiation, for each n there is a y_n satisfying $d(y_n, y) < \frac{1}{n}$ and $y_n \succ y$. Since $y_n \succ y$ and $y \succcurlyeq x$, transitivity implies $y_n \succ x$, so $y_n \in P(x)$. But $y_n \to y$, so $y \in \overline{P(x)}$. ∎

5 Proposition *If $\succcurlyeq$ is an upper semicontinuous and monotonic preference on $\boldsymbol{R}^{\mathrm{n}}_{+}$, then $x \geqq y$ implies $x \succcurlyeq y$.*

Proof: Put $x_n = x + (1/n)\mathbf{1}$ and note that $x_n \gg x \geqq y$, so by monotonicity $x_n \succ y$, and thus $x_n \in U(y)$. Now $x_n \to x$ and by upper semicontinuity $U(y)$ is closed, so $x \succcurlyeq y$. ∎

Existence of utility

The following theorem is easy to prove, but the hypotheses are stronger than needed.

6 Theorem *Let $X = \boldsymbol{R}^{\mathrm{n}}_{+}$ and let $\succcurlyeq$ be continuous and monotonic. Then $\succcurlyeq$ has a continuous utility function u.*

Proof: Let $\mathbf{1} \in \boldsymbol{R}^{\mathrm{n}}$ be the vector whose components are all one, and let $D = \{\lambda \mathbf{1} : \lambda \geqslant 0\}$ be the "diagonal" of $\boldsymbol{R}^{\mathrm{n}}_{+}$. For $x \in \boldsymbol{R}^{\mathrm{n}}_{+}$, let $M(x) = \max_i x_i$ and let $m(x) = \min_i x_i$. Then $M(x)\mathbf{1} \geqq x \geqq m(x)\mathbf{1}$, so $M(x)\mathbf{1} \succcurlyeq x \succcurlyeq m(x)\mathbf{1}$. Now let

$$L(x) = \{\lambda \geqslant 0 : x \succcurlyeq \lambda \mathbf{1}\} \qquad H(x) = \{\lambda \geqslant 0 : \lambda \mathbf{1} \succcurlyeq x\}.$$

These sets are intervals (by monotonicity), closed (by continuity), and nonempty (since $M(x)\mathbf{1} \in H(x)$ and $m(x)\mathbf{1} \in L(x)$). Moreover since $\succcurlyeq$ is total and reflexive $H(x) \cup L(x) = [0, \infty)$. Thus $H(x) \cap L(x)$ consists of a single point $\lambda(x)\mathbf{1} \sim x$. Define the function $u \colon \boldsymbol{R}^{\mathrm{n}}_{+} \to \boldsymbol{R}_{+}$ by

$$u(x) \text{ is the unique number satisfying } x \sim u(x)\mathbf{1}.$$

See Figure 1.

We now show that u is a utility for $\succcurlyeq$. First assume that $u(x) \geqslant u(y)$. Then $u(x)\mathbf{1} \geqq u(y)\mathbf{1}$, so by monotonicity $u(x)\mathbf{1} \succcurlyeq u(y)\mathbf{1}$. But then $x \sim u(x)\mathbf{1} \succcurlyeq u(y)\mathbf{1} \sim y$, so by transitivity $x \succcurlyeq y$. For the converse, assume $x \succcurlyeq y$. Then $u(x)\mathbf{1} \sim x \succcurlyeq y \sim u(y)\mathbf{1}$, so $u(x)\mathbf{1} \succcurlyeq u(y)\mathbf{1}$. Then by monotonicity, $u(x) \geqslant u(y)$. This completes the proof that u is a utility function for $\succcurlyeq$.

We shall prove that u is continuous by proving that it is both upper and lower semicontinuous.[3] For $\alpha \geqslant 0$, $\{x \in \boldsymbol{R}^{\mathrm{n}}_{+} : u(x) \geqslant \alpha\} = U(\alpha\mathbf{1})$, which is closed since $\succcurlyeq$ is continuous. Similarly, the set $\{x \in \boldsymbol{R}^{\mathrm{n}}_{+} : u(x) \leqslant \alpha\} = U^{-1}(\alpha\mathbf{1})$, which is also closed. For $\alpha < 0$, monotonicity implies that $\{x \in \boldsymbol{R}^{\mathrm{n}}_{+} : u(x) \geqslant \alpha\} = \boldsymbol{R}^{\mathrm{n}}_{+}$ and $\{x \in \boldsymbol{R}^{\mathrm{n}}_{+} : u(x) \leqslant \alpha\} = \varnothing$, again both closed. Thus u is both upper and lower semicontinuous and therefore continuous. ∎

Debreu [1, §4.6] proves that a continuous utility exists for a continuous preference on any connected subset of $\boldsymbol{R}^{\mathrm{n}}$, without assuming monotonicity.

As a reminder, there are (total, reflexive, transitive) preferences on $\boldsymbol{R}^{\mathrm{n}}_{+}$ that have no utility, but which are discontinuous. Everyone's favorite example is the lexicographic preference.[4]

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[3] There are many equivalent definitions of semicontinuity for functions. The easiest to use here is that u is **upper semicontinuous** if for every $\alpha \in \boldsymbol{R}$, the set $\{x \in \boldsymbol{R}^{\mathrm{n}}_{+} : u(x) \geqslant \alpha\}$ is closed; and u is **lower semicontinuous** if for every $\alpha \in \boldsymbol{R}$, the set $\{x \in \boldsymbol{R}^{\mathrm{n}}_{+} : u(x) \leqslant \alpha\}$ is closed. See section 17 of my on-line notes on metric spaces for other characterizations.

