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Notes on the Structure of Common Knowledge*

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Abstract

These notes are intended to fill in some of details in the formal analysis that are labeled "obvious" in [1, 4]. They really are obvious if you think about them long enough the right way. I omit virtually all aspects of interpretation, but try to make the formal results self-contained. These results really are elementary. In fact, almost everything follows from the simple fact that $A \subset B$ if and only if $A = A \cap B$. (Well maybe I'm exaggerating.)

Contents

1 Partitions

Let Ω be a set. Points in Ω are interpreted as **states of nature** or **states of the world**, and subsets of Ω are sometimes called **events**. (Although later on we may wish to restrict what we call events.)

A **partition** of Ω is a family $\mathcal{P}$ of pairwise disjoint nonempty subsets of Ω whose union includes Ω. That is,

*These notes were prepared as a supplement to a seminar conducted by Paolo Ghirardato.

P1. $\bigcup \mathcal{P} = \Omega$,[1] and

P2. $A, B \in \mathcal{P}$ and $A \cap B \neq \varnothing$ implies $A = B$.

Property P1 says that any $\omega \in \Omega$ belongs to at least one element of the partition, while P2 says it belongs to at most one. Let $P(\omega)$ denote the unique element of $\mathcal{P}$ that contains ω. We can view P as a correspondence from Ω to subsets of Ω, which we denote by $P\colon \Omega \twoheadrightarrow \Omega$. The correspondence P is called the **partition correspondence** associated with $\mathcal{P}$. Clearly the partition correspondence satisfies

$$\omega \in P(\omega)$$

for all ω. Also immediate is that

$$\text{if } \gamma \in P(\omega), \text{ then } \omega \in P(\gamma) \text{ and } P(\gamma) = P(\omega).$$

These properties characterize partition correspondences.

1 Lemma *Let $P\colon \Omega \twoheadrightarrow \Omega$ be a correspondence from Ω to subsets of Ω. Assume that for all $\omega \in \Omega$,*

$$\omega \in P(\omega)$$

and for all $\omega, \gamma \in \Omega$,

$$\gamma \in P(\omega) \implies \omega \in P(\gamma).$$

Then $\{P(\omega) : \omega \in \Omega\}$ is a partition of Ω, and P is its partition correspondence.

As usual with correspondences, for any set A, $P(A)$ denotes $\bigcup\{P(\omega) : \omega \in A\}$. Thus P defines a function from 2^Ω into itself, and as such, the partition correspondence P is **idempotent**. That is, for any A, $P(P(A)) = P(A)$. (See Proposition 5 below.) However, not every idempotent correspondence is a partition correspondence for a partition. For instance, let $\Omega = \{0, 1\}$ and set $P(0) = \{0\}$ and $P(1) = \{0, 1\}$. Then P is idempotent, but does not define a partition.

There is another way of thinking about partitions—namely as equivalence relations. An **equivalence relation** on Ω is a total, reflexive, and transitive binary relation on Ω. Given an equivalence relation $\sim$ on Ω, the **equivalence class** $[\omega]$ of ω is $\{\gamma \in \Omega : \gamma \sim \omega\}$. The collection $\{[\omega] : \omega \in \Omega\}$ of equivalence classes of an equivalence relation $\sim$ is a partition. Conversely given a partition $\mathcal{P}$, the binary relation $\sim$ defined by $\omega \sim \gamma$ if $\gamma \in P(\omega)$ is an equivalence relation. The proof of these assertions is left as an exercise.

[1] The notation $\bigcup \mathcal{P}$ is a simple but perhaps unfamiliar way of writing $\bigcup\limits_{A \in \mathcal{P}} A$ or $\bigcup\{A : A \in \mathcal{P}\}$. See, e.g., Halmos [6, p. 13]

2 Fields of sets

Partitions may be natural and intuitive to Paolo, but I find them unwieldy for proofs. Fortunately there is an equivalent way of looking at things that I find much more tractable.

A **field** $\mathcal{F}$ of subsets of Ω is a family of subsets satisfying

F1. $\varnothing \in \mathcal{F}$ and $\Omega \in \mathcal{F}$.

F2. If $A \in \mathcal{F}$, then $A^{c} \in \mathcal{F}$.

