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# Core of a replica economy

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Consider a pure exchange economy $\mathcal{E}$ with $m$ consumers and $\ell$ goods. (Each consumption set is $\boldsymbol{R}^\ell_+$.) The endowment of consumer $i$ is $\omega^i$ and his preference relation is $\succsim_i$.

A **coalition** is a nonempty subset of consumers. An allocation $(x^1, \dots, x^m)$ is **blocked by coalition $S$** if there is a partial allocation $(\tilde{x}^i)_{i \in S}$ such that

1. $\sum_{i \in S} \tilde{x}^i = \sum_{i \in S} \omega^i$.
2. For each $i \in S$, $\tilde{x}^i \underset{i}{\succ} x^i$.

The allocation is **weakly blocked** if (2) is replaced by

2′. For each $i \in S$, $\tilde{x}^i \succsim_i x^i$, and for some $k \in S$, $\tilde{x}^k \underset{k}{\succ} x^k$.

The **core** of the economy is the set of unblocked allocations.

**Lemma 1** *If each preference relation is continuous and strictly monotonic, an allocation is blocked if and only if it is weakly blocked.*

The core is a generalization of the *contract curve* that was introduced by Francis Y. Edgeworth [12]. The term core goes back to Gillies [14] in his 1963 dissertation on cooperative games. Its use in economics goes back to Shubik [20] in 1959. Scarf [18], Debreu [10], and Debreu and Scarf [11] proved the first "limit theorem" for the core, and Aumann [7] applied the concept to "nonatomic" economies. An excellent monograph on the relation of the core to the set of Walrasian equilibria is Kirman and Hildenbrand [16].

**Theorem 2** *Assume each preference relation is locally nonsatiated. Then every Walrasian equilibrium allocation is in the core.*

*These notes are based on Ket Richter's variation on the Debreu–Scarf paper. I thank Federico Echenique for pointing out an error in an earlier version.

*Proof*: Let $(\bar{x}^1, \dots, \bar{x}^m, p)$ be a Walrasian equilibrium, and suppose by way of contradiction that the allocation $(\bar{x}^1, \dots, \bar{x}^m)$ is blocked. Then there is a coalition $S$ and $(\tilde{x}^i)_{i \in S}$ satisfying

$$\tilde{x}^i \underset{i}{\succ} \bar{x}^i$$

for each $i \in S$ and

$$\sum_{i \in S} \tilde{x}^i = \sum_{i \in S} \omega^i. \tag{1}$$

Since preferences are locally nonsatiated, in equilibrium, all income is spent so $p \cdot \bar{x}^i = p \cdot \omega^i$. Also, by utility maximization subject to the budget constraint, we have

$$\tilde{x}^i \underset{i}{\succ} \bar{x}^i \implies p \cdot \tilde{x}^i > p \cdot \bar{x}^i = p \cdot \omega^i$$

for each $i \in S$. Summing over $S$ yields

$$p \cdot \sum_{i \in S} \tilde{x}^i > p \cdot \sum_{i \in S} \bar{x}^i = p \cdot \sum_{i \in S} \omega^i,$$

which contradicts (1). ∎

## Replica economies

**Definition 3** *The $n^{\text{th}}$ replica $\mathcal{E}_n$ of $\mathcal{E}$ has $n \times m$ consumers, $n$ of each of $m$* ***types****. Consumers of type $i$ have the same endowment $\omega^i$ and the same preference relation $\succsim_i$.*

**Lemma 4 (Equal treatment property)** *Assume preferences are strictly monotonic, strictly convex, and continuous. Then in the core of a replica economy, consumers of the same type receive the same consumption.*

