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Introductory Notes on Preference and Rational Choice

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Contents

*These notes owe a heavy debt to classes I took from Charlie Plott, Jim Quirk, and Ket Richter, and to conversations with Chris Chambers, Fede Echenique, and Paolo Ghirardato.

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I am in the process of being expanding these notes, so certain sections are just place holders that will be filled in later.

1 The concept of preference

Economists and political scientists conceive of *preference* as a binary relation. That is, we do not attach meaning to a proposition such as, "I prefer to ride my motorcycle," as this raises the question, "Prefer it to what?" Rather an expression of preference takes a form such as, "I prefer riding my motorcycle along the Angeles Crest Highway to riding my bicycle along the Los Angeles River," which expresses a relation between two activities. The collection of all such mental relations is referred to as my **preference relation**, or more simply, my **preference**, or even occasionally as my **preferences**.[1] We may also informally refer to the relation of a particular pair as a preference.

But what does a preference relation mean operationally? Since we cannot observe mental states directly (at least not yet, but neuroscience may yet render this assertion obsolete), we interpret it to mean that if I prefer x to y, then given a choice between x and y, I will choose x. Indeed it is almost impossible to discuss preference meaningfully without referring to choice, but one can easily imagine making choices without considering preference, for example, choosing by tossing a coin. It is also possible (likely?) that whatever cognitive processes are involved in making choices, there is no need to appeal to the notion of preference to make a choice, or to predict someone's choices. However, were neuroscientists to claim to have a measure of my "utility" or "ophelimity" associated with various activities, such as motorcycling, and that measure did not predict my choices, I would argue strongly that they were measuring something other than preferences. Wouldn't you?

So what then is the rôle of preference? The naïve response is that it seems to be a real phenomenon. That is, individuals really do "feel" a preference for some alternatives over others. But this feeling may not apply to all pairs of alternatives. For instance, if offered a choice between a vacation in Cork, Ireland or Ayr, Scotland, I may not have a feeling that I can invoke to make a choice. This is different from my being *indifferent*

[1] Occasionally some economists and political scientists may give consideration to the degree or intensity of this relationship, but for many purposes this is irrelevant, a point stressed by Fisher [23], and reiterated by Hicks and Allen [30].

between Cork and Ayr. Indifference is also a felt mental relation between Cork and Ayr, and I simply may have no feelings to relate the two.

But why do we care about preferences even if they do exist? I think the main reason lies in policy evaluation. If individuals have preferences, then they evaluate policies and institutions using their preferences over the outcomes. But for an argument that things are not so simple, see Camerer, Loewenstein, and Prelec [12], especially section 4.4, where they distinguish between "liking" and "wanting." That is, it is possible to want something without any feeling of pleasure resulting from it. It is true that many nineteenth century economists believed that "pleasure" was what motivated choices, but it is not clear to me that that is relevant to the discussion of whether choices are motivated by "preferences." But if choices reveal what people want rather than what gives them "pleasure," then giving them what they want may not make them "better off."

How can we tell if an individual has a preference relation? The answer must be by observing his or her choice behavior. What kind of choice behavior is consistent with being motivated by preference? That is the subject of these notes.

The guiding principle, perhaps first articulated by Paul Samuelson [51, 52], is that if you choose x when you could have chosen y, then it is reasonable for me to infer that you prefer x to y or are at least indifferent between x and y. We call this the **principle of revealed preference**. In adopting this position I am placing the burden on the side of those who would argue that individuals do not have a preference relation, or if they do, it is not the basis for choice. Gul and Pesedorfer [25] seem to argue that preferences should be regarded as theoretical conveniences rather than as "real" mental states, and as mentioned above Camerer, Loewenstein, and Prelec [12] argue that the the results of the brain apparatus that chooses should not be interpreted as an expression of "liking." Nevertheless, the feeling of preference is one that everyone I know has experienced and even small children can articulate. See the paper by Dietrich and List [17] for a further discussion of these issues.

There is still the issue of what it means to "observe a choice," particularly since the theory we develop may often require the possibility of observing a chosen *set*. That is, if I offer you the choice of x, y, or z, and if you prefer x and y to z, but are indifferent between x and y, then I need to find a way to allow to choose "x or y." One way to do this is to ask you to eliminate the options that you have not chosen, and to treat what is left as what you have chosen. This is fine for laboratory settings, but field observations of consumers' purchases or voters' ballots do not allow for this.