F3. If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$ and $A \cup B \in \mathcal{F}$.

It follows by induction that $\mathcal{F}$ is closed under finite intersections and finite unions. A **σ-field** is a field that is closed under countable intersections and countable unions. (Think of σ as standing for "sequence.") A **complete field** is a field closed under arbitrary intersections and unions. The power set 2^{Ω} of Ω is a complete field. If Ω is finite, every field is automatically complete.

The next lemma is a simple consequence of the definitions.

2 Lemma *The intersection of a family of fields is a field.*

The intersection of a family of σ-fields is a σ-field.

The intersection of a family of complete fields is a complete field.

Consequently, given a family $\mathcal{A}$ of subsets, there is a smallest field $\mathcal{F}(\mathcal{A})$ that includes $\mathcal{A}$, a smallest σ-field $\sigma(\mathcal{A})$ that includes $\mathcal{A}$, and a smallest complete field $\mathcal{F}^*(\mathcal{A})$ that includes $\mathcal{A}$.

An **atom** of a field is an element of the field that cannot be written as the disjoint union of two nonempty elements of the field. The atoms of the power set of Ω are the singleton sets.

The relationship between partitions and complete fields is given in the next two propositions.

3 Proposition *A set A belongs to $\mathcal{F}^*(\mathcal{P})$ if and only if $P(A) = A$. The set A is an atom of $\mathcal{F}^*(\mathcal{P})$ if and only if $A \in \mathcal{P}$.*

4 Proposition *If $\mathcal{F}$ is a complete field of subsets of Ω, the atoms of $\mathcal{F}$ constitute a partition of Ω.*

3 Knowledge and partitions

Let us say that the partition **$\mathcal{P}$ knows the event A at ω** if $P(\omega) \subset A$. The set of points ω at which $\mathcal{P}$ knows A is denoted $K(A)$, i.e.,

$$\omega \in K(A) \iff P(\omega) \subset A. \qquad (\star)$$

The function K from 2^{Ω} into 2^{Ω} is called the **knowledge operator induced by $\mathcal{P}$.**

The following proposition summarizes many of the frequently used properties of P and K. We may apply these properties without specific reference to the proposition.

5 Proposition (Summary of properties of P and K) *Let P be the partition correspondence for a partition $\mathcal{P}$ and let K be the knowledge operator induced via ($\star$). Then they satisfy the following properties. (The universal quantifiers "for all events $A, B \subset \Omega$" have been suppressed.)*

1. $K(\Omega) = \Omega$ *and* $P(\Omega) = \Omega$.
2. $K(\varnothing) = \varnothing$ *and* $P(\varnothing) = \varnothing$.
3. $K(A) \subset A$ *and* $A \subset P(A)$.
4. $K(A) \cap K(B) = K(A \cap B)$ *and* $P(A) \cap P(B) = P(A \cap B)$.
5. $P\big(P(A)\big) = P(A)$ *and* $P\big(K(A)\big) = K(A)$.
6. $K\big(K(A)\big) = K(A)$ *and* $K\big(P(A)\big) = P(A)$.
7. $K(A)^c = K\big(K(A)^c\big)$ *and* $P(A)^c = P\big(P(A)^c\big)$.
8. $A \subset K\big(P(A)\big)$ *and* $P\big(K(A)\big) \subset A$.
9. *If* $A \subset B$*, then* $K(A) \subset K(B)$ *and* $P(A) \subset P(B)$
10. $A = K(A) \iff A = P(A) \iff P(A) = K(A)$.

Proof: Properties 1–4 are immediate consequences of the definitions.

Property 5: The proof that $P\big(P(A)\big) = P(A)$ is straightforward. To see the second claim that $P\big(K(A)\big) = K(A)$, note that since ω always belongs to $P(\omega)$, we have $K(A) \subset P\big(K(A)\big)$. On the other hand, suppose ω belongs to $P\big(K(A)\big)$. That is, $\omega \in P(\gamma)$ where $\gamma \in K(A)$. Then by ($\star$), $P(\gamma) \subset A$. But P is a partition correspondence, so $\omega \in P(\gamma)$ implies $P(\omega) = P(\gamma)$, so ω belongs to $K(A)$ too. That is, $P\big(K(A)\big) \subset K(A)$, which completes the proof.

