Caltech Division of the Humanities and Social Sciences

Robinson Crusoe Walrasian Examples

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Two goods: x and y. Good y is produced from good x using the production function f. Normalize so that

$$p_x = 1 \qquad \text{and} \qquad \bar{x} = 1,\ \bar{y} = 0.$$

Drop the subscript on p_y and just call it p, the price of output relative to the price of input. For these examples,

$$f(x) = \gamma x^{\frac{1}{2}}.$$

Let RC have the Cobb–Douglas utility

$$u(x, y) = x^{1-\alpha} y^{\alpha},$$

where $0 < \alpha < 1$.

Profit maximization

The profit maximization problem is to maximize

$$pf(x) - x = p\gamma x^{\frac{1}{2}} - x.$$

This is a strictly concave function of x, so the first order condition determines the maximum. It is

$$\frac{1}{2} p\gamma x^{-\frac{1}{2}} - 1 = 0,$$

which implies

$$\hat{x}(p) = \frac{\gamma^2 p^2}{4} \tag{1}$$

is the factor demand function, so the supply function is

$$\hat{y}(p) = \gamma \hat{x}(p)^{\frac{1}{2}} = \frac{\gamma^2}{2} p, \tag{2}$$

and the optimal profit function is

$$\hat{\pi}(p) = \frac{\gamma^2 p^2}{2} - \frac{\gamma^2 p^2}{4} = \frac{\gamma^2 p^2}{4}. \tag{3}$$

Utility maximization

Recall that for a Cobb–Douglas utility, the expenditure on a good is proportional to the exponent. Thus demand as a function income m and price p is

$$y^*(p,m) = \frac{\alpha m}{p}, \qquad x^*(p,m) = (1-\alpha)m.$$

Market clearing

In equilibrium RC gets all the profits plus the value of the endowment, so

$$m(p) = p_x \bar{x} + \hat{\pi}(p) = 1 + \frac{\gamma^2 p^2}{4},$$

so the demands are given by

$$x^*(p) = (1-\alpha)m = (1-\alpha)\left(1 + \frac{\gamma^2 p^2}{4}\right), \tag{4}$$

and

$$y^*(p) = \frac{\alpha m}{p} = \frac{\alpha\left(1 + \frac{\gamma^2 p^2}{4}\right)}{p}. \tag{5}$$

Clearing the market for y

Equating supply (2) and demand (5) for y gives

$$\frac{\gamma^2}{2} p = \frac{\alpha\left(1 + \frac{\gamma^2 p^2}{4}\right)}{p}. \tag{6}$$

Rewriting (6) yields

$$\begin{aligned}
\frac{\gamma^2}{2} p^2 &= \alpha\left(1 + \frac{\gamma^2 p^2}{4}\right) \\
2\gamma^2 p^2 &= 4\alpha + \alpha\gamma^2 p^2 \\
(2-\alpha)\gamma^2 p^2 &= 4\alpha \\
p^2 &= \frac{4\alpha}{(2-\alpha)\gamma^2},
\end{aligned}$$

so

$$\boxed{p^* = \frac{2}{\gamma}\sqrt{\frac{\alpha}{2-\alpha}}.} \tag{7}$$

Clearing the market for x

Equating supply (one unit) and demand [(1)+(4)] for x gives

$$1 = \hat{x}(p) + x^*(p) = \frac{\gamma^2 p^2}{4} + (1-\alpha)\left(1 + \frac{\gamma^2 p^2}{4}\right). \tag{8}$$

Rewriting (8) yields

$$\begin{aligned} 4 &= 4(1-\alpha) + (2-\alpha)\gamma^2 p^2 \\ 4\alpha &= (2-\alpha)\gamma^2 p^2 \\ p^2 &= \frac{4\alpha}{(2-\alpha)\gamma^2} \\ p &= \frac{2\sqrt{\alpha}}{\gamma\sqrt{2-\alpha}}. \end{aligned}$$

and again we get

$$\boxed{p^* = \frac{2}{\gamma}\sqrt{\frac{\alpha}{2-\alpha}}.} \tag{7$'$}$$

The complete equilibrium

Substituting the value for p^* given by (7) or (7′) into (1)–(5) gives

$$\boxed{\begin{array}{cc} p^* = \dfrac{2}{\gamma}\sqrt{\dfrac{\alpha}{2-\alpha}}, & y^* = \hat{y} = \gamma\sqrt{\dfrac{\alpha}{2-\alpha}}, \\ x^* = \dfrac{2-2\alpha}{2-\alpha}, & \hat{x} = \dfrac{\alpha}{2-\alpha}. \end{array}} \tag{9}$$

Examples

For example, setting $\gamma = 1$ and $\alpha = \frac{2}{5}$ gives the equilibrium

$$p^* = 1, \quad y^* = \hat{y} = \frac{1}{2}, \quad x^* = \frac{3}{4}, \quad \hat{x} = \frac{1}{4}.$$

For another example, set $\alpha = \frac{1}{2}$ and $\gamma = 2$. Then

$$p^* = \frac{1}{\sqrt{3}}, \quad y^* = \hat{y} = \frac{2}{\sqrt{3}}, \quad x^* = \frac{2}{3}, \quad \hat{x} = \frac{1}{3}.$$

Nice values

The key to getting nice (e.g., rational) values for the equilibrium is to choose α so that $\sqrt{\alpha/(2-\alpha)}$ is nice. Now if $\sqrt{\alpha/(2-\alpha)} = x$, then $\alpha = 2x^2/(1+x^2)$. So if you want $\sqrt{\alpha/(2-\alpha)} = a/b$, then choose $\alpha = 2a^2/(a^2+b^2)$. Note that if α is between 0 and 1, then so is $\sqrt{\alpha/(2-\alpha)}$.

Optimality

Suppose Robinson simply maximized his utility subject to the resource constraint and technology. Let x denote his consumption of good x, so that $1-x$ is used to produce y. The amount of y produced is the $\gamma(1-x)^{1/2}$. Thus he will choose x to

$$\text{maximize } \; x^{1-\alpha}\left(\gamma(1-x)^{1/2}\right)^{\alpha}.$$

The first order condition for a maximum is

$$(1-\alpha)x^{-\alpha}\left(\gamma(1-x)^{1/2}\right)^{\alpha} + x^{1-\alpha}\alpha\left(\gamma(1-x)^{1/2}\right)^{\alpha-1}\gamma\tfrac{1}{2}(1-x)^{-1/2}(-1) = 0.$$

Multiply by $2x^{\alpha}\left(\gamma(1-x)^{1/2}\right)^{-\alpha}$ and rearrange to get

$$\begin{aligned}
2(1-\alpha) &= x\alpha\left(\gamma(1-x)^{1/2}\right)^{-1}\gamma(1-x)^{-1/2} \\
2(1-\alpha) &= \alpha x(1-x)^{-1} \\
2(1-\alpha)(1-x) &= \alpha x \\
2(1-\alpha) &= (2-\alpha)x
\end{aligned}$$

which implies

$$x = \frac{2-2\alpha}{2-\alpha}.$$

That is, RC will choose the same consumption of x (and thus also of y) as the market equilibrium described in (9). In other words, the market equilibrium is efficient.