Another problem with interpreting choice observations is this. Suppose I go to the cafeteria every day for lunch, and on Monday I have the burrito, Tuesday I have the pasta, and Wednesday I have the sushi. Is this a set of three observations of choice from the same fixed menu? If so, may you conclude that I have chosen the set {burrito, pasta, sushi} from the menu? Or is it a single observation, namely "burrito on Monday, pasta on Tuesday, and sushi on Wednesday?" There are some interpretations of economic models in which an individual chooses one incredibly detailed contingent plan for his or her entire life. That is, there is only one observation, and almost nothing useful can be inferred from it. These are deep philosophical questions, and I do not wish to debate them now, and indeed never.

2 A digression on binary relations

I stated above that we shall regard preference as a binary relation on a set, that is, the relation of one element of the set to another (not necessarily different) element. I shall typically use R to denote a binary relation on the set X. The statement $x\ R\ y$ means "x bears the relation R to y." It is often useful to specify R via its **graph**, that is, $\{(x, y) \in X \times X : x\ R\ y\}$. The graph is a set-theoretic way of representing the relation, but it is not the relation itself, see, for instance, K. J. Devlin [16].[2] Nonetheless, many authors do identify a relation with its graph.

There are many binary relations on a set with a variety of properties, and they can be illustrated using various kinship relations on the set of people, or more specifically my family. For instance, R= "is the mother of" is a binary relation that is easily understood. Note that this relation is one-way, that is, if Virginia is the mother of Kim, then it is not the case that Kim is the mother of Virginia. We say that this relation is **asymmetric** in that $x\ R\ y \implies \neg\, y\ R\ x$. A **symmetric** binary relation R satisfies $x\ R\ y \implies y\ R\ x$. For example, the relation "is a sibling of" is symmetric. A relation need be neither symmetric nor asymmetric. For example, "is a brother of" is neither, as Kim is the brother of David and David is the brother of Kim, while Kim is the brother of Sandra, but Sandra is not the brother of Kim (she's his sister).

A relation R is **transitive** if $(x\ R\ y\ \ \&\ \ y\ R\ z) \implies x\ R\ z$. The mother-of relation is not transitive: Evelyn is the mother of Virginia, and Virginia is the mother of Sandra, but Evelyn is not the mother of Sandra. The relation "is an ancestor of" is transitive.

The relation "is a sibling of" is also not complete. That is, William is not a sibling of Kim and Kim is not a sibling of William. A binary relation on a set X is **complete**, if for every x and y in the set X, either $x\ R\ y$ or $y\ R\ x$ or both. The relation "was born no later than" is a complete binary relation on members of my family. (It is also transitive.)

Finally, a relation R is **reflexive** if for every x, it must be that $x\ R\ x$. For instance, the was-born-no-later-than relation is reflexive. A relation is **irreflexive** if it is always the case that $\neg\, x\ R\ x$, where $\neg$ is the negation operator.

I think that transitivity and asymmetry are properties that must be true of any sensible notion of strict preference. I also think that symmetry and reflexivity are properties of indifference. Ideally, transitivity would be a property of indifference, but sensory limitations may make indifference intransitive. That is, I may be indifferent between n and $n + 1$ micrograms of sugar in my coffee for every n, but I am definitely not indifferent between one microgram and ten million micrograms in my coffee.

[2]Devlin (p. viii) goes so far as to say (in capital letters) that you should not buy a textbook that states, "A relation is a set of ordered pairs."

By the way, I hope these examples make clear the remark that a relation and its graph are not the same thing. The relation "is the mother of" defines a statement that may be true or false and is not a set of ordered pairs of people. I may refer to other properties of binary relations. The definitions may be found in Appendix A.

3 Greatest and maximal elements of a relation

Given a binary relation R on a set X and a nonempty subset $B \subset X$, we say that x is an ***R*-greatest element** of B if (i) $x \in B$ and (ii) for every $y \in B$ we have $x \mathrel{R} y$. By this definition for any R-greatest element x we must have $x \mathrel{R} x$, so it makes sense to insist that R be reflexive. In terms of preference, we would want to use the weak notion of "at least as good as."

We say that x is an ***R*-maximal element** of B if (i) $x \in B$ and (ii) for every $y \in B$ we have $\neg\, y \mathrel{R} x$. It makes most sense to refer to maximal elements of irreflexive relations. For preferences, we think of looking for maximal elements of the strict preference relation.