Property 6: To show that $K\big(K(A)\big) = K(A)$, by property 3 it suffices to show the inclusion $K\big(K(A)\big) \supset K(A)$. But this is immediate from ($\star$) and the fact that $\omega \in P(\omega)$ for all ω.

To see that $K\big(P(A)\big) = P(A)$, note that $\omega \in K\big(P(A)\big)$ if and only if $P(\omega) \subset P(A)$. But $\omega \in P(\omega)$ implies this happens if and only if $\omega \in P(A)$.

Property 7: We first show $K(A)^{\mathrm{c}} = K\big(K(A)^{\mathrm{c}}\big)$: By property 2 it suffices to show the inclusion $K\big(K(A)^{\mathrm{c}}\big) \supset K(A)^{\mathrm{c}}$, or equivalently $K\big(K(A)^{\mathrm{c}}\big)^{\mathrm{c}} \subset K(A)$. So suppose ω does not belong to $K\big(K(A)^{\mathrm{c}}\big)$. That is, $P(\omega) \not\subset K(A)^{\mathrm{c}}$, so $P(\omega) \cap K(A) \neq \varnothing$. Let γ belong to this intersection. Then $\gamma \in K(A)$, so $P(\gamma) \subset A$. But $\gamma \in P(\omega)$, so $P(\gamma) = P(\omega)$. Therefore $P(\omega) \subset A$, or in other words, $\omega \in K(A)$.

The proof that $P(A)^{\mathrm{c}} = P\big(P(A)^{\mathrm{c}}\big)$ is easy.

Property 8 is straightforward.

Property 9: If $A \subset B$, then $A = A \cap B$, so by property 4, $K(A) = K(A) \cap K(B)$, which shows $K(A) \subset K(B)$. The case of P is straightforward.

Property 10: First assume $A = K(A)$, so $P(A) = P\big(K(A)\big) = K(A) \subset A$, where the second equality follows from property 5 and the inclusion from property 3. Thus $P(A) \subset A$, but again by property 3, $A \subset P(A)$, so $A = P(A)$.

Now assume $A = P(A)$, so $K(A) = K\big(P(A)\big) = P(A) \supset A$, where the second equality follows from property 5 and the inclusion from property 3. Thus $K(A) \supset A$, but again by property 3, $K(A) \subset A$, so $A = K(A)$.

Property 3 now gives the equivalence of either of these with the condition that $P(A) = K(A)$. ∎

Say that an event A is **self evident** if $A = K(A)$. Thus every self evident set is in the range of K. On the other hand Proposition 5 (6) say that every set in the range of K is self evident. Thus:

6 Corollary *The range of K is the set of self evident events.*

The next result is a simple consequence of from Proposition 3 and Proposition 5 (10).

7 Corollary *The collection of self evident events (and hence the range of K) coincides with $\mathcal{F}^*(\mathcal{P})$, the complete field generated by $\mathcal{P}$.*

4 S5 and partitions

Let $L\colon 2^\Omega \to 2^\Omega$. The following properties are collectively known as **S5**.

S5.1 $L(\Omega) = \Omega$.

For all events A and B,

S5.2 $L(A) \cap L(B) = L(A \cap B)$.

For all events A,

S5.3 $L(A) \subset A$.

S5.4 $L\big(L(A)\big) = L(A)$.

S5.5 $L(A)^{\mathrm{c}} = L\big(L(A)^{\mathrm{c}}\big)$.

It follows from Proposition 5 that if K is the knowledge operator derived from a partition correspondence P, then K satisfies S5. The converse is true—any L that satisfies S5 is the knowledge operator induced by some partition. The conditions of S5 are not independent.

For instance S5.3 and S5.5 readily imply S5.1. The terminology dates back to axiom systems studied by Lewis [7].

8 Proposition *Let* $L\colon 2^\Omega \to 2^\Omega$ *satisfy S5. Then there is a unique partition* $\mathcal{S}$ *such that* L *is the knowledge operator induced by* $\mathcal{S}$*.*

The proof makes use of the following lemmas.