[4] The lexicographic preference on the plane is given by $(x, y) \succcurlyeq (x', y')$ if $\big[x > x' \text{ or } (x = x' \text{ and } y \geqslant y')\big]$. To

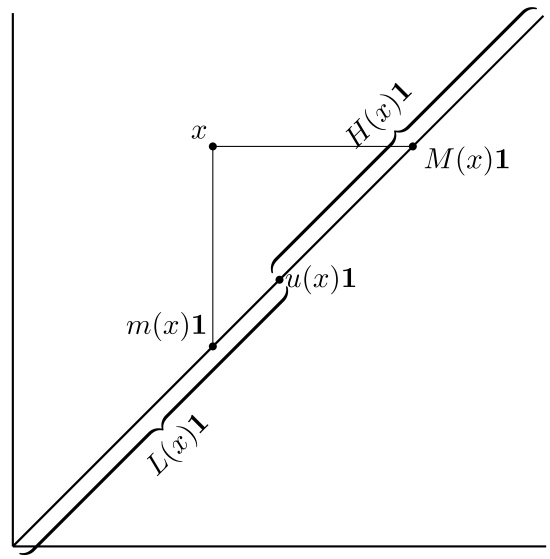

Figure 1. Construction of utility for a monotonic continuous preference.

Convexity

When X is a subset of a linear space, we say that $\succcurlyeq$ is

$$\begin{array}{rrcl} \textbf{convex} \text{ if} & y \succ x & \implies & (1-\lambda)x + \lambda y \succ x, \\ \textbf{weakly convex} \text{ if} & y \succcurlyeq x & \implies & (1-\lambda)x + \lambda y \succcurlyeq x, \\ \textbf{strictly convex} \text{ if} & y \neq x \ \&\ y \succcurlyeq x & \implies & (1-\lambda)x + \lambda y \succ x, \end{array}$$

for all $0 < \lambda < 1$.

7 Lemma *Let X be convex, and let $\succcurlyeq$ be a preference on X. The following are equivalent.*

1. *$\succcurlyeq$ is weakly convex.*
2. *For each x, the strict upper contour set $P(x)$ is a convex set.*
3. *For each x, the weak upper contour set $U(x)$ is a convex set.*

Proof: (1) $\implies$ (2) Assume $\succcurlyeq$ is weakly convex, and let y_1 and y_2 belong to $P(x)$. Since $\succcurlyeq$ is total and reflexive, without loss of generality we may assume that

$$y_2 \succcurlyeq y_1 \succ x.$$

see that no utility exists for this preference relation, let $x > x'$. Then any utility u would imply the existence of rational numbers q_x and $q_{x'}$ satisfying

$$u\big((x,1)\big) > q_x > u\big((x,0)\big) > u\big((x',1)\big) > q_{x'} > u\big((x',0)\big).$$

This defines a one-to-one correspondence $x \longleftrightarrow q_x$ between the reals and a subset of the rational numbers. But Cantor proved long ago via his famous "diagonal procedure" that no such correspondence can exist.

Let $0 \leqslant \lambda \leqslant 1$. By weak convexity then

$$(1-\lambda)y_1 + \lambda y_2 \succcurlyeq y_1 \succ x,$$

where the second relation follows from transitivity. Thus $(1-\lambda)y_1 + \lambda y_2$ belongs to $P(x)$, which proves that $P(x)$ is convex.

(2) $\implies$ (3) Assume that $P(x)$ is convex for each x. Fix some x, and suppose by way of contradiction, that $U(x)$ is not convex. Then for some y_1 and y_2 belonging to $U(x)$, that is, $y_1 \succcurlyeq x$ and $y_2 \succcurlyeq x$, and for some $0 < \bar\lambda < 1$, the point $z = (1-\bar\lambda)y_2 + \bar\lambda y_1$ does not belong to $U(x)$. Then since $\succcurlyeq$ is total, we must have $x \succ z$. Thus $y_1 \succcurlyeq x \succ z$ and $y_2 \succcurlyeq x \succ z$, so by transitivity $y_1 \succ z$ and $y_2 \succ z$. By hypothesis $P(z)$ is convex, we must have $(1-\bar\lambda)y_2 + \bar\lambda y_1 = z \succ z$, a contradiction. Therefore $U(x)$ must be convex.

(3) $\implies$ (1) Assume that $U(x)$ is convex, and let $y \succcurlyeq x$. Then for any $0 < \lambda < 1$, we have $(1-\lambda)x + \lambda y \in U(x)$, that is, $(1-\lambda)x + \lambda y \succcurlyeq x$, so $\succcurlyeq$ is weakly convex. ∎

Despite its name, the property of weak convexity is not actually weaker than convexity.

8 Exercise (Convexity does not imply weakly convexity) Let $X = [-1, 1]$ and define $\succcurlyeq$ by means of the utility function

$$u(x) = \begin{cases} 1 & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Prove that $\succcurlyeq$ is convex, but not weakly convex. □

The preference relation in the example above is not continuous, which brings up the next lemma. To simplify the discussion of these properties let say that ***z* is between *x* and *y*** if (i) $x \neq y$, and (ii) $z = (1-\lambda)x + \lambda y$ for some $0 < \lambda < 1$.