*That is, let $(x^{1,1}, \dots, x^{1,n}, \dots, x^{m,1}, \dots, x^{m,n})$ belong to the core of $\mathcal{E}_n$. Then for each type $i$, and each $j, k = 1, \dots, n$ we have*

$$x^{i,j} = x^{i,k}.$$

*Proof*: Let $(x^{1,1}, \dots, x^{1,n}, \dots, x^{m,1}, \dots, x^{m,n})$ belong to the core of $\mathcal{E}_n$. Since every consumer of type $i$ has the same preference relation, they can all agree on which of them has the worst consumption allocation $x^{i,j}$. (They may be indifferent, in which case any of them qualifies as having the worst allocation.) Form a coalition $S$ that has one consumer of each type, that consumer having the worst allocation for his type. Consider the partial allocation $(\tilde{x}^i)_{i \in S}$ (here we are indexing members of $S$ solely by their type) defined by

$$\tilde{x}^i = \frac{\sum_{j=1}^n x^{i,j}}{n}$$

Now by definition of an allocation

$$\sum_{i=1}^m \sum_{j=1}^n x^{i,j} = \sum_{i=1}^m \sum_{j=1}^n \omega^{i,j} = \sum_{i=1}^m n\omega^i.$$

Dividing by $n$ we get

$$\sum_{i=1}^m \tilde{x}^i = \sum_{i=1}^m \frac{\sum_{j=1}^n x^{i,j}}{n} = \sum_{i=1}^m \omega^i.$$

Now suppose by way of contradiction that for some type $i$, we have unequal treatment. Then by strict convexity of preference, $\tilde{x}^i = \frac{1}{n} \sum_{j=1}^n x^{i,j} \underset{i}{\succ} x^{i,j^*(i)}$, where $(i, j^*(i))$ is the worst off of type $i$. Then $S$ weakly blocks via $(\tilde{x}^1, \dots, \tilde{x}^m)$, a contradiction. Thus we must have equal treatment. ∎

Given equal treatment, we can treat every core allocation in a replica economy, as if it were an allocation the original economy. (This is not true of general allocations, since an allocation in $\mathcal{E}_n$ actually belongs to $\boldsymbol{R}^{mn\ell}$, not $\boldsymbol{R}^{m\ell}$.)

**Theorem 5 (Limit of the core)** *Assume preferences are strictly monotonic, continuous, and strictly convex. Suppose the allocation $(\bar{x}^1, \dots, \bar{x}^m)$ belongs to the core of $\mathcal{E}_n$ for each $n$. Then there exists a nonzero price vector $p \in \boldsymbol{R}^\ell$ such that $(\bar{x}^1, \dots, \bar{x}^m, p)$ is a Walrasian quasi-equilibrium.*

*Proof*: (This treatment is based on Debreu [10] and lectures by Ket Richter.) The proof is similar to the proof of the second welfare theorem, but involves the initial endowment. For each $i = 1, \dots, m$ define

$$P_i = \{z \in \boldsymbol{R}^\ell : \omega^i + z \underset{i}{\succ} \bar{x}^i\}.$$

That is, $P_i$ is the set of net trades from $\omega^i$ that make a consumer of type $i$ better off than his core allocation $\bar{x}^i$. Define

$$P = \text{convex hull} \bigcup_{i=1}^m P_i.$$

That is, $P$ is the set of all vectors of the form $\sum_{i=1}^m \alpha_i z^i$ where each $z^i \in P_i$, $\alpha_i \geqslant 0$, and $\sum_{i=1}^m \alpha_i = 1$.

I claim that $0 \notin P$. To see why, note that the continuity of preferences implies that each $P_i$ is open, so that their union is open, which in turn implies that the convex hull is open. So assume by way of contradiction that 0 belongs to $P$. Then there is some strictly negative vector $\hat{v} \ll 0$ that also belongs to $P$. We can thus write $\hat{v} = \sum_{i=1}^m \hat{\alpha}_i z^i$ where each $z^i \in P_i$, $\hat{\alpha}_i \geqslant 0$, and $\sum_{i=1}^m \hat{\alpha}_i = 1$. Moreover, since the mapping $(\beta_1, \ldots, \beta_m) \to \sum_{i=1}^m \beta_i z^i$ is continuous, we can find $\alpha_i$ close enough to $\hat{\alpha}_i$ such that each $\alpha_i$ is rational, $\sum_{i=1}^m \alpha_i = 1$, and