4 Choice functions

Following Arrow [6], we start the formal theory of choice and preference with a nonempty set X of "alternatives" and a nonempty family $\mathcal{B}$ of nonempty subsets of X. Following Richter [48], members of $\mathcal{B}$ are called **budgets** or **budget sets**. The pair $(X, \mathcal{B})$ is called a **budget space**. Each budget in $\mathcal{B}$ is a nonempty subset of X, but not every nonempty subset of X need belong to $\mathcal{B}$. The term **menu** has recently become popular among decision theorists as a synonym for budget, as some authors prefer to reserve the term *budget* for a budget defined by prices and income.

1 Definition *A **choice** (or **choice function** or **choice rule** or **choice correspondence**) is a mapping c that assigns to each budget B in $\mathcal{B}$ a* subset *$c(B)$ of B. The subset $c(B)$ is called the **choice for *B*** and if $x \in c(B)$ we say that x is **chosen** from B. For $y \in B$, we say that **y could have been chosen from *B***, or that **y was available**. The triple $(X, \mathcal{B}, c)$ is called a **choice structure** or a **choice space**.*

Note that this definition does not require that the choice set be a singleton, nor even nonempty. A choice c is **decisive** if $c(B)$ is nonempty for every budget B. We shall also say that c is **univalent** if $c(B)$ is a singleton for every B. Political scientists Austen-Smith and Banks [7] call a univalent choice **resolute**.

One interpretation of the choice function and budget space is that it represents a set of **observations** of an individual's (or group's) choices. Many results in the literature assume that $\mathcal{B}$ includes the set of all nonempty finite subsets of X, but never do social scientists have a set of observations so detailed. For lack of a better term, let us agree to say that $\mathcal{B}$ is **saturated** if it contains every nonempty finite subset of X. Not all results rely on such rich budget spaces, and indeed results about arbitrary budget spaces are to be preferred, since we rarely get to choose our data.

My colleague Federico Echenique insists that real budgets spaces (sets of observations) can contain only finitely many budget sets, but most economists are willing to consider thought experiments involving infinitely many budget sets.

4.1 Competitive budgets

Of special interest in neoclassical economics is the infinite budget space of **competitive budgets**. In this case, $X = \boldsymbol{R}^{\mathrm{n}}_+$ for some n, and $\mathcal{B}$ is the collection of budgets $\beta(p, w)$ of the form

$$\beta(p, w) = \{x \in \boldsymbol{R}^{\mathrm{n}}_+ : p \cdot x \leqslant w\}, \qquad p \gg 0, w > 0.$$

That is, the set of budgets determined by a price vector p and an income or wealth w. A choice defined on this budget space is traditionally called a **demand correspondence**. A **demand function** is a singleton-valued demand correspondence. Demand correspondences play a central rôle in most of economics, but they should be viewed as an idealization, and not the sort of observations that can ever be collected. Still there is value in contemplating the nature of such data if it could be collected.

A demand correspondence satisfies **budget exhaustion** if [3]

$$x \in c\Big(\beta(p, w)\Big) \quad \Longrightarrow \quad p \cdot x = w.$$

It is customary to write a demand function (or even a demand correspondence) as $x(p, w)$ instead of $c\Big(\beta(p, w)\Big)$. Note that there are some hidden restrictions on demand functions. For any $\lambda > 0$, we have $\beta(p, w) = \beta(\lambda p, \lambda w)$, so it must be that $x(p, w) = x(\lambda p, \lambda w)$.

5 Revealed preference

Given a choice c on the budget space $(X, \mathcal{B})$, there are (at least) two revealed preference relations that seem to naturally embody the revealed preference principle.

2 Definition (Revealed preference) *We say that* ***x is (directly) revealed (weakly) preferred to y*** *or* ***x is revealed as good as y*** *if there is some budget B in $\mathcal{B}$ for which x is chosen and y could have been chosen. We denote this relation by V. That is,*

$$x \; V \; y \iff (\, \exists B \in \mathcal{B}\,)\; [\, y \in B \;\;\&\;\; x \in c(B)\,].$$

[3] Mas-Colell, Whinston, and Green [43, Definition 2.E.2, p. 23] refer to this property as Walras' Law, which I think is unfortunate, as the term Walras' Law was previously used in regard to aggregate excess demand functions. It is true that what I call Walras' Law is a simple consequence of budget exhaustion, and so many young economists have been molded by MWG, so I suspect their usage will prevail. By the way, the term Walras' Law was coined by Oskar Lange [40]. While we're on the subject, my thirteenth edition of *Chicago Manual of Style* [61, § 6.15, 6.23] still claims that it should be written as Walras's Law, but notes that others may disagree.