9 Lemma (Convexity and upper semicontinuity imply weak convexity) *If $\succcurlyeq$ is convex and upper semicontinuous, then it is weakly convex.*

Proof: Assume that $y \succcurlyeq x$. In case $y \succ x$, then by convexity $(1-\lambda)x + \lambda y \succ x$ for $0 < \lambda < 1$, so a fortiori $(1-\lambda)x + \lambda y \succcurlyeq x$.

So now consider the case $y \sim x$ and assume by way of contradiction that for some $0 < \bar\lambda < 1$ the point $z = (1-\bar\lambda)x + \bar\lambda y$ satisfies $x \succ z$. By upper semicontinuity, $P^{-1}(x)$ is open, so we may choose $\tilde\lambda$ close to $\bar\lambda$, but with $\tilde\lambda > \bar\lambda$ so that $w = (1-\tilde\lambda)x + \tilde\lambda y$ satisfies $x \succ w$. See Figure 2. But this means that z is between w and x, and since $x \succ w$, convexity implies $z \succ w$. On the other hand, w is between y and z, and $y \sim x \succ z$, so convexity implies $w \succ z$, a contradiction. ∎

10 Corollary *Let X be convex, and let $\succcurlyeq$ be a weakly convex regular preference on X. Any utility function for $\succcurlyeq$ is quasiconcave. If in addition, $\succcurlyeq$ is convex, any utility function for $\succcurlyeq$ is*

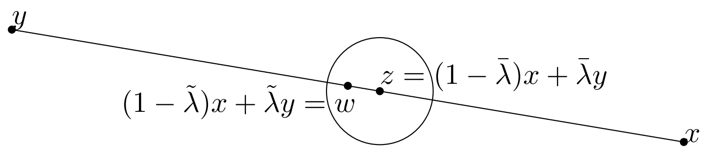

Figure 2. $(x \succ w \implies z \succ w)$ and $(y \succ z \implies w \succ z)$. Oops.

explicitly quasiconcave.[5]

The next result gives conditions that rules out "thick" indifference classes.

11 Lemma *If X is convex, and $\succcurlyeq$ is convex, continuous, and nonsatiated, then $P(x)$ is the interior of $U(x)$.*

Proof: Since $P(x) \subset U(x)$ and $P(x)$ is open by lower semicontinuity, we have $P(x) \subset \operatorname{int} U(x)$. For the reverse inclusion, let y belong to the interior of $U(x)$, so there is some $\varepsilon > 0$ such that the ε-ball centered at y lies wholly in $U(x)$. Assume by way of contradiction that $y \notin P(x)$. Then since $y \in U(x)$, it must be that $y \sim x$. Since $\succcurlyeq$ is nonsatiated, there is a point $z \in X$ with $z \succ y$. Choose $\alpha < 0$ but close enough to zero, so that the point $w = (1-\alpha)y + \alpha z$ is within ε of y and also so that $z \succ w$, which can be done by upper semicontinuity of $\succcurlyeq$. See Figure 3. Then $z \succ w \succcurlyeq x \sim y$. But since y lies between z and w, by convexity we must have $y \succ w$, a contradiction. ∎

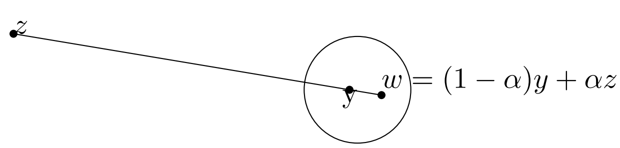

Figure 3. $w \succcurlyeq x \sim y$ and $z \succ y \succ w$, oops.

12 Example (Lemma 11 may fail without convexity) Let $X = \boldsymbol{R}$ and let $\succcurlyeq$ be defined by the utility $u(x) = x^2$. Then $\succcurlyeq$ is locally nonsatiated and continuous, but $P(0) = \boldsymbol{R} \setminus \{0\} \neq \boldsymbol{R} = \operatorname{int} U(0)$. □

[5] Recall that a function $u\colon X \to \boldsymbol{R}$ is **quasiconcave** if for each $\alpha \in \boldsymbol{R}$, the set $\{x \in X : u(x) \geqslant \alpha\}$ is convex. This is equivalent to the following condition: for all x, y in X and all $0 < \lambda < 1$

$$f\big((1-\lambda)x + \lambda y\big) \geqslant \min\{f(x), f(y)\}.$$

A function u is **explicitly quasiconcave** if it is quasiconcave and in addition, for all x, y in X and all $0 < \lambda < 1$

$$f(x) > f(y) \implies f\big((1-\lambda)x + \lambda y\big) > \min\{f(x), f(y)\} = f(y).$$

The next result is a riff on the one above, where convexity is replaced by weak convexity and and nonsatiation is replaced by local nonsatiation.

13 Lemma *If X is convex, and $\succcurlyeq$ is weakly convex, lower semicontinuous, and locally nonsatiated, then $P(x)$ is the interior of $U(x)$.*

Proof: Since $P(x) \subset U(x)$ and $P(x)$ is open by lower semicontinuity, we have $P(x) \subset \operatorname{int} U(x)$.

For the reverse inclusion, let y belong to the interior of $U(x)$, so there is some $\delta > 0$ such that the δ-ball V centered at y lies wholly in $U(x)$. Assume by way of contradiction that $y \notin P(x)$. Then since $y \in U(x)$, it must be that $y \sim x$. Since $\succcurlyeq$ is locally nonsatiated, there is a point $\bar{z} \in X$ with $\bar{z} \succ y$. Moreover by lower semicontinuity, there is some $\varepsilon > 0$ such that the ε-ball B centered at $\bar{z}$ is included in $P(y)$.