9 Lemma *If* L *satisfies S5, then* L *is monotonic, i.e.,*

$$A \subset B \implies L(A) \subset L(B).$$

Proof: If $A \subset B$, then $A = A \cap B$, so by S5.2, $L(A) = L(A) \cap L(B)$, which shows $L(A) \subset L(B)$. ∎

Let us say that an event A is self evident if $A = L(A)$. If L is indeed a knowledge operator induced from a partition, this accords with our earlier terminology. By property S5.4, the range of L is the collection of self evident events.

10 Lemma *The complement of a self evident event is self evident.*

Proof: By property S5.5, $L(A)^c = L\big(L(A)^c\big)$, so if $L(A) = A$, this reduces to $A^c = L(A^c)$, which says that A^c is self evident. ∎

11 Lemma *The union of any family of self evident events is self evident.*

Proof: Let A_α be self evident for each α in some suitable index set, and put $A = \bigcup_\alpha A_\alpha$. Then for each α, $A_\alpha = A \cap A_\alpha$, so by S5.2, $L(A_\alpha) = L(A) \cap L(\alpha)$. Since A_α is self evident, we have $A_\alpha = L(A) \cap A_\alpha$, so $A_\alpha \subset L(A)$. Therefore $A = \bigcup_\alpha A_\alpha \subset L(A)$, so S5.3 implies $A = L(A)$. ∎

The preceding results and de Morgan's laws clearly imply the next lemma.

12 Lemma *The intersection of any family of self evident events is self evident.*

Clearly Ω is self evident, so the collection of self evident events is a complete field. By Proposition 4 its atoms constitute a partition $\mathcal{S}$ of Ω. We now show that this partition induces the knowledge operator L.

Proof of Proposition 8: We wish to show that for any event A, $L(A) = \{\omega : S(\omega) \subset A\}$, where $S(\omega)$ is that atom of the field of self evident events that contains ω. As we saw in the proof of Proposition 4, this means that $S(\omega)$ is the intersection of all the self evident events containing ω. That is,

$$S(\omega) = \bigcap \{B : \omega \in B \text{ and } L(B) = B\}.$$

Now by property S5.4, $L(A)$ is self evident, so it is one of the sets B in the above intersection. Therefore if $\omega \in L(A)$, by construction $S(\omega) \subset L(A)$. That is, $L(A) \subset \{\omega : S(\omega) \subset A\}$. For the reverse inclusion suppose $S(\omega) \subset A$. Then by Lemma 9, $L(S(\omega)) \subset L(A)$. But $S(\omega)$ is by construction self evident, so $S(\omega) = L(S(\omega)) \subset L(A)$. ∎

5 Public Events

We now consider several partitions $\mathcal{P}_i$, $i \in I$ and their associated partition correspondences P_i, and knowledge operators K_i.

An event E is **public** if it is self evident for all $i \in I$. That is, if $K_i(E) = E$ for each $i \in I$. Equivalently, E is public if and only if $P_i(E) = E$ for all $i \in I$. In other words, by Corollary 7:

13 Lemma *An event is public if and only if it belongs to the complete field $\bigcap_{i \in I} \mathcal{F}^*(\mathcal{P}_i)$.*

We know that this complete field corresponds to some partition, so we now identify its partition correspondence.

For any ω define the event $M(\omega)$ by

$$M(\omega) = \bigcup_{i_1, \dots i_m} P_{i_1}(\cdots P_{i_m}(\omega) \cdots),$$

where the union is taken over all finite sequences $i_1, \dots, i_m$ in I. Clearly M is idempotent as a function, that is, $M\big(M(\omega)\big) = M(\omega)$.