We say that ***x is (directly) strict-sense revealed preferred to y*** *if there is some budget B in $\mathcal{B}$ for which x is chosen and y could have been chosen, but is not. We denote this by $x\ S\ y$. That is,*

$$x\ S\ y \iff (\exists B \in \mathcal{B})\ [\, y \in B \ \ \&\ \ x \in c(B) \ \ \&\ \ y \notin c(B)\,].$$

We follow Richter [48] for our notation. You can think of V as a mnemonic for "reVealed preference," and S as a mnemonic for either "strict" or "Samuelson." Mas-Colell [43, Definition 1.C.2, p. 11] uses the somewhat awkward notation $\succsim*$ to denote the relation V. Samuelson [51, p. 65] defines the S relation in the context of univalent choice on the competitive budget space. Uzawa [62] and Arrow [6] consider the more abstract framework we are working in. Note that $x\ S\ y \implies x\ V\ y$ and also that $x\ S\ y \implies x \neq y$.

6 Rational choice

We are interested in how the preference relation, if it exists, defines and can be inferred from choice functions. The traditional definition has been given the unfortunate name "rationalization," and choice derived from a preference relation in this way is called a "rational choice." This terminology is unfortunate because the term "rational" is loaded with connotations, and choice functions that are rational in the technical sense may be nearly universally regarded as irrational by reasonable people. There is also the common belief that if people make choices that are influenced by emotions, then they cannot be rational, but emotions are irrelevant to our definition. Here is the technical meaning of rationality adopted by economists and political scientists.

3 Definition (Rational choice) *A binary relation R on X* ***rationalizes*** *the choice function c over the budget space $(X, \mathcal{B})$ if for every $B \in \mathcal{B}$, the choice set $c(B)$ is the set of R-greatest elements of B, that is,*

$$c(B) = \Big\{x \in B : (\forall y \in B)\ [\, x\ R\ y\,]\Big\}.$$

In this case we say that c is a ***rational choice****.*

Note that this allows for some patently irrational choice behavior to be considered rational in our technical sense. Suppose X is the set $\{\$1, \$2, \$1001\}$ and you have a great fear of odd numbers. The result is that you prefer \$2 to \$1001 to \$1. This leads to the choice function $c\big(\{\$1, \$1001\}\big) = \{\$1001\}$ and $c\big(\{\$1, \$2, \$1001\}\big) = \{\$2\}$, which is perfectly rational by our definition, but most sane people would find this behavior "irrational."

Even so, not all choice functions are rational in our technical sense.

4 Example (A non-rational choice) Let $X = \{x, y, z\}$ and $\mathcal{B} = \{B_1, B_2\}$, where $B_1 = X$ and $B_2 = \{x, y\}$. Define

$$c(B_1) = c\big(\{x, y, z\}\big) = \{x\} \quad c(B_2) = c\big(\{x, y\}\big) = \{y\}.$$

The choice c is not rational. □

5 Exercise Explain why the choice c in the above example is not rational. □

The next lemma is immediate from the definitions.

6 Lemma *If R rationalizes c, then $x \mathrel{V} y \implies x \mathrel{R} y$.*

Proof: By definition, $x \mathrel{V} y$ means there is some B with $y \in B$ and $x \in c(B)$, but $c(B)$ is the set of R-greatest elements of B, so since $y \in B$, we must have $x \mathrel{R} y$. ∎

Rational choices are characterized by the V-axiom.

7 Definition (V-axiom) *The choice c satisfies the V-axiom if for every $B \in \mathcal{B}$ and every $x \in X$,*

$$\big(x \in B \;\; \& \;\; (\forall y \in B\,)\,[\,x \mathrel{V} y\,]\big) \implies x \in c(B).$$

If c is decisive and satisfies the V-axiom, it is what Sen [57] calls a **normal** choice function. The following theorem is taken from Richter [48, Theorem 2, p. 33].