Consider the open sets of the form

$$A_\alpha = (1-\alpha)y + \alpha B, \quad \text{where } \alpha < 0.$$

See Figure 4. The set A_α is a ball of radius $|\alpha|\varepsilon$ centered at $\bar{w} = (1-\alpha)y + \alpha\bar{z}$. Moreover,

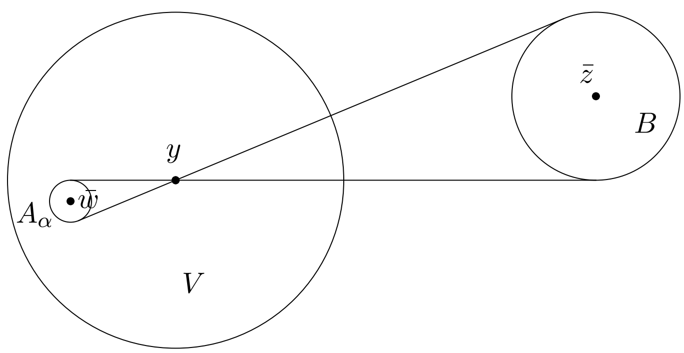

Figure 4. The set $A = (1-\alpha)y + \alpha B$, where $\alpha < 0$.

$\|\bar{w} - y\| = |\alpha|\|\bar{z} - y\|$, so

$$w \in A_\alpha \implies \|w - y\| \leqslant \|w - \bar{w}\| + \|\bar{w} - y\| < |\alpha|\big(\varepsilon + \|\bar{z} - y\|\big).$$

So for $|\alpha| < \delta/\big(\varepsilon + \|\bar{z} - y\|\big)$, we have $\|w - y\| < \delta$, so $A_\alpha \subset V \subset U(x)$. So fix such an $\alpha < 0$. Now for any point $w \in A_\alpha$ there is by definition some $z \in B$ such that

$$w = (1-\alpha)y + \alpha z, \quad \text{or} \quad y = \frac{1}{1-\alpha}w + \frac{-\alpha}{1-\alpha}z.$$

I claim that this implies that $w \sim y$. To see why, first note that since $A_\alpha \subset B_\delta(y) \subset U(x) = U(y)$ we have $w \succcurlyeq y$. But if $w \succ y$, since $z \succ y$, and y is a proper convex combination of z and w, weak convexity implies $y \succcurlyeq w \succ y$, a clear contradiction. This means that $w \sim y$. But w is an arbitrary element of the open set A_α, so local nonsatiation is violated on A_α.

This contradiction shows that $\operatorname{int} U(x) \subset P(x)$, completing the proof that $\operatorname{int} U(x) = P(x)$. ∎

Lemma 9 gives conditions under which convexity implies weak convexity. The next one gives conditions under which weak convexity implies convexity.

14 Lemma (Weak convexity, lower semicontinuity, and local nonsatiation imply convexity) *If X is convex, and $\succcurlyeq$ is weakly convex, lower semicontinuous, and locally nonsatiated, then it is convex.*

Proof: Assume $y \succ x$. Then $y \in P(x) = \operatorname{int} U(x)$ and $x \in U(x)$, so by a well known property of convex sets, the entire segment $[y, x)$ lies in the interior of the convex set $U(x)$. By Lemma 13, $\operatorname{int} U(x) = P(x)$, so for $1 \geqslant \lambda > 0$, we have $(1-\lambda)x + \lambda y \succ x$. That is, $\succcurlyeq$ is convex. ∎

Preference maximization

15 Definition *The point x^* is a* ***$\succcurlyeq$-greatest point*** *in the set B if $x^* \in B$ and for every $x \in B$, we have $x^* \succcurlyeq x$.*

16 Lemma *Let X be a finite nonempty set, and assume $\succcurlyeq$ is a total, reflexive, and transitive binary relation on X. Then X has a $\succcurlyeq$-greatest element.*

Proof: This provides an excellent opportunity to demonstrate proof by induction. Let $\mathbb{P}[n]$ denote the proposition: If X has n elements, then X has a $\succcurlyeq$-greatest element.

Clearly $\mathbb{P}[1]$ is valid, that is if X has a single element x, then by reflexivity, $x \succcurlyeq x$, so x is $\succcurlyeq$-greatest in X.

We now show that $\mathbb{P}[n]$ implies $\mathbb{P}[n+1]$. So assume that X has $n+1$ elements. Pick some $x \in X$, and let $A = X \setminus \{x\}$. By the induction hypothesis $\mathbb{P}[n]$, since the set A has n elements, it has a $\succcurlyeq$-greatest element y. Since $\succcurlyeq$ is total, there are two (overlapping) cases: $x \succcurlyeq y$ and $y \succcurlyeq x$.

Case 1: $x \succcurlyeq y$. Since y is $\succcurlyeq$-greatest in A, we have $y \succcurlyeq z$ for all $z \in A = X \setminus \{x\}$. By transitivity we must have $x \succcurlyeq z$ for $z \in A$, and by reflexivity $x \succcurlyeq x$. Thus $x \succcurlyeq z$ for all $z \in X$ so x is $\succcurlyeq$-greatest in X.

Case 2: $y \succcurlyeq x$. We already have $y \succcurlyeq z$ for all $z \in A = X \setminus \{x\}$. Therefore $y \succcurlyeq z$ for all $z \in X$, so y is $\succcurlyeq$-greatest in X.