14 Proposition *Every event $M(\omega)$ is public.*

Proof: We need to show that $P_i\big(M(\omega)\big) = M(\omega)$ for any i. Now it is easy to see that

$$\begin{aligned} P_i(M(\omega)) &= P_i\Big(\bigcup_{i_1, \dots i_m} P_{i_1}(\cdots P_{i_m}(\omega) \cdots)\Big) \\ &= \bigcup_{i_1, \dots i_m} P_i\left(P_{i_1}(\cdots P_{i_m}(\omega) \cdots)\right), \end{aligned}$$

but $i, i_1, \dots, i_m$ is a finite sequence in I, so $P_i\left(P_{i_1}(\cdots P_{i_m}(\omega) \cdots)\right) \subset M(\omega)$. Therefore $P_i\big(M(\omega)\big) \subset M(\omega)$, which implies $P_i\big(M(\omega)\big) = M(\omega)$. ∎

15 Proposition *If E is public and $\omega \in E$, then $M(\omega) \subset E$.*

Proof: Let A be any subset of the public event E. Clearly $P_i(A) \subset P_i(E) = E$ for any i. Thus any $P_{i_1}(\cdots P_{i_m}(\omega) \cdots)$ is a subset of E, so $M(\omega) \subset E$. ∎

We now verify the hypotheses of Lemma 1 to show that M defines a partition.

16 Lemma *The correspondence M is a partition correspondence.*

Proof: Clearly $\omega \in M(\omega)$ for any ω so it remains to show that if $\gamma \in M(\omega)$ that $\omega \in M(\gamma)$. If γ belongs to $M(\omega)$, there is some finite sequence satisfying

$$\gamma \in P_{i_1}(\cdots P_{i_m}(\omega)\cdots).$$

That is, there are states $\gamma_0, \gamma_1, \ldots, \gamma_m$ such that $\gamma_m = \omega$, $\gamma_0 = \gamma$, and for each $1 \leqslant j < m$, $\gamma_j \in P_{i_{j+1}}(\gamma_{j+1})$. But $\gamma_j \in P_{i_{j+1}}(\gamma_{j+1})$ implies $P_{i_{j+1}}(\gamma_j) = P_{i_{j+1}}(\gamma_{j+1})$, so it follows that $\gamma_{j+1} \in P_{i_{j+1}}(\gamma_j)$. This implies

$$\omega \in P_{i_m}(\cdots P_{i_1}(\gamma)\cdots),$$

so $\omega \in M(\gamma)$. ∎

These results imply:

17 Corollary *The correspondence M is the partition correspondence for the complete field $\bigcap_{i\in I} \mathcal{F}^*(\mathcal{P}_i)$ of public events.*

The proof of Lemma 16 suggest the following definition. Define the relation r on Ω by $\omega r \gamma$ if for some $i \in I$, $\gamma \in P_i(\omega)$. Let R denote the transitive closure of r. Then R is clearly reflexive and transitive, and the proof of Lemma 16 shows that R is symmetric. Thus R is an equivalence relation and the equivalence class of ω is $M(\omega)$. Aumann [1] calls this relation "**reachability**."

6 Common Knowledge

The **event *E* is common knowledge at *ω*** if for every finite sequence $i_1, \ldots, i_m$ in I,

$$\omega \in K_{i_m}(K_{i_{m-1}}(\cdots K_{i_1}(E)\cdots).$$

This corresponds to the everyone knows that everyone knows E, etc. notion of common knowledge. By ($\star$), this is equivalent to the condition

$$P_{i_1}(\cdots P_{i_m}(\omega)\cdots) \subset E.$$

Written this way, the following is immediate:

18 Lemma *The event E is common knowledge at ω if and only if $M(\omega) \subset E$.*

19 Corollary *An event E is common knowledge at ω if and only if there is a public event A satisfying $\omega \in A \subset E$.*

Proof: Suppose first that E is common knowledge at ω. Then $M(\omega)$ is the desired public event.

Now suppose that $A \subset E$ is a public event containing ω, then $M(\omega) \subset A \subset E$, so E is common knowledge at ω. ∎

By the way, nowhere have we assumed that I is finite.

7 The general "agreement" theorem

Following Bacharach [2] and Geanakoplos [4], an **external action rule** or more simply a **decision rule** is a function ψ that assigns to each nonempty event $A \subset \Omega$ an **action** in some nonempty set $\mathcal{A}$. The decision rule ψ satisfies the **sure thing principal** if for any family $\{A_\alpha : \alpha \in I\}$ of pairwise disjoint sets,

$$(\forall \alpha \in I,\ \psi(A_\alpha) = a) \implies \psi\Big(\bigcup_{\alpha \in I} A_\alpha\Big) = a.$$

Given a partition $\mathcal{P}$ and a decision rule ψ, the **behavior** induced by them is the function f defined by

$$f(\omega) = \psi(P(\omega)). \tag{1}$$

As usual, we adopt the statisticians' notational convention that

$$[f = a] \text{ denotes } \{\omega \in \Omega : f(\omega) = a\}.$$

Now consider a family $(\mathcal{P}_i, \psi_i)$, $i \in I$ of partition-decision rule pairs. (It is not necessary for all the action spaces $\mathcal{A}_i$ to be identical.)

20 General "Agreement" Theorem *Assume that for each $i \in I$, ψ_i satisfies the sure thing principle. If for each $i \in I$, the event $[f_i = a_i]$ is common knowledge at ω, then there exists a public event E containing ω such that for all $i \in I$, $\psi_i(E) = a_i$.*

Proof: Set $E = M(\omega)$, the smallest public event containing ω. Since each event $[f_i = a_i]$ is common knowledge at ω, $E \subset [f_i = a_i]$ for each i (Lemma 18). That is, for each i, $f_i(\gamma) = \psi_i(P_i(\gamma)) = a_i$ for all $\gamma \in E$. Since E is public, $E = P_i(E) = \bigcup\{P_i(\gamma) : \gamma \in E\}$ for each i. Then by the sure thing principle,[2] $\psi_i(E) = a_i$ for each i. ∎

8 Aumann's theorem as a special case

We now show how the general agreement theorem implies Aumann's [1] result on posteriors. Aumann's theorem asserts that if two people have common priors, and if their posteriors probabilities of an event A are common knowledge, the their posterior probabilities of A are equal.

So for this section (Ω, Σ, μ) is a probability space and each partition $\mathcal{P}_i$ consists of nonnull events. That is, for each i, if $A \in P_i$, then $A \in \Sigma$ and $\mu(A) > 0$. In particular this implies that each $\mathcal{P}_i$ is countable. Consequently, the complete field $\mathcal{F}^*(\mathcal{P}_i)$ coincides with the σ-field $\sigma(\mathcal{P}_i)$

[2] The family $\{P_i(\gamma) : \gamma \in E\}$ is not generally pairwise disjoint, but can easily be replaced by a pairwise disjoint family—select one representative from each $P_i(\gamma)$.

and is included in Σ. Fix some event $A \in \Sigma$ with $\mu(A) > 0$, and for each i suppose $\mathcal{A}_i = [0, 1]$ and

$$\psi_i(B) = \mu(B|A) = \frac{\mu(B \cap A)}{\mu(A)},$$

for any $B \in \Sigma$. Note that ψ_i is the same function for each i, call it ψ. Observe that ψ satisfies the sure thing principle, at least with respect to members of Σ, which is adequate. The behavior f_i is given by

$$f_i(\gamma) = \psi_i(P_i(\gamma)) = \mu(P_i(\gamma)|A).$$

So if for each i, the event $[f_i = a_i]$ is common knowledge at ω, by the General Agreement Theorem there is a public event E (which necessarily belongs to Σ and is nonnull) containing ω such that for each i

$$\psi_i(E) = \psi(E) = a_i.$$

In particular, all a_i are identical.

A Proofs of some of the simple propositions

Proof of Proposition 3: Clearly, A belongs to $\mathcal{F}^*(\mathcal{P})$ if and only if A is a union of elements of $\mathcal{P}$. So if $\omega \in A$ and $A \in \mathcal{F}^*(\mathcal{P})$, then $P(\omega) \subset A$, which shows $A = P(A)$. On the other hand, if $A = P(A)$, then A is a union of elements of $\mathcal{P}$, namely $A = \bigcup\{P(\omega) : \omega \in A\}$.

Clearly each $A \in \mathcal{P}$ is an atom. Now suppose A is an atom. Then $A = P(\omega)$ for any $\omega \in A$. To see this, note that since $A = P(A)$, $A \supset P(\omega)$ for every $\omega \in A$. Now suppose there are $\omega, \gamma \in A$ with $P(\omega) \neq P(\gamma)$. Then $A = P(\gamma) \cup \bigcup\{P(\omega) : \omega \in A \setminus P(\gamma)\}$, which contradicts the assumption that A is an atom. Therefore $A = P(\omega)$. ∎

Proof of Proposition 4: For each $\omega \in \Omega$, set

$$A(\omega) = \bigcap\{A \in \mathcal{F} : \omega \in A\}.$$

Note that since $\Omega \in \mathcal{F}$, each $A(\omega)$ is nonempty. Furthermore, being an intersection of members of $\mathcal{F}$, each $A(\omega)$ is also a member of $\mathcal{F}$. I claim that the sets $A(\omega)$ are precisely the atoms of $\mathcal{F}$.

To see that $A(\omega)$ is an atom, suppose $A(\omega) = A \cup B$, where $A, B \in \mathcal{F}$ and $A \cap B = \varnothing$. Suppose without loss of generality that $\omega \in A$. Then $\omega \in B^c \in \mathcal{F}$, so $A(\omega) \subset B^c$, which implies $A(\omega) = A$ and $B = \varnothing$. Thus $A(\omega)$ is an atom of $\mathcal{F}$.

Now suppose A is an atom of $\mathcal{F}$ and $\omega \in A$. I claim $A = A(\omega)$. To see this, suppose $B \in \mathcal{F}$ and $\omega \in B$. Write $A = (A \setminus B) \cup (A \cap B)$, and observe that $A \setminus B$ and $A \cap B$ are disjoint, and $A \cap B$ is nonempty (since it contains ω). Since A is an atom, it follows that $A \setminus B = \varnothing$, i.e., $A \subset B$. Thus $\omega \in B \in \mathcal{F}$ implies $A \subset B$, so $A(\omega) = A$.

To see that the atoms are pairwise disjoint, suppose $\gamma \in A(\omega)$. Then ω belongs to $A(\gamma)$, for otherwise $A(\gamma)^c$ belongs to $\mathcal{F}$ and contains ω. This would imply $A(\omega) \subset A(\gamma)^c$, contradicting

$\gamma \in A(\omega)$. Thus $\omega \in A(\gamma)$. It follows that A belonging to $\mathcal{F}$ contains ω if and only if it contains γ, which in turn implies that $A(\gamma) = A(\omega)$. This establishes pairwise disjointness.

Now clearly $\bigcup\{A(\omega) : \omega \in \Omega\} = \Omega$, so we have a partition. ∎

References

[1] R. J. Aumann. 1976. Agreeing to disagree. *Annals of Statistics* 4(6):1236–1239. http://www.jstor.org/stable/2958591

[2] M. Bacharach. 1985. Some extensions of a claim of Aumann in an axiomatic model of knowledge. *Journal of Economic Theory* 37(1):167–190.

[3] J. D. Geanakoplos. 1992. Common knowledge. *Journal of Economic Perspectives* 6(4):53–82. http://www.jstor.org/stable/2138269

[4] ——— . 1993. Common knowledge. Discussion Paper 1062, Cowles Foundation, Yale University, New Haven, Connecticut. http://cowles.yale.edu/sites/default/files/files/pub/d10/d1062.pdf

[5] ——— . 1993. Common knowledge. Working Paper 93-09-059, Santa Fe Institute Economics Research Program.

[6] P. R. Halmos. 1974. *Naive set theory*. New York: Springer–Verlag. Reprint of the edition published by Litton Educational Printing, 1960.

[7] C. I. Lewis. 1918. *A survey of symbolic logic*. Berkeley: University of California Press.

[8] R. D. McKelvey and T. Page. 1986. Common knowledge, consensus, and aggregate information. *Econometrica* 54(1):109–128. http://www.jstor.org/stable/1914160

[9] R. D. McKelvey and T. Page. 1987. A simple proof of a theorem on common knowledge and consensus. Mimeo.

[10] P. Milgrom. 1981. An axiomatic characterization of common knowledge. *Econometrica* 49(1):219–222. http://www.jstor.org/stable/1911137

[11] L. T. Nielsen, A. Brandenburger, J. Geanakoplos, R. McKelvey, and T. Page. 1990. Common knowledge of an aggregate of expectations. *Econometrica* 58(5):1235–1239. http://www.jstor.org/stable/2938308