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Introduction to Correspondences

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In these particular notes, all spaces are metric spaces. The set of extended real numbers, $\boldsymbol{R} \cup \{\infty, -\infty\}$, is denoted $\boldsymbol{R}^\sharp$.

1 Correspondences

There are many instances when a set of points depends on a parameter. For instance:

- In economics, the budget set $\beta(p, w) = \{x \in \boldsymbol{R}^n_+ : p \cdot x \leqslant w\}$ is the set of "shopping lists" x that can be purchased with wealth w at the price list p. This set depends on the parameter vector $(p, w) \in \boldsymbol{R}^n_+ \times \boldsymbol{R}_+$.
- In a metric space (X, d), the open ball $B_\varepsilon(x) = \{y \in X : d(y, x) < \varepsilon\}$ is a set that depends on the parameter list $(x, \varepsilon) \in X \times \boldsymbol{R}_{++}$.
- For a family of constrained optimization problems

$$\text{maximize } f(x) \text{ subject to } g(x) = \alpha,$$

the constraint set $\{x \in X : g(x) = \alpha\}$ depends on the parameter α.

In order to capture this dependence we would like to have a notion of a set-valued function. The seemingly obvious idea of a function $f\colon X \to 2^Y$ from the set X into the power set (the set of subsets) of Y may not be the best choice. The problem comes when we try to imagine its graph, which is a subset of $X \times 2^Y$. A simpler object is what we call a correspondence, which is just another name for a binary relation from X to Y.

Definition 1 *A* ***correspondence*** *φ from X to Y associates to each point in X a subset $\varphi(x)$ of Y. We write this as $\varphi\colon X \twoheadrightarrow Y$. For a correspondence $\varphi\colon X \twoheadrightarrow Y$, let $\operatorname{gr}\varphi$ denote the* ***graph*** *of φ, which we define to be*

$$\operatorname{gr}\varphi = \{(x, y) \in X \times Y : y \in \varphi(x)\}.$$

Let $\varphi\colon X \twoheadrightarrow Y$, and let $F \subset X$. The ***image*** *$\varphi(F)$ of F under φ is defined to be*

$$\varphi(F) = \bigcup_{x \in F} \varphi(x).$$

The value $\varphi(x)$ is allowed to be the empty set, but we call $\{x \in X : \varphi(x) \neq \varnothing\}$, the ***domain*** *of φ, denoted $\operatorname{dom}\varphi$.*

The terms ***multifunction***, ***point-to-set mapping***, *and* ***set-valued function*** *are also used for a correspondence.*

Depending on your spatial relation skills, it may be possible to visualize the graph of a correspondence as we have defined it, especially if X and Y are subsets of the line or plane.

Remark 2 Technically a singleton-valued correspondence $\varphi\colon X \to Y$ is not a function from X to Y, since the value of φ is a subset of Y with only element, e.g., $\{y\}$, whereas the value of a function is point y in Y. But the graph of a singleton-valued correspondence φ is also the graph of a function f, where $\varphi(x) = \{f(x)\}$, and we may find it convenient to identify φ and f (that is, treat them as though they are the same object). We may also use an expression like "the correspondence φ is a function" to mean that it is singleton-valued.

2 Inverse images

For correspondences there are two natural notions of inverse.

Definition 3 *The* ***upper*** *(or* ***strong****)* ***inverse*** *of E under φ, denoted $\varphi^u[E]$, is defined by*

$$\varphi^u[E] = \{x \in X : \varphi(x) \subset E\}.$$

The ***lower*** *(or* ***weak****)* ***inverse*** *of E under φ, denoted $\varphi^\ell[E]$, is defined by*

$$\varphi^\ell[E] = \{x \in X : \varphi(x) \cap E \neq \varnothing\}.$$

For a single y in Y, define

$$\varphi^{-1}(y) = \{x \in X : y \in \varphi(x)\}.$$

Note that $\varphi^{-1}(y) = \varphi^\ell[\{y\}]$.

Berge [8] uses the notation φ^+ and φ^-, and Hildenbrand [11] uses φ^{-*} and φ^{-1} for the upper and lower inverses respectively.

3 Notions of continuity for functions

Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f\colon X \to Y$ be a *function*. When it is clear to which space a pair of points belongs, I may drop the subscripts on d_X and d_Y and simply write d for the metric. I will also use the notion $B_\varepsilon(x)$ to denote the open ball of radius ε centered at x when the metric is obvious.

Recall that the Euclidean (or ℓ_2) metric on $\boldsymbol{R}^{\mathrm{n}}$ is given by $d_2(x, y) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$. Other useful metrics on $\boldsymbol{R}^{\mathrm{n}}$ include the max (or ℓ_∞) metric $d_\infty(x, y) = \max_i |x_i - y_i|$ and the taxicab (or ℓ_1) metric $d_1(x, y) = \sum_{i=1}^n |x_i - y_i|$. All of these metrics give rise to the same topology, that is, the same family of open sets.

For a review of the important concepts (such as open sets, neighborhoods, closed sets, compact sets, and (semi)continuous functions) relating to metric spaces see my on-line notes or Hildenbrand [11, Chapter 1]. The three standard, equivalent, definitions of continuity are these.

Definition 4 (Continuity using neighborhoods) *A function $f\colon X \to Y$ is **continuous at x** if for every neighborhood G of $f(x)$, its inverse image $f^{-1}[G]$ is a neighborhood of x. That is, for every open set G with $f(x) \in G$, there exists an open set U with $x \in U$ such that*

$$z \in U \implies f(z) \in G.$$

Note that this definition is *topological*, that is, it uses only the notion of neighborhoods and does not mention the metrics. The next definition makes explicit use of the metrics.