8 Theorem (V-axiom characterizes rational choice) *A choice c is rational if and only if it satisfies the V-axiom. In this case the relation V rationalizes c.*

9 Exercise Prove Theorem 8. □

But usually we are interested in more than just rationality, we are interested in rationalizations with "nice" properties.

7 Regular preference relations

In this section we introduce a special class of binary relations that we shall call "regular" preference relations. The members of this class satisfy the properties one might reasonably expect in preference relation, plus the additional assumption of completeness. There are two approaches to defining regularity, one is in terms of the weak preference-or-indifference relation, the other uses the strict preference relation. In the social choice literature, weak preference is usually denoted by R and strict preference by P, and in the economics literature on consumer choice weak preference is typically denoted by something like $\succcurlyeq$ and strict preference by $\succ$. There are other notions of regularity for preference relations. Hansson [28] and Sen [56] discuss intuitive notions of consistency of choices that may or may not be captured by properties of binary relations.

10 Definition (Regularity) *A* ***regular (weak) preference*** $\succcurlyeq$ *on* X *is a total, transitive, and reflexive binary relation on* X*. The statement* $x \succcurlyeq y$ *is interpreted to mean "*x *is as good as* y*" or "*x *is preferred or indifferent to* y*."*

A ***U-regular strict preference*** $\succ$ *on* X *is an asymmetric and negatively transitive binary relation on* X*. The statement* $x \succ y$ *is interpreted to mean "*x *is strictly preferred to* y*."*

Richter [48] may have introduced the term "regular" in this context.[4] Mas-Colell [43, Definition 1.B.1, p. 6] calls such a relation a **rational preference**. We shall not however refer to a binary relation as rational, only choice functions. The U in U-regularity is for Uzawa [62], who may have introduced the notion.[5]

There is a one-to-one correspondence between regular weak preferences and U-regular strict preferences on a set X.

11 Proposition (cf. Kreps [39, Props. 2.4, 2.5, pp. 10–11])

I. *Given a U-regular strict preference* $\succ$*, define the relation* $\not\prec$ *by*

$$x \not\prec y \iff \neg\, y \succ x.$$

Then $\not\prec$ *is a regular preference (total, transitive, and reflexive). Moreover* $\succ$ *is the asymmetric part of* $\not\prec$*, that is,* $x \succ y$ *if and only if* $x \not\prec y$ *and* $\neg\, y \not\prec x$*. Call* $\not\prec$ *the regular preference induced by* $\succ$*.*

II. *Given a regular preference* $\succcurlyeq$*, define* $\succ$ *and* $\sim$ *to be the asymmetric and symmetric parts of* $\succcurlyeq$*. That is,*

$$x \succ y \iff (x \succcurlyeq y \;\;\&\;\; \neg\, y \succcurlyeq x) \quad \text{and} \quad x \sim y \iff (y \succcurlyeq x \;\;\&\;\; x \succcurlyeq y).$$

Then $\succ$ *is a U-regular strict preference and* $\sim$ *is an equivalence relation (reflexive, symmetric, and transitive). Call* $\succ$ *the strict preference induced by* $\succcurlyeq$*.*

In addition, $\succ$ *is transitive, and we have the following relations among* $\succcurlyeq$*,* $\succ$*, and* $\sim$*: For all* $x, y, z \in X$*,*

$$\begin{aligned} x \succ y &\implies x \succcurlyeq y, & x \sim y &\implies x \succcurlyeq y, \\ (x \succcurlyeq y \;\;\&\;\; y \succ z) &\implies x \succ z, & (x \succcurlyeq y \;\;\&\;\; y \sim z) &\implies x \succcurlyeq z, \\ (x \succ y \;\;\&\;\; y \succcurlyeq z) &\implies x \succ z, & (x \sim y \;\;\&\;\; y \succcurlyeq z) &\implies x \succcurlyeq z, \end{aligned}$$
$$\textit{etc.}$$

[4] In a testament to how nonstandardized and confusing terminology in this area can be, Richter [47] uses the simple term "rationality" to mean regular-rationality, but in [48] he does not.

[5] Unfortunately Uzawa's paper [62] fails to explain that a bar over an expression indicates negation, and accidentally omits a number of these bars. (At least the copy I have seen omits these bars, which could be just a scanning problem.) I am relying on Arrow's [6] reading of Uzawa's paper.

III. *Further, given a U-regular preference $\succ$, let $\rho(\succ)$ denote the regular preference $\not\prec$ induced by $\succ$; and given a regular preference $\succcurlyeq$, let $\sigma(\succcurlyeq)$ denote the U-regular strict preference $\succ$ induced by $\succcurlyeq$. Then*

$$\succ = \sigma\Big(\rho(\succ)\Big) \quad \text{and} \quad \succcurlyeq = \rho\Big(\sigma(\succcurlyeq)\Big).$$

12 Exercise Prove Proposition 11. (This is easy, but incredibly tedious.) □

8 Regular-rationality

13 Definition (Regular-rationality) *A choice function c is **regular-rational** if it can be rationalized by a regular preference relation.*

14 Example (A rational choice that is not regular-rational) Let $X = \{x, y, z\}$, and let $\mathcal{B} = \Big\{\{x, y\}, \{y, z\}, \{x, z\}\Big\}$. Let

$$c\Big(\{x, y\}\Big) = \{x\}, \quad c\Big(\{y, z\}\Big) = \{y\}, \quad c\Big(\{x, z\}\Big) = \{z\}.$$

Then c is rational, but not regular-rational. □

The problem in the example above is that the revealed preference relation V is not transitive, so we introduce a revealed preference relation that is transitive.

15 Definition (Indirect revelation) *Given a choice c on the budget space $(X, \mathcal{B})$ we say that **x is indirectly revealed preferred to y**, denoted $x\ W\ y$, if it is directly revealed preferred to y or there is some finite sequence $u_1, \dots, u_n$ in X for which*

$$x\ V\ u_1\ V\ \cdots\ V\ u_n\ V\ y.$$

*That is, the relation W is the **transitive closure** of the relation V.*

16 Definition (W-axiom) *The choice c satisfies the W-axiom if for every $B \in \mathcal{B}$ and every $x \in X$,*

$$\Big(x \in B \ \ \& \ \ (\forall y \in B\,)\,[\,x\ W\ y\,]\Big) \implies x \in c(B).$$

The W-axiom characterizes rationalization by a regular preference. The following theorem is taken from Richter [48, Theorem 8, p. 37]. See also Hansson [27].

17 Theorem (The W-axiom characterizes regular-rationality) *A decisive choice is regular-rational if and only if it satisfies the W-axiom.*

The proof is given in Richter [47], and uses Szpilrajn's Theorem 54 below. The idea is that W is a transitive relation that rationalizes c, and by Szpilrajn's Theorem it can be compatibly extended to a regular preference. The fact that the extension is compatible implies that it also rationalizes c. (Compatibility requires that the extension preserve strict preference. See the discussion of Szpilrajn's Theorem below.)

Richter also gives the following axiom, which is equivalent to the W-axiom for decisive choices. Sen [57] refers to it as the **Strong Congruence Axiom**

18 The Congruence Axiom *The choice c satisfies the Congruence Axiom if for every $B \in \mathcal{B}$ and every $x, y \in B$,*

$$\Big(y \in c(B) \;\;\&\;\; x \; W \; y\Big) \implies x \in c(B).$$

19 Proposition *If a choice function satisfies the W-axiom, then it satisfies the Congruence Axiom.*

A decisive choice satisfies the Congruence Axiom if and only if it satisfies the W-Axiom.

20 Exercise Prove Proposition 19. □

9 Samuelson's Weak Axiom

The "weak axiom of revealed preference" is a "consistency" condition on a demand function introduced by Paul Samuelson [51]. Let $x(p, w)$ be a demand function (not correspondence) on the competitive budget space that satisfies budget exhaustion, that is, for every (p, w), we have $p \cdot x(p, w) = w$ [51, eqn. 1.1]. Let $x = x(p, w)$ and $x' = x(p', w')$. Samuelson argues [51, bottom of p. 64 through top of p. 65],

> Suppose now that we combine the prices of the first position with the batch of goods bought in the second. ... If this cost is less than or equal to the actual expenditure in the first period when the first batch of goods was actually bought, then it means that the individual could have purchased the second batch of goods with the first price and income situation, but did not choose to do so. That is, the first batch x was selected over x'. We may express this symbolically by saying ... $x' \circledless x$. The last symbol is merely an expression of the fact that the first batch was selected over the second.

Equations 6.01–6.02 then state his Postulate III as

$$x' \circledless x \implies \neg\, x \circledless x'.$$

(The circles were inadvertently left out of