We have just shown that $\mathbb{P}[n]$ implies $\mathbb{P}[n+1]$. By the principle of induction, every finite set has a greatest element. ∎

Alternative proof: Let $|X| = m$. Define $y_1 = x_1$ and inductively define

$$y_{n+1} = \begin{cases} x_{n+1} & \text{if } x_{n+1} \succcurlyeq y_n, \\ y_n & \text{if } y_n \succ x_{n+1}. \end{cases}$$

Since preferences are total, each y_n is well defined, and since preferences are transitive, the last term y_m satisfies $y_m \succcurlyeq x_i$ for $i = 1, \dots, m$. (This actually requires a simple proof by induction.) That is, y_m is $\succcurlyeq$-greatest in X. ∎

17 Proposition *Let X be nonempty and compact, and assume $\succcurlyeq$ is upper semicontinuous (and complete and transitive). Then X has a $\succcurlyeq$-greatest element.*

Proof: The family $\mathcal{U} = \{U(x) : x \in X\}$ consists of closed sets by upper semicontinuity. We now show that $\mathcal{U}$ has the finite intersection property.

Let $\{x_1, x_2, \dots, x_m\}$ be a nonempty finite subset of X. By Lemma 16, $\{x_1, x_2, \dots, x_m\}$ has a $\succcurlyeq$-greatest element y. I claim that $\bigcap_{i=1}^m U(x_i) = U(y)$. To see this note that $y \succcurlyeq x_i$ for each i, so if $z \in U(x_i)$, that is, if $z \succcurlyeq y$, then by transitivity $z \succcurlyeq x_i$. That is $U(y) \subset U(x_i)$ for each i, so $U(y) \subset \bigcap_{i=1}^m U(x_i)$. On the other hand $y = x_i$ for some i, so $U(y) \supset \bigcap_{i=1}^m U(x_i)$. Thus $\bigcap_{i=1}^m U(x_i) = U(y)$, as claimed. Since $y \in U(y)$, this intersection is nonempty. In other words, $\mathcal{U}$ has the finite intersection property.

Since X is compact, and $\mathcal{U}$ is a family of closed sets with the finite intersection property, it has nonempty intersection. That is, there is some y belonging to $\bigcap_{x \in X} U(x)$. But this just says that $y \succcurlyeq x$ for all $x \in X$. ∎

18 Proposition *Let X be convex, and assume that the regular preference $\succcurlyeq$ is strictly convex. Then a $\succcurlyeq$-greatest element is unique (if it exists).*

Proof: Suppose that both x and y are $\succcurlyeq$-greatest elements of X. Suppose by way of contradiction that if $x \neq y$. Then by strict convexity, $\frac{1}{2}y + \frac{1}{2}x \succ x$, contradicting the hypothesis that x is $\succcurlyeq$-greatest. Therefore $x = y$. ∎

Preference maximization and expenditure minimization

Let X be a subset of $\boldsymbol{R}^{\mathrm{n}}$, and set $\beta(p, w) = \{x \in X : p \cdot x \leqslant w\}$, where $p \in \boldsymbol{R}^{\mathrm{n}}$ and $w \in \boldsymbol{R}$.

19 Lemma *Assume $\succcurlyeq$ is locally nonsatiated and transitive.*

Assume that

$$x^* \text{ is } \succcurlyeq\text{-greatest in } \beta(p, w),$$

that is,

$$x^* \in \beta(p, w) \quad \text{and} \quad p \cdot x \leqslant w \implies x^* \succcurlyeq x.$$

The contrapositive of the last statement is

$$x \succ x^* \implies p \cdot x > w \geqslant p \cdot x^*.$$

Then

1. *$p \cdot x^* = w$ (all income is spent).*

2. *x^* minimizes $p \cdot x$ over $U(x^*)$. That is,*

$$x \succcurlyeq x^* \implies p \cdot x \geqslant p \cdot x^*.$$

Proof: (1.) Assume by way of contradiction that $p \cdot x^* < w$. Then there is some $\varepsilon > 0$ such that for all y with $d(x^*, y) < \varepsilon$, we have $p \cdot y < w$, so $y \in \beta(p, w)$. By local nonsatiation, at least one such one y satisfies $y \succ x$, which contradicts the hypothesis that x^* is $\succcurlyeq$-greatest in $\beta(p, w)$.

Therefore $p \cdot x^* = w$.

(2.) Assume by way of contradiction that there is some $x \in U(x^*)$ with $p \cdot x < p \cdot x^*$. Then there is some $\varepsilon > 0$ such that for all y with $d(x, y) < \varepsilon$, we have $p \cdot y < p \cdot x^* \leqslant w$, so $y \in \beta(p, w)$. See Figure 5. By local nonsatiation, at least one such one y satisfies $y \succ x$. But $x \in U(x^*)$ means $x \succcurlyeq x^*$, so by transitivity, $y \succ x^*$, which contradicts the hypothesis that x^* is $\succcurlyeq$-greatest in $\beta(p, w)$.

Therefore x^* minimizes $p \cdot x$ over $U(x^*)$. ∎

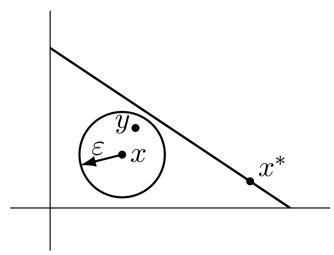

Figure 5. Preference maximization subject to a budget implies expenditure minimization.