Definition 5 (Continuity using metrics) *A function $f\colon X \to Y$ is **continuous at x** if for every $\varepsilon > 0$ there exists some $\delta > 0$ such that*

$$d_X(x,z) < \delta \implies d_Y\big(f(x), f(z)\big) < \varepsilon.$$

Definition 6 (Continuity using sequences) *A function $f\colon X \to Y$ is **continuous at x** if whenever $x_n \to x$, then $f(x_n) \to f(x)$.*

The following proposition is a simple consequence of the definitions open sets and convergence in a metric space, but let's walk through it to see where problems arise when trying to define continuity for correspondences.

Proposition 7 *A function f is continuous at x in the topological sense of Definition 4 if and only if it is continuous at x in the metric sense of Definition 5 if and only if it is continuous at x in the sequential sense of Definition 6.*

Proof: ($\implies$) Assume f is continuous at x in the topological sense (Definition 4), and let $\varepsilon > 0$ be given. Then $G = B_\varepsilon(f(x)) = \{y : d(y, f(x)) < \varepsilon\}$ is an open set with $f(x) \in G$, so by topologcial continuity there is some open set U containing x such that $z \in U \implies f(z) \in G$. By openness of U, there is a $\delta > 0$ such that $d_X(z,x) < \delta \implies z \in U$. Thus

$$d_X(z,x) < \delta \implies d_Y\big(f(z), f(x)\big) < \varepsilon,$$

that is, f is continuous at x in the metric sense (Definition 5).

($\impliedby$) Assume f is continuous at x in the metric sense (Definition 5), and let G be an open set containing $f(x)$. Since G is open, there is some $\varepsilon > 0$ such that $d_Y\big(y, f(x)\big) < \varepsilon \implies y \in G$. Then by metric continuity, there is a $\delta > 0$ such that $d_X(x,z) < \delta \implies d_Y\big(f(z), f(x)\big) < \varepsilon$. So letting U denote the open set $B_\delta(x)$, we have

$$z \in U \implies f(z) \in G,$$

that is, f is continuous at x in the topological sense (Definition 4). ∎

4 Notions of continuity for correspondences

To apply Definition 4 or Definition 5 to correspondences we have to confront the problem that there are two natural notions of inverse image, so there are two natural versions of each definition. But there is another problem, namely, the topological and metric versions are not equivalent. But first we have to figure out the appropriate statement of the metric ε-δ definition. Define the **distance function** $d(x, A)$, where A is a set and x is a point, by

$$d(x, A) = \inf_{z \in A} d(x, z).$$

Then d is continuous. In fact it satisfies the Lipschitz condition

$$\bigl|d(x, A) - d(z, A)\bigr| \leqslant d(x, z).$$

Definition 8 *Let A be a set in a metric space (X, d). The* ***ε-neighborhood*** *$N_\varepsilon(A)$ of A is defined by*[1]

$$N_\varepsilon(A) = \bigcup_{x \in A} B_\varepsilon(x) = \{x \in X : d(x, A) < \varepsilon\}.$$

Note that as the union of the open balls $B_\varepsilon(x)$, the set $N_\varepsilon(A)$ is open. If X is also a vector space with translation-invariant metric[2] d, then we also have $N_\varepsilon(A) = A + B_\varepsilon(0)$.

Now let's start with the upper inverse. The ε-δ definition of upper hemicontinuity is this:

Definition 9 (Metric upper hemicontinuity) *A correspondence $\varphi \colon X \twoheadrightarrow Y$ satisfies* ***Property U_M at x*** *if the upper inverse of an ε-neighborhood of $\varphi(x)$ is a neighborhood of x. In other words, φ satisfies Property U_M at x if for every $\varepsilon > 0$, there is $\delta > 0$ such that*

$$d(z, x) < \delta \implies \varphi(z) \subset N_\varepsilon\bigl(\varphi(x)\bigr).$$

We say that φ satisfies Property U_M if it satisfies Property U_M at each $x \in X$.

The topological version is this:

Definition 10 (Upper hemicontinuity) *A correspondence $\varphi \colon X \twoheadrightarrow Y$ satisfies* ***Property U_T at x*** *if the upper inverse of an open neighborhood of $\varphi(x)$ is a neighborhood of x. In other words, φ satisfies Property U_T at x if for every open set G with $\varphi(x) \subset G$, there is an open set U with $x \in U$ such that*

$$z \in U \implies \varphi(z) \subset G.$$

We say that φ satisfies Property U_T if it satisfies Property U_T at each $x \in X$.

What is the relationship between these properties?

Proposition 11 *If $\varphi \colon X \twoheadrightarrow Y$ satisfies Property U_T at x, then it satisfies Property U_M at x.*

If $\varphi \colon X \twoheadrightarrow Y$ satisfies Property U_M at x and if $\varphi(x)$ is compact, then it satisfies Property U_T at x.

The first half of the proposition is straightforward and similar to Proposition 7, the second half relies on the following lemma.

[1] Borwein and Zhu [10, p. 3] call this set, with strict inequality replaced by weak inequality, the **ε-enlargement** of A. It is possible to put useful topologies on 2^X, in which case the term neighborhood might be confusing. But I don't intend to topologize 2^X here.

[2] A metric d on a vector space is **translation-invariant** if for all x, y, z, we have $d(x, y) = d(x + z, y + z)$.

Lemma 12 *Let G be an open subset of a metric space (X, d) and let K be a nonempty compact subset of G. Then there is an $\varepsilon > 0$ such that*

$$K \subset N_\varepsilon(K) \subset G.$$

Proof: If $K \subset G$, where G is open, then G^c is closed and disjoint from K. The distance function $x \mapsto d(x, G^c)$ is continuous, so it achieves a minimum on the compact set K at some point $\bar{x} \in K$. But we must have $d(\bar{x}, G^c) > 0$, because if $d(\bar{x}, G^c) = 0$, then since G^c is closed, we would have $\bar{x} \in G^c \cap K$ contradicting the fact that K and G^c are disjoint. Any ε with $0 < \varepsilon < d(\bar{x}, G^c)$ satisfies the conclusion. ∎

Thus Property U_T is stronger than Property U_M for correspondences that do not have compact values, and the next example shows that it may be so strong as to rule out many natural correspondences.