$$v = \sum_{i=1}^m \alpha_i z^i \ll 0.$$

Putting all the coefficients over a common denominator $n$ we get

$$0 \gg v = \sum_{i=1}^m \frac{k_i}{n} z^i, \tag{2}$$

where $\sum_{i=1}^m k_i = n$. Consider now a coalition $S$ that has $n$ members, $k_i$ members of each type $i$, and consider the partial equal treatment allocation where each consumer in $S$ of type $i$ receives

$$\tilde{x}^i = \omega^i + z^i - v.$$

By monotonicity, since $v \ll 0$ we have

$$\tilde{x}^i \succ_i \omega^i + z^i.$$

$$\omega^i + z^i \succ_i \bar{x}^i,$$

so

$$\tilde{x}^i \succ_i \bar{x}^i.$$

I now need to show that this partial allocation $\tilde{x}$ is feasible for the coalition $S$. But

$$\sum_{i \in S} k_i \tilde{x}^i = \sum_{i \in S} k_i(\omega^i + z^i - v) = \sum_{i \in S} k_i \omega^i + \sum_{i \in S} k_i z^i - \sum_{i \in S} k_i v = \sum_{i \in S} k_i \omega^i,$$

where the last equality follows from (2). The upshot is that $(\tilde{x}^i)$ blocks the allocation $(\bar{x}^i)$ in the $n$-replica economy $\mathcal{E}_n$, a contradiction. Therefore

$$0 \notin P.$$

We now use the separating hyperplane theorem to find the existence of a nonzero $p \in \boldsymbol{R}^\ell$ such that $p \cdot z \geqslant 0$ for all $z \in P$. Since each $P_i \subset P$, for each $i$,

$$z \in P_i \implies p \cdot z \geqslant 0. \tag{3}$$

Now suppose $x \underset{i}{\succ} \bar{x}^i$. Setting $z = x - \omega^i$ we have $\omega^i + z = x \underset{i}{\succ} \bar{x}^i$, so $z \in P_i$. Thus (3) implies $p \cdot (x - \omega^i) = p \cdot z \geqslant 0$. Thus

$$x \underset{i}{\succ} \bar{x}^i \quad \implies \quad p \cdot x \geqslant p \cdot \omega^i.$$

Since preferences are locally nonsatiated, if $x \succcurlyeq_i \bar{x}^i$ there is a sequence $x_n \to x$ with $x_n \underset{i}{\succ} x \succcurlyeq_i \bar{x}^i$. Thus $p \cdot x_n \geqslant p \cdot \omega^i$, so by continuity,

$$x \succcurlyeq_i \bar{x}^i \implies p \cdot x \geqslant p \cdot \omega^i.$$

In particular, $p \cdot \bar{x}^i \geqslant p \cdot \omega^i$ for each $i$, and since $\sum_{i=1}^m \bar{x}^i = \sum_{i=1}^m \omega^i$, we conclude that for each $i$,

$$p \cdot \bar{x}^i = p \cdot \omega^i.$$

Thus, $p \cdot \bar{x}^i = p \cdot \omega^i$ and $x \succcurlyeq_i \bar{x}^i$ implies $p \cdot x \geqslant p \cdot \bar{x}^i$, which proves that we have a Walrasian quasi-equilibrium. ∎

## Edgeworth equilibria

**Definition 6** *An* ***Edgeworth equilibrium*** *for the economy $\mathcal{E}$ is an allocation $(x^1, \ldots, x^m)$ such that for every $n \geqslant 1$, the $n^{\text{th}}$ replica*

$$(x^{1,1}, \ldots, x^{1,n}, \ldots, x^{m,1}, \ldots, x^{m,n})$$

*of the allocation belongs to the core of the $n^{\text{th}}$ replica economy $\mathcal{E}_n$.*

You can show that under the assumptions of the previous section, every Edgeworth equilibrium is a Walrasian quasi-equilibrium.

I believe the term was coined by Aliprantis, Brown, and Burkinshaw [1].

## Suggested reading

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