20 Lemma *Assume X is convex, and $\succcurlyeq$ is lower semicontinuous.*

Assume x^* minimizes $p \cdot x$ over $U(x^*)$, and the ***cheaper-point assumption*** *holds, that is, there exists $\tilde{x} \in X$ satisfying $p \cdot \tilde{x} < p \cdot x^*$. Then x^* is $\succcurlyeq$-greatest in $B = \beta(p, p \cdot x^*)$.*

Proof: Suppose by way of contradiction that there is some $y \in B$ satisfying $y \succ x^*$, that is, $y \in P(x^*) \subset U(x^*)$. Then $p \cdot y \geqslant p \cdot x^*$, as x^* minimizes expenditure over $U(x^*)$. But y is in the budget B, so we conclude $p \cdot y = p \cdot x^*$.

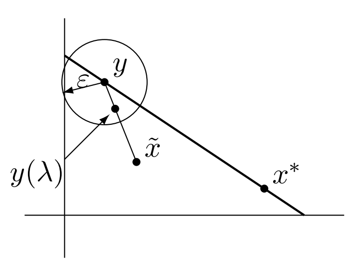

Figure 6. Expenditure minimization implies preference maximization

For λ satisfying $0 < \lambda < 1$, define $y(\lambda) = (1 - \lambda)y + \lambda\tilde{x}$. Then $p \cdot \tilde{x} < p \cdot y(\lambda) < p \cdot x^*$. Since X is convex, $y(\lambda) \in B$ for all $0 < \lambda \leqslant 1$.

But $y(\lambda) \to y$ as $\lambda \to 0$, and y belongs to the open set $P(x^*)$ (lower semicontinuity), so for some $\varepsilon > 0$, for every $\lambda < \varepsilon$ we have $y(\lambda) \in P(x^*) \subset U(x^*)$. See Figure 6. But for such λ we have $p \cdot y(\lambda) < p \cdot x^*$, which contradicts the hypothesis that x^* minimizes $p \cdot x$ over $U(x^*)$.

Therefore x^* is $\succcurlyeq$-greatest in $B = \beta(p, p \cdot x^*)$. ∎

To see what may happen if the cheaper-point assumption is violated, consider the following example.

21 Example (Why the cheaper point is needed) Let $X = \boldsymbol{R}^2_+$. Let preferences be defined by the utility function $u(x_1, x_2) = x_1 + x_2$. (This preference relation is continuous, convex, and locally nonsatiated.) Let $x^* = (1, 0)$ and $p = (0, 1)$. Then x^* minimizes $p \cdot x$ over $U(x^*)$. But $\beta(p, p \cdot x^*) = \beta(p, 0)$, which is just the x_1-axis. This budget set is unbounded and no $\succcurlyeq$-greatest element exists. See Figure 7.

If you don't like the fact that I resorted to using a zero price, consider the case where $X = \{x \in \boldsymbol{R}^2_+ : x_1 + x_2 \geqslant 2\}$. Let $u(x_1, x_2) = 2x_1 + x_2$, $p = (1, 1)$, and $\hat{x} = (1, 1)$. Again $\hat{x}$ minimizes expenditure over $U(\hat{x})$, but $\bar{x} = (2, 0)$ is $\succcurlyeq$-greatest in the budget set $\beta(p, p \cdot \hat{x}) = \beta(p, 2)$, which is the southwest boundary of the consumption set. See Figure 8. □

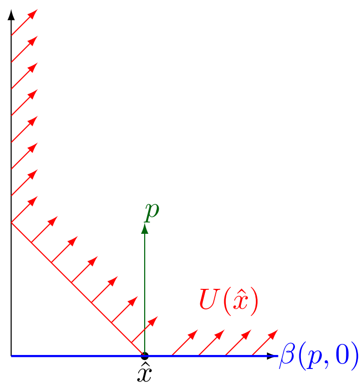

Figure 7. Cheaper-point violation 1.

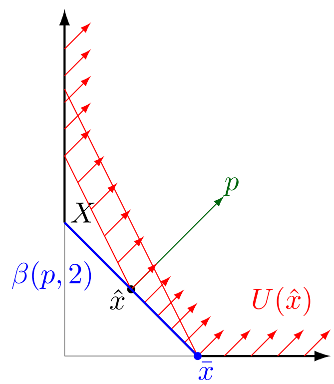

Figure 8. Cheaper-point violation 2.

22 Corollary *Assume X is convex, and $\succcurlyeq$ is lower semicontinuous and locally nonsatiated. Let p be given and set $w = p \cdot x^*$. Assume there is a point $\tilde{x} \in X$ satisfying $p \cdot \tilde{x} < w$.*

Then x^* is $\succcurlyeq$-greatest in $\beta(p, w)$ if and only if x^* minimizes $p \cdot x$ over $U(x^*)$.

Local nonsatiation, monotonicity, and demand

On the face of it local nonsatiation is a weaker condition than monotonicity for preference relations. Clearly monotonicity implies local nonsatiation, provided the consumption set is increasing. (A set A in $\boldsymbol{R}^{\mathrm{n}}$ is **increasing** if $(x \in A \;\&\; z \gg x) \implies z \in A$.) But the preference relation with utility $u(x, y) = y - (x - 1)^2$ is locally nonsatiated without being monotone. See Figure 9. But this preference relation generates the same demand as the preference relation with utility given by

$$\hat{u}(x, y) = \begin{cases} y - (1 - x)^2 & x \leqslant 1 \\ y & x \geqslant 1 \end{cases} \tag{1}$$

See Figure 9. I leave it to you to verify this claim.

This is true more generally. That is, if $\succcurlyeq$ is a locally nonsatiated upper semicontinuous preference, then there is a monotonic preference $\succcurlyeq^*$ that generates the same demand.

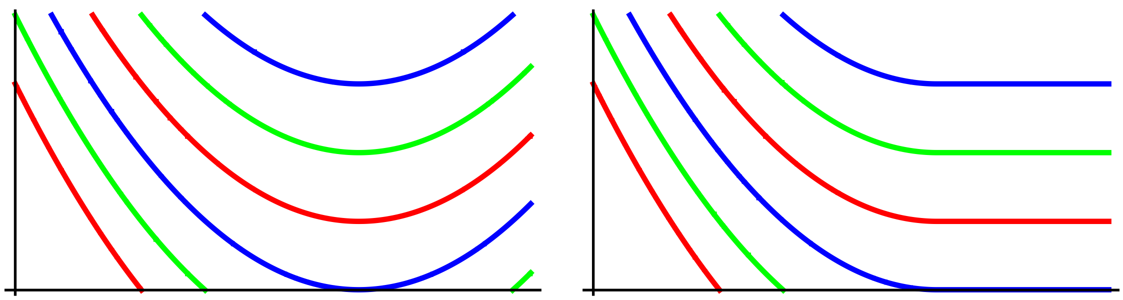

Figure 9. Indifference curves for locally nonsatiated utility $u(x, y) = y - (1 - x)^2$ and the monotonic utility given by $\hat{u}(x, y) = y - (1 - x)^2$ for $x \leqslant 1$ and $\hat{u}(x, y) = y$ for $x \geqslant 1$.

To simplify the explicit description of such a monotonic relation, let us introduce the following notation. Given a set A, write $x \succcurlyeq A$ to mean "$x \succcurlyeq y$ for all $y \in A$," and let $D(x) = \{y \in \boldsymbol{R}^{\mathrm{n}}_{+} : y \leqq x\}$.

23 Proposition *Let $\succcurlyeq$ be locally nonsatiated and upper semicontinuous regular preference on $\boldsymbol{R}^{\mathrm{n}}_{+}$. Then the binary relation $\overset{*}{\succcurlyeq}$ defined by*

$$x \overset{*}{\succcurlyeq} y \quad \textit{if there exists } v \textit{ such that } x \geqq v \succcurlyeq D(y)$$

is a monotonic upper semicontinuous preference relation that generates the same demand for $(p, w) \in \boldsymbol{R}^{\mathrm{n}}_{++} \times \boldsymbol{R}_{+}$. Moreover, if $\succcurlyeq$ is convex, then $\overset{*}{\succcurlyeq}$ is convex.

Proof: Note that for each y the set $D(y)$ is compact, and since $\succcurlyeq$ is upper semicontinuous, the set $\mu(y)$ of $\succcurlyeq$-greatest element of $D(y)$ is nonempty. So $x \overset{*}{\succcurlyeq} y$ if and only if there exist $\underline{x} \leqq x$ and $\underline{y} \in \mu(y)$ with $\underline{x} \succcurlyeq \underline{y}$.

First we show that $\overset{*}{\succcurlyeq}$ is

i. reflexive: Let $v \in \mu(x)$. Then $x \geqq v \succcurlyeq D(x)$, so $x \overset{*}{\succcurlyeq} x$.

ii. transitive: Assume $x \overset{*}{\succcurlyeq} y$ and $y \overset{*}{\succcurlyeq} z$. Then there exist v_y, v_z, u_y, and u_z such that $x \geqq v_y \succcurlyeq u_y \in \mu(y)$ and $y \geqq v_z \succcurlyeq u_z \in \mu(z)$.

   Since $v_z \leqslant y$ and $u_y \succcurlyeq D(y)$ we have $u_y \succcurlyeq v_z$. Thus $x \geqq v_y \succcurlyeq u_y \succcurlyeq v_z \succcurlyeq D(z)$, so $x \overset{*}{\succcurlyeq} z$.

iii. total: If $x \not\overset{*}{\succcurlyeq} y$, then by definition, for every $u \leqq x$, there is some $v \leqq y$ with $v \succ u$.

   Let $u^* \in \mu(x)$, and let $v^* \leqq y$ satisfy $v^* \succ u^*$. Thus $y \geqq v^* \succ u^* \succcurlyeq D(x)$. Thus $y \overset{*}{\succcurlyeq} x$.

We can now characterize the strict relation $\overset{*}{\succ}$ by

$$y \overset{*}{\succ} x \text{ if and only if } \exists v \text{ such that } y \geqq v \succ D(x).$$

Next we show that $\overset{*}{\succcurlyeq}$ is monotonic, that is, if $x \gg y$, then $x \overset{*}{\succ} y$. So assume $x \gg y$ and let $z \in \mu(y) \subset D(y)$, so $x \gg z$. Then for $\varepsilon > 0$ small enough, $d(v, z) < \varepsilon$ implies $x \gg v$. By local nonsatiation, at least one such v satisfies $v \succ z \succcurlyeq D(y)$. Thus $x \gg z \succ D(y)$, so $x \overset{*}{\succ} y$.

To see that $\overset{*}{\succcurlyeq}$ is upper semicontinuous, I shall prove that if $y \overset{*}{\succ} x$, then there is an $\varepsilon > 0$ such that $d(x, x') < \varepsilon$ implies $y \overset{*}{\succ} x'$ too. So assume $y \overset{*}{\succ} x$. Then there exists $v \leqq y$ such that $v \succ D(x)$. Since $D(x)$ is compact, I claim there is some $\varepsilon > 0$ such that $v \succ N_\varepsilon\big(D(x)\big)$.[6] Then if $d(x, x') < \varepsilon$, we have $D(x') \subset N_\varepsilon\big(D(x)\big)$ too, so $v \succ D(x')$. But this implies $y \overset{*}{\succ} x'$.

Finally we show that for $p \gg 0$, a point x^* is $\succcurlyeq$-greatest in $\beta(p, w)$ if and only if x^* is also $\overset{*}{\succcurlyeq}$-greatest.

Assume first that x^* is $\succcurlyeq$-greatest in $\beta(p, w)$. Let $y \in \beta(p, w)$. Then $D(y) \subset \beta(p, w)$, so $x^* \geqq x^* \succcurlyeq D(y)$. Thus $x^* \overset{*}{\succcurlyeq} y$. Therefore x^* is $\overset{*}{\succcurlyeq}$-greatest in $\beta(p, w)$.

Now assume that x^* is $\overset{*}{\succcurlyeq}$-greatest in $\beta(p, w)$, and let $\bar{x}$ be $\succcurlyeq$-greatest. Since $x^* \overset{*}{\succcurlyeq} \bar{x}$ there is some $z \leqq x^*$ with $z \succcurlyeq D(\bar{x})$. In particular, $z \succcurlyeq \bar{x} \in D(\bar{x})$, so z too is $\succcurlyeq$-greatest. But by local nonsatiation $p \cdot z = w$, so $z \leqq x^* \in \beta(p, w)$ implies $z = x^*$, so x^* is also $\succcurlyeq$-greatest.

To see that $\overset{*}{\succcurlyeq}$ is convex if $\succcurlyeq$ is convex, let $x, x' \overset{*}{\succcurlyeq} y$, where $x \geqq v \succcurlyeq D(y)$ and let $x' \geqq v' \succcurlyeq D(y)$. Then $(1-\lambda)x + \lambda x' \geqq (1-\lambda)v + \lambda v'$ and assuming $\succcurlyeq$ is convex, $(1-\lambda)v + \lambda v' \succcurlyeq D(y)$. Thus $(1-\lambda)x + \lambda x' \overset{*}{\succcurlyeq} y$. ∎

Finally we mention that if u is an upper semicontinuous locally nonsatiated function, then its **monotonic hull** v, defined by $v(x) = \max\{u(y) : 0 \leqq y \leqq x\}$, is the smallest monotonic function that dominates u. The Berge Maximum Theorem implies that it is upper semicontinuous. If u is a utility that represents $\succcurlyeq$, then its monotonic hull represents $\overset{*}{\succcurlyeq}$.

24 Example Consider the quasilinear utility function for two goods x and y defined by

$$u(x, y) = y - (1-x)^2$$

which gives a linear demand function for x. It is locally nonsatiated but not monotone. It has the property that the demand for x never exceeds 1. It has the same demand behavior as the monotone utility

$$v(x, y) = \begin{cases} y - (1-x)^2 & x \leqslant 1 \\ y & x \geqslant 1 \end{cases}$$

which is its monotonic hull. See Figure 10. □

References

[1] G. Debreu. 1959. *Theory of value: An axiomatic analysis of economic equilibrium*. Number 17 in Cowles Foundation Monographs. New Haven: Yale University Press.
cowles.econ.yale.edu/P/cm/m17/m17-all.pdf

[6] Here $N_\varepsilon(A)$ denotes the ε-neighborhood of A, that is, $\big\{x : (\exists y \in A)\ [d(x, y) < \varepsilon]\big\}$. It is an open set, being the union of the open balls of radius ε centered on points of A. To see why this claim is true, let F denote the closed upper contour set $\{u : u \succcurlyeq v\}$. Then the distance function $d(z, F) = \inf\{d(z, u) : u \in F\}$ is (Lipschitz) continuous and so achieves its minimum over the compact set $D(x)$. Since F and $D(x)$ are disjoint closed sets this minimum is strictly greater than zero. Choose $\varepsilon > 0$ less than this minimum.

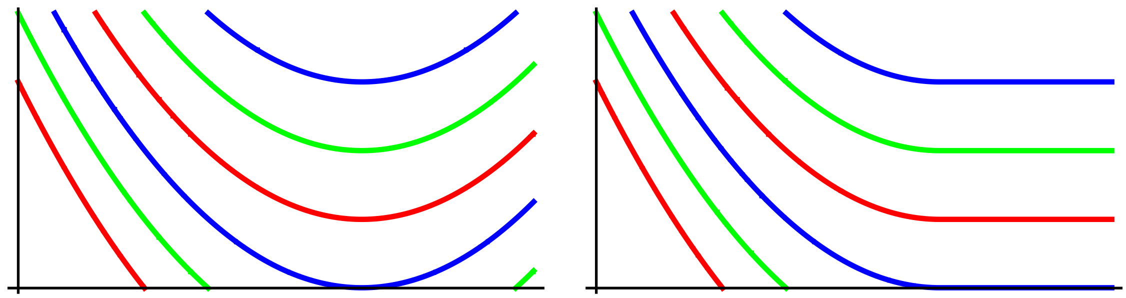

Figure 10. Indifference curves for locally nonsatiated utility $u(x,y) = y - (1-x)^2$ and monotone utility with same demand.

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[3] M. K. Richter. 1971. Rational choice. In J. S. Chipman, L. Hurwicz, M. K. Richter, and H. F. Sonnenschein, eds., *Preferences, Utility, and Demand: A Minnesota Symposium*, chapter 2, pages 29–58. New York: Harcourt, Brace, Jovanovich.