Example 13 (Cf. Aubin and Ekeland [3, p. 108] for a related example.) The correspondence $\varphi\colon \boldsymbol{R}^{\mathrm{m}}_+ \twoheadrightarrow \boldsymbol{R}^{\mathrm{m}}$ defined by

$$\varphi(x) = \{y \in \boldsymbol{R}^{\mathrm{m}} : y \leqq x\}$$

satisfies Property U_M, but does not satisfy Property U_T.

To see that this violates Property U_T, let $n = 2$ and let

$$G = \{(x, y) \in \boldsymbol{R}^2 : y < 1/|x| \text{ if } x < 0\}.$$

Note that this set includes the entire right half-plane as well as $\varphi((0, 0))$. But for any $\delta > 0$, the set $\varphi\big((\delta, \delta)\big)$ contains $(x, \delta/2)$, which does not belong to G when $x < 0$ and $|x| > 2/\delta$. Thus φ violates Property U_T.

I'll leave it to you to show that φ does satisfy Property U_M. Hint: If $\|x' - x\| < \varepsilon$ and $y \leqq x$, then $y' = y + x' - x$ satisfies $\|y' - y\| < \varepsilon$ and $y' = x' - (x - y) \leqq x'$.

This example shows that it might be impossible for a correspondence satisfy Property U_T if is not compact-valued. Observe that a constant correspondence (that is, $\varphi(x) = A$ for all x) does satisfy Property U_T, even if A is not compact. □

So either Property U_T or U_M can be used as the basis for a definition of continuity. In the past I've preferred U_T (e.g., [1, 9]), but now I may wish to join other authors, e.g., Aubin [2] or Aubin and Ekeland [3, p. 108], who prefer U_M. Nevertheless, until I get around around to rewriting everything, I will stick with the following definition.

Definition 14 *A correspondence $\varphi\colon X \twoheadrightarrow Y$ is* ***upper hemicontinuous (uhc) at x*** *if it satisfies Property U_T at x; and φ is* ***lower hemicontinuous (lhc) at x*** *if whenever x is in the lower inverse of an open set so is a neighborhood of x.*

In other words, φ is uhc at x if for any open set $G \subset Y$, if

$$\varphi(x) \subset G,$$

then there is an open set $U \subset X$ containing x such that

$$z \in U \implies \varphi(z) \subset G.$$

And φ is lhc at x if for any open set $G \subset Y$, if

$$\varphi(x) \cap G \neq \varnothing,$$

then there is an open set $U \subset X$ containing x such that

$$z \in U \implies \varphi(z) \cap G \neq \varnothing.$$

The correspondence $\varphi\colon X \twoheadrightarrow Y$ is ***upper hemicontinuous*** *if it is upper hemicontinuous at every $x \in X$. The correspondence $\varphi\colon X \twoheadrightarrow Y$ is* ***lower hemicontinuous*** *if it is lower hemicontinuous at every $x \in X$. Thus φ is upper hemicontinuous if the upper inverses of open sets are open and φ is lower hemicontinuous if the lower inverses of open sets are open.*

A correspondence is ***continuous*** *if it is both upper and lower hemicontinuous.*

The term **semicontinuity** is used by many authors to mean hemicontinuity (indeed Aubin [2] or Aubin and Ekeland [3], Phelps [16], and Borwein and Zhu [10] use it). The term *hemicontinuity* has been prevalent in mathematical economics since the appearance of Hildenbrand [11].

Warning! The definition of upper hemicontinuity is not fully agreed upon. For example, Berge [7, 8] requires in addition that φ have non-empty compact values in order to be called upper hemicontinuous. Hildenbrand [11, Definition 1, p. 21] requires nonempty values. See Moore [15] for a catalog of related definitions. It is true that the most interesting results apply to compact-valued correspondences (e.g., compare Examples 13 and 26 below), but it seems useful to me and others, including Beer [6, Definition 6.2.4, p. 193], Borwein and Zhu [10, Definition 5.1.15, p. 173], Hildenbrand [11, Definition 1, p. 21], and Phelps [16, Definition 7.2, p. 102], to make upper hemicontinuity and compact values separate properties. These authors use the term **usco** as an adjective to describe a correspondence that is upper hemicontinuous and compact-valued.

If $\varphi\colon X \twoheadrightarrow Y$ is singleton-valued, the upper and lower inverses of a set coincide and agree with the inverse regarded as a function. Either form of hemicontinuity is equivalent to continuity as a function. Thus a semicontinuous real-valued function is not a hemicontinuous correspondence unless it is also a continuous function. It is also possible to put topologies on the space of compact subsets of Y that capture hemicontinuity by regarding φ as a function into 2^Y. See Klein and Thompson [14, Theorems 7.1.4 and 7.1.7, pp. 73–75].

5 Sequential notions of continuity

A function f is continuous at x if

$$x_n \to x \text{ implies } f(x_n) \to f(x). \qquad (\mathbf{S})$$

How do we formulate condition (**S**) for correspondences? Let $\varphi\colon X \twoheadrightarrow Y$ be a correspondence, let $x_n \to x$ and let $y_n \in \varphi(x_n)$ for each n. It is asking too much for the sequence y_n to be convergent,[3] but we can ask that it have a convergent subsequence, in which case we would want that the limit of the subsequence to belong to $\varphi(x)$. That is one way to reformulate (**S**), but there is another. We might also ask that every point in $\varphi(x)$ be the limit of such a sequence y_n.

Check this terminology against Hiriart-Urruty and Lemaréchal [12].

[3]Consider the case where $X = Y = [0, 1]$ and $\varphi(x) = [0, 1]$ for each x. This ought to be continuous by any reasonable definition, if we let $y_n = 0$ for n even and $y_n = 1$ for n odd, then the sequence y_n does not converge.

Definition 15 *Let X and Y be metric spaces, and let $\varphi\colon X \twoheadrightarrow Y$ be a correspondence. We say that φ is **outer hemicontinuous** if whenever $x_n \to x$ and $y_n \in \varphi(x_n)$ for each n, there is a convergent subsequence $y_{n_k} \to y$ with $y \in \varphi(x)$. We say that φ is **inner hemicontinuous** if whenever $x_n \to x$ and $y \in \varphi(x)$, there is a sequence $y_n \in \varphi(x_n)$ for each n such that $y_n \to y$.*

There is also a useful intermediate condition.

Definition 16 *The correspondence $\varphi\colon E \twoheadrightarrow F$ is **closed at x** if whenever $x_n \to x$, $y_n \in \varphi(x_n)$, and $y_n \to y$, then $y \in \varphi(x)$. A correspondence is **closed** if it is closed at every point of its domain, that is, if its graph is closed.*

Fact 17 *If φ is closed at x, then $\varphi(x)$ is a closed set.*

Proof: Let y_n be a sequence in $\varphi(x)$ with $y_n \to y$. Define $x_n = x$ for all n, so $x_n \to x$ and $y_n \in \varphi(x_n)$. Since φ is closed at x, we have $y \in \varphi(x)$. This shows that $\varphi(x)$ is a closed set. ∎

Example 18 (Closedness vs. hemicontinuity) In general, a correspondence may be closed without being upper or lower or inner or outer hemicontinuous.

Define $\varphi\colon \boldsymbol{R} \twoheadrightarrow \boldsymbol{R}$ via

$$\varphi(x) = \begin{cases} \{\frac{1}{x}\} & x \neq 0 \\ \{0\} & x = 0. \end{cases}$$

Then φ is closed but neither lower nor upper nor inner nor outer hemicontinuous. (This correspondence is also a function.)

Define $\mu\colon \boldsymbol{R} \twoheadrightarrow \boldsymbol{R}$ via $\mu(x) = (0, 1)$. Then μ is upper hemicontinuous in my sense, and also lower hemicontinuous, and inner and outer hemicontinuous, but not closed.[4] □

However, a closed-valued upper hemicontinuous correspondence is a closed correspondence, and if the range space Y is compact, then a closed correspondence is upper hemicontinuous.

Proposition 19 *Let $\varphi\colon X \twoheadrightarrow Y$ be a upper hemicontinuous at x and assume that $\varphi(x)$ is a closed set. Then φ is closed at x.*

Moreover, if Y is compact, and φ is closed at x, then φ is upper hemicontinuous at x.

Draw pictures.

Proof: Let φ be upper hemicontinuous at x and assume that $\varphi(x)$ is closed. Let $x_n \to x$, $y_n \in \varphi(x_n)$, and $y_n \to y$. Suppose by way of contradiction that $y \notin \varphi(x)$. Since $\varphi(x)$ is closed, there are disjoint open sets G and U in Y with $\varphi(x) \subset G$ and $y \in U$. Since φ is upper hemicontinuous at x, the upper inverse $\varphi^{\mathrm{u}}[G]$ is an open neighborhood of x. Since $x_n \to x$, there is some N such that $n \geqslant N$ implies $x_n \in \varphi^{\mathrm{u}}[G]$ or $\varphi(x_n) \subset G$. In particular, $y_n \in \varphi(x_n) \subset G$, so $y_n \notin U$. Thus $y_n \not\to y$, a contradiction. This establishes that φ is closed at x.

Now assume that Y is compact and φ is closed at x. Assume by way of contradiction that there is some open set G with $\varphi(x) \subset G$ but $\varphi^{\mathrm{u}}[G]$ is not a neighborhood of x. This means that for every n there is some x_n in $B_{1/n}(x)$, the ball of radius $1/n$ centered at x, such that $\varphi(x_n) \not\subset G$. Thus there exists $y_n \in \varphi(x_n)$ with $y_n \notin G$. Since Y is compact, the sequence y_n has a subsequence converging to some $y \in Y$. Along this subsequence we have $x_n \to x$, $y_n \in \varphi(x_n)$,

[4] Again, under Berge's definition, an upper hemicontinuous correspondence is automatically closed.

$y_n \to y$ so since φ is closed at x we must have $y \in \varphi(x) \subset G$. But no $y_n \in G$, which contradicts $y_n \to y$. This contradiction establishes that for every open set G with $\varphi(x) \subset G$ it must be the case that $\varphi^{\mathrm{u}}[G]$ is a neighborhood of x. That is, φ is upper hemicontinuous at x. ∎

Remark 20 We can weaken the assumption that the range space Y is compact. What we need is for each x there is a neighborhood whose image is included in some compact set. So let us say that φ is **locally bounded at *x*** if there is a neighborhood U of x in X and a compact set $K \subset Y$ such that $\varphi(U) \subset K$. Then the argument above proves the following result.

Corollary 21 *Let $\varphi\colon X \twoheadrightarrow Y$ be locally bounded and closed at x. Then φ is upper hemicontinuous at x.*

Corollary 22 *Let $\varphi\colon X \twoheadrightarrow Y$.*

1. *If φ is upper hemicontinuous and closed-valued, then φ is closed.*

2. *If φ is closed and locally bounded everywhere, then φ is upper hemicontinuous.*

Proposition 23 (Upper/lower and outer/inner hemicontinuity) *Let $\varphi\colon X \twoheadrightarrow Y$.*

1. *If φ has nonempty compact values, then φ is upper hemicontinuous at x if and only if it is outer hemicontinuous at x.*

2. *Then φ is lower hemicontinuous at x if and only if it is inner hemicontinuous at x.*

Proof: (1.) ($\Longrightarrow$) Suppose φ is upper hemicontinuous at x, $x_n \to x$ and $y_n \in \varphi(x_n)$. Since φ is compact-valued, $\varphi(x)$ has a bounded neighborhood U. Since φ is upper hemicontinuous, there is a neighborhood V of x such that $\varphi(V) \subset U$. Thus y_n is eventually in U, thus bounded, and so has a convergent subsequence. Since compact sets are closed, this limit belongs to $\varphi(x)$.

($\Longleftarrow$) Now suppose that for every sequence $x_n \to x$, $y_n \in \varphi(x_n)$, there is a subsequence of y_n with limit in $\varphi(x)$. Suppose φ is not upper hemicontinuous; then there is a neighborhood U of x and a sequence $z_n \to x$ with $y_n \in \varphi(z_n)$ and $y_n \notin U$. Such a sequence y_n can have no subsequence with limit in $\varphi(x)$, a contradiction.

(2.) ($\Longrightarrow$) Assume φ is lower hemicontinuous at x and that $x_n \to x$ and $y \in \varphi(x)$. Then for each natural number k, the set $G_k = \{z \in Y : d(z,y) < 1/k\}$ is an open set with $y \in G_k \cap \varphi(x) \neq \varnothing$. Since φ is lower hemicontinuous at x, the lower inverse $\varphi^{\ell}[G_k]$ is a neighborhood of x. Since $x_n \to x$, there is some $n(k)$ such that for all $n \geqslant n(k)$, we have $x_n \in \varphi^{\ell}[G_k]$, that is, $\varphi(x_n) \cap G_k \neq \varnothing$. Without loss of generality we may take $n(k+1) > n(k)$ for all k. For $n < n(1)$ pick any $y_n \in \varphi(x_n)$. For $n(k) \leqslant n < n(k+1)$ pick $y_n \in \varphi(x_n) \cap G_k$. Then $n \geqslant n(k)$ implies $d(y_n, y) < 1/k$, so $y_n \to y$ as desired.

($\Longleftarrow$) Assume that whenever $x_n \to x$ and $y \in \varphi(x)$, there is a sequence $y_n \in \varphi(x_n)$ with $y_n \to y$. Let G be an open subset of Y with $\varphi(\bar{x}) \cap G \neq \varnothing$, and let $\bar{y} \in \varphi(\bar{x}) \cap G$. Let $U = \varphi^{\ell}[G]$. We need to show that U is a neighborhood of $\bar{x}$. Suppose by way of contradiction that U is not a neighborhood of $\bar{x}$. Then for each natural number n, the ball $B_{1/n}(\bar{x})$ is not a subset of U, so pick $x_n \in B_{1/n}(\bar{x}) \cap U^c$. Then $x_n \to \bar{x}$, but $\varphi(x_n) \cap G = \varnothing$ for each n. On the other hand, by hypothesis there are $y_n \in \varphi(x_n)$ with $y_n \to \bar{y}$. Thus there is some n_0 such that $n \geqslant n_0$ implies $y_n \in G$. But then for $n \geqslant n_0$, we have $y_n \in \varphi_n \cap G = \varnothing$, a contradiction. Therefore $U = \varphi^{\ell}[G]$ is a neighborhood of $\bar{x}$. ∎

Proposition 24 *Let $\varphi, \mu \colon X \twoheadrightarrow Y$, and define $(\varphi \cap \mu) \colon X \twoheadrightarrow Y$ pointwise by $(\varphi \cap \mu)(x) = \varphi(x) \cap \mu(x)$. Suppose $\varphi(x) \cap \mu(x) \neq \varnothing$.*

1. *If φ and μ are upper hemicontinuous at x and closed-valued, then the correspondence $(\varphi \cap \mu)$ is upper hemicontinuous at x.*

2. *If μ is closed at x and φ is upper hemicontinuous at x and $\varphi(x)$ is compact, then $(\varphi \cap \mu)$ is upper hemicontinuous at x.*

Proof: Let U be an open neighborhood of $\varphi(x) \cap \mu(x)$. Put $C = \varphi(x) \cap U^c$.

(1) Note that C is closed and $\mu(x) \cap C = \varnothing$. Thus there are disjoint open sets V_1 and V_2 with $\mu(x) \subset V_1$ and $C \subset V_2$. Since μ is upper hemicontinuous at x, there is a neighborhood W_1 of x with $\mu(W_1) \subset V_1 \subset V_2^c$. Now $\varphi(x) \subset U \cup V_2$, which is open and so x has a neighborhood W_2 with $\varphi(W_2) \subset U \cup V_2$, as φ is upper hemicontinuous at x. Put $W = W_1 \cap W_2$. Then for $z \in W$, $\varphi(z) \cap \mu(z) \subset V_2^c \cap (U \cup V_2) \subset U$. Thus $(\varphi \cap \mu)$ is upper hemicontinuous at x.

(2) Note that in this case C is compact and $\mu(x) \cap C = \varnothing$. Since μ is closed at x, if $y \notin \mu(x)$ then we cannot have $y_n \to y$, where $y_n \in \mu(x_n)$ and $x_n \to x$. Thus there is a neighborhood U_y of y and W_y of x with $\mu(W_y) \subset U_y^c$. Since C is compact, we can write $C \subset V_2 = U_{y^1} \cup \cdots \cup U_{y_n}$; so setting $W_1 = W_{y^1} \cap \cdots \cap W_{y_n}$, we have $\mu(W_1) \subset V_2^c$. The rest of the proof is as in part (1). ∎

6 Exercises and examples

Example 25 Every constant correspondence, $\varphi(x) = A$ for all x, is both upper and lower hemicontinuous. □

Example 26 Let $\boldsymbol{R}^{\mathrm{m}}_+ = \{x \in \boldsymbol{R}^{\mathrm{m}} : x \geqq 0\}$. The compact-valued correspondence $\varphi \colon \boldsymbol{R}^{\mathrm{m}}_+ \twoheadrightarrow \boldsymbol{R}^{\mathrm{m}}_+$ defined by

$$\varphi(x) = \{y \in \boldsymbol{R}^{\mathrm{m}}_+ : 0 \leqq y \leqq x\}$$

is continuous. (Compare this with Example 13.)

For this argument it is most convenient to use the max metric on $\boldsymbol{R}^{\mathrm{m}}$. (But the Euclidean metric also works.) We start by observing the following fact.

> **Fact 27** *Let $x, x' \in \boldsymbol{R}^{\mathrm{m}}_+$ satisfy $d(x, x') < \varepsilon$ (recall that d is the max metric), $0 \leqq y \leqq x$ and set*
>
> $$y' = (y + x' - x)^+$$
>
> *(where for any vector $z \in \boldsymbol{R}^{\mathrm{m}}$, the vector z^+ is defined to be the coordinate-wise maximum of z and 0). See Figure 1. Then*
>
> $$d(y, y') < \varepsilon \quad \text{and} \quad y' \leqq x'.$$
>
> *Proof*: Consider the i^{th} coordinate y'_i. There are two cases:
>
> Case (i): $y'_i > 0$. Then $y'_i = y_i + x'_i - x_i$, so $|y'_i - y_i| = |x'_i - x_i| < \varepsilon$ and $y'_i = x'_i + (y_i - x_i) \leqslant x'_i$.

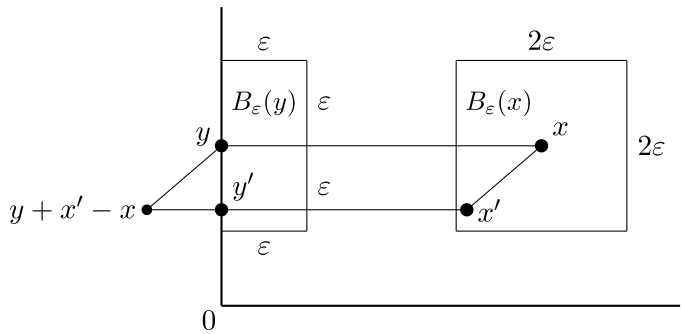

Figure 1. $0 \leqq y' \leqq x'$ and $d(y, y') < \varepsilon$.

Case (ii): $y_i' = 0$ and $y_i + x_i' - x_i \leqslant 0$. The latter inequality only occurs if $x_i' \leqslant x_i$, in which case we have $\varepsilon > x_i - x_i' \geqslant y_i = |y_i - y_i'|$. Also, $y_i' = 0 \leqslant x_i'$.

So in either case $|y_i' - y_i| < \varepsilon$ and $y_i' \leqslant x_i'$. ∎

To see that φ is upper hemicontinuous, let G be an open subset of $\boldsymbol{R}^{\mathrm{m}}_+$ and assume $x \in \varphi^{\mathrm{u}}[G]$. We need to find a $\delta > 0$ so that if $d(x, x') < \delta$, then $\varphi(x') \subset G$.

Just as in the proof of Lemma 12, the distance function $d(z, G^c)$ achieves a minimum value m on the compact set $\varphi(x)$. Since $\varphi(x)$ and G^c are disjoint closed sets the minimum value m satisfies $m > 0$. Set $\delta = m/2$. Then the neighborhood $N_\delta(\varphi(x))$ of $\varphi(x)$ is disjoint from G^c, that is, $N_\delta(\varphi(x)) \subset G$.

By the above fact (interchanging x and x'), for $x' \in B_\delta(x)$, if $z' \leqq x'$, then z' lies within δ of some point $z \leqq x$ (namely $z = (z' + x - x')^+$), so $z' \in G$. Thus $\varphi(x') \subset G$.

To prove the lower hemicontinuity of φ, let G be an open subset of $\boldsymbol{R}^{\mathrm{m}}_+$ and assume $x \in \varphi^{\ell}[G]$. That is, there is some $y \in G$ with $0 \leqq y \leqq x$. We need to find a $\delta > 0$ so that if $d(x, x') < \delta$, then $\varphi(x') \cap G \neq \varnothing$.

Since G is open relative to $\boldsymbol{R}^{\mathrm{m}}_+$, there is some $\varepsilon > 0$ such that $B_\varepsilon(y) \cap \boldsymbol{R}^{\mathrm{m}}_+ \subset G$, where

$$B_\varepsilon(y) = \{z \in \boldsymbol{R}^{\mathrm{m}}_+ : \max_i |y_i - z_i| < \varepsilon\}.$$

By the fact above there is some $y' \in G \cap \varphi(x')$. This proves the lower hemicontinuity of φ at x. □

Exercise 28 If the graph of φ is open, then φ is lower hemicontinuous. □

Exercise 29 Let $f, g \colon X \to \boldsymbol{R}$ be continuous and assume that $f(x) \geqslant g(x)$ for all x. Show that

$$x \mapsto\!\!\!\to \{\alpha \in \boldsymbol{R} : f(x) \geqslant \alpha \geqslant g(x)\}$$

is continuous. □

Example 30 The classical budget space is

$$\mathcal{B} = \{(p, w) \in \boldsymbol{R}^{\mathrm{n}} \times \boldsymbol{R} : p \gg 0,\ w > 0\}.$$

Define $\mathcal{B}_0 = \{(p, w) \in \boldsymbol{R}^{\rm n} \times \boldsymbol{R} : p \geqq 0,\ w \geqslant 0\}$. The budget correspondence

$$\beta(p, w) = \{x \in \boldsymbol{R}^{\rm n}_+ : p \cdot x \leqslant w\}$$

is compact-valued and continuous on $\mathcal{B}$. It is closed, but not lower hemicontinuous on $\mathcal{B}_0$.

Upper hemicontinuity on $\mathcal{B}$ follows from the fact that the graph of β is clearly closed, and it is easy to see that β is locally bounded on $\mathcal{B}$.

For lower hemicontinuity at a point $(\bar{p}, \bar{w}) \in \mathcal{B}$, assume G is open and let $\bar{x} \in \beta(\bar{p}, \bar{w}) \cap G$.

There are two cases: Case 1 is that $\bar{x} \neq 0$. In this case, pick some $\hat{x} \neq \bar{x}$ satisfying $\hat{x} \leqq \bar{x}$ and $\hat{x} \in G$. Then $\bar{p} \cdot \hat{x} < \bar{w}$, since $\bar{p} \gg 0$. Thus $U = \{(p, w) \in \mathcal{B} : p \cdot \hat{x} < w\}$ is an open neighborhood of $(\bar{p}, \bar{w})$ such that for every $(p, w) \in U$, we have $\hat{x} \in \beta(p, w) \bigcap G \neq \varnothing$, which shows that β is lhc at $(\bar{p}, \bar{w})$. Case 2 is that $\bar{x} = 0$. In this case, $U = \mathcal{B}$ is an open neighborhood of $(\bar{p}, \bar{w})$ satisfying $\bar{x} \in \beta(p, w) \bigcap G \neq \varnothing$, for all $(p, w) \in U$. This proves that β is lhc at $(\bar{p}, \bar{w})$.

Now in case $\bar{w} = 0$ and some $\bar{p}_j = 0$, then for $\bar{x} = e^j$ (the $j^{\rm th}$ unit coordinate vector) we have $\bar{x} \in \beta(\bar{p}, 0)$. Now consider the sequence $(p_n, w_n) \to (\bar{p}, 0)$ defined by $p_n = \bar{p} + (1/n)e^j$ and $w_n = 1/n^2$. If $x \in \beta(p_n, w_n)$, then $x_j \leqslant 1/n$, so if $x_n \in \beta(p_n, w_n)$ we cannot have $x_n \to \bar{x}$, so by Proposition 23, β is not lhc at $(\bar{p}, 0)$. □

7 The Berge maximum theorem

The main reason for writing these notes is to present the "maximum theorem" due to Berge [7], one of the most useful and powerful theorems employed in mathematical economics. It deals with the continuity of the solution and value of a parametrized family of constrained optimization problems. We start with a set P of *parameters* or *state variables*, and a set X of *controls* or *actions* or *choice variables*. To each parameter value p, there is a *constraint set* or *feasible set* $\varphi(p)$ of controls,

$$\varphi \colon P \twoheadrightarrow X.$$

The goal is to maximize an *objective function* f over the feasible set. We allow f to depend on both the parameter and the control. The domain of f need not be all of $P \times X$, but it must include the graph of φ,

$$f \colon \operatorname{gr} \varphi \to \boldsymbol{R}.$$

We define the *optimal value function* $V \colon P \to \boldsymbol{R}^\sharp$ by

$$V(p) = \sup\{f(p, x) : x \in \varphi(p)\}.$$

Note that we allow this to be extended real-valued (the supremum may be ∞). And recall that $\sup \varnothing = -\infty$.

Definition 31 *An extended real-valued function $f \colon Z \to \boldsymbol{R}^\sharp$ on a metric space is* ***upper semicontinuous*** *if for every $\alpha \in \boldsymbol{R}$, the upper contour set $\{z \in Z : f(z) \geqslant \alpha\}$ is closed (or equivalently $\{z \in Z : f(z) < \alpha\}$ is open). The function f is* ***lower semicontinuous*** *if for every $\alpha \in \boldsymbol{R}$, the lower contour set $\{z \in Z : f(z) \leqslant \alpha\}$ is closed (or equivalently $\{z \in Z : f(z) > \alpha\}$ is open).*

We can localize the concept and say that f is upper semicontinuous at z if $f(z) < \alpha$ implies that there is a neighborhood U of z such that $x \in U$ implies $f(x) < \alpha$. Lower semicontinuity at a point is defined similarly.

You should prove the following.

Fact 32 *A real-valued function is continuous if and only if it is both upper and lower semicontinuous.*

Proposition 33 (Lower semicontinuity of the optimal value function) *Assume the feasibility correspondence $\varphi\colon P \twoheadrightarrow X$ is lower hemicontinuous at p_0. Let $f\colon \operatorname{gr}\varphi \to \boldsymbol{R}$ be lower semicontinuous. Then the optimal value function V is lower semicontinuous at p_0.*

Proof: We need to prove that if $V(p_0) > \alpha$, then there is a neighborhood W of p_0 such that $p \in W$ implies $V(p) > \alpha$. So assume that $V(p_0) > \alpha$. Then there is some $x_0 \in \varphi(p_0)$ satisfying

$$f(p_0, x_0) > \alpha.$$

Since f is lower semicontinuous on $\operatorname{gr}\varphi$, there is an open neighborhood $U \times V$ of (p_0, x_0) such that for all $(p, x) \in \operatorname{gr}\varphi \cap U \times V$ we have $f(p, x) > \alpha$. Since φ is lower hemicontinuous, $W = \varphi^{\ell}[V] \cap U$ is a neighborhood of p_0. By the definition of lower inverse, for each $p \in W$ there is some $x \in \varphi(p) \cap V$, which implies $f(p, x) > \alpha$. Therefore $V(p) = \sup_{x\in\varphi(p)} f(p, x) > \alpha$ too. ∎

Proposition 34 (Upper semicontinuity of the optimal value function) *Assume that the feasibility correspondence $\varphi\colon P \twoheadrightarrow X$ has nonempty compact values. Let $f\colon \operatorname{gr}\varphi \to \boldsymbol{R}$ be upper semicontinuous. Then the optimal value function V is actually a maximum,*

$$V(p) = \max\{f(p, x) : x \in \varphi(p)\}.$$

If φ is upper hemicontinuous at p_0, then V is upper semicontinuous at p_0.

Proof: We need to show that if $V(p_0) < \alpha$, then there is a neighborhood W of p_0 such that for all $p \in W$ we have $V(p) < \alpha$. Since $V(p_0) < \alpha$, for every $x \in \varphi(p_0)$ we have $f(p_0, x) < \alpha$. Since f is upper semicontinuous, for each $x \in \varphi(p_0)$, there is a neighborhood $W_x \times U_x$ of (p_0, x) such that for every $(p, z) \in W_x \times U_x$ we have $f(p, z) < \alpha$. Since $\varphi(p)$ is compact, there are finitely many $x_1, \ldots, x_k$ such that $U = \bigcup_{j=1}^{k} U_{x_j}$ includes $\varphi(p_0)$. Put $W = \bigcap_{j=1}^{k} W_{x_j}$. Then for any $(p, x) \in W \times U$ we have $f(p, x) < \alpha$. Moreover $\varphi(p_0) \subset U$, so $\varphi^{\mathrm{u}}[U]$ is a neighborhood of p_0. Now for any $p \in W \cap \varphi^{\mathrm{u}}[U]$ and any $x \in \varphi(p)$ we have $x \in U$ so $f(p, x) < \alpha$ which implies $V(p) < \alpha$. That is, V is upper semicontinuous at p_0. ∎

Berge Maximum Theorem *Let $\varphi\colon P \twoheadrightarrow X$ be a compact-valued correspondence. Let $f\colon \operatorname{gr}\varphi \to \boldsymbol{R}$ be continuous. Define the "argmax" correspondence $\mu\colon P \twoheadrightarrow X$ by*

$$\mu(p) = \{x \in \varphi(p) : x \textit{ maximizes } f(p, \cdot) \textit{ on } \varphi(p)\},$$

and the optimal value function $V\colon P \to \boldsymbol{R}$ by

$$V(p) = f(p, x) \qquad \textit{for any } x \in \mu(p).$$

If φ is continuous at p, then μ is closed and upper hemicontinuous at p and V is continuous at p. Furthermore, μ is compact-valued.

Proof: Given the previous Propositions 33 and 34, all that remains is to show that μ is closed at p, for then $\mu = \varphi \cap \mu$ and Proposition 24(2) implies that μ is upper hemicontinuous at p. Let $p_n \to p$, $x_n \in \mu(p_n)$, $x_n \to x$. We wish to show $x \in \mu(p)$. Since φ is upper hemicontinuous and compact-valued, Proposition 23(1) implies that indeed $x \in \varphi(p)$. Suppose $x \notin \mu(p)$. Then there is $z \in \varphi(p)$ with $f(p,z) > f(p,x)$. Since φ is lower hemicontinuous at p, by Proposition 23 there is a sequence $z_n \to z$ with $z_n \in \varphi(p_n)$. Since $z_n \to z$, $x_n \to x$, and $f(p,z) > f(p,x)$, the continuity of f implies that eventually $f(p_n, z_n) > f(p_n, x_n)$, contradicting $x_n \in \mu(p_n)$. ∎

8 More about correspondences

Proposition 35 (Upper hemicontinuous image of a compact set) *Let K be a compact set and assume that $\varphi\colon K \twoheadrightarrow Y$ is upper hemicontinuous and compact-valued. Then $\varphi(K)$ is compact.*

Proof: Let $\{U_\alpha\}$ be an open covering of $\varphi(K)$. Since $\varphi(x)$ is compact, there is a finite subcover $U_x^1, \dots, U_x^{n_x}$, of $\varphi(x)$. Put $V_x = U_x^1 \cup \dots \cup U_x^{n_x}$. Then since φ is upper hemicontinuous, $\varphi^{\mathrm{u}}[V_x]$ is open and contains x. Hence K is covered by a finite number of $\varphi^{\mathrm{u}}[V_x]$s and the corresponding U_x^is are a finite cover of $\varphi(K)$. ∎

Proposition 36 (Sums of correspondences) *Let $\varphi_i\colon X \twoheadrightarrow \boldsymbol{R}^{\mathrm{m}}$, $i = 1, \dots, k$.*

(a) *If each φ_i is upper hemicontinuous at x and compact-valued, then*

$$\sum_i \varphi_i \colon z \mapsto\!\!\!\!\rightarrow \sum_i \varphi_i(z)$$

is upper hemicontinuous at x and compact-valued.

(b) *If each φ_i is lower hemicontinuous at x, then the sum $\sum_i \varphi_i$ is lower hemicontinuous at x.*

Proof: Exercise. Hint: Use Proposition 23(1). ∎

Proposition 37 (Products of correspondences) *Let $\gamma_i\colon E \twoheadrightarrow Y_i$, $i = 1, \dots, k$.*

(a) *If each γ_i is upper hemicontinuous at x and compact-valued, then*

$$\prod_i \gamma_i \colon z \mapsto\!\!\!\!\rightarrow \prod_i \gamma_i(z)$$

is upper hemicontinuous at x and compact-valued.

(b) *If each γ_i is lower hemicontinuous at x, then the product $\prod_i \gamma_i$ is lower hemicontinuous at x.*

(c) *If each γ_i is closed at x, then $\prod_i \gamma_i$ is closed at x.*

Proof: Exercise. ∎

9 More on inner/outer hemicontinuity

In Section 5, inner and outer hemicontinuity were described in terms of limits of sequences y_n where each y_n belongs to $\varphi(x_n)$. In this section, I'll describe some notions of limits for a sequence of sets and relate them to inner and outer hemicontinuity.

So let E_n be a sequence of sets in a metric space. How can we describe the set L of points that can are limits of a sequence of points x_n where each x_n belongs to E_n? By definition, if $x \in L$, then for every $\varepsilon > 0$, there must be some n such that for all $m \geqslant n$, $d(x_n, x) < \varepsilon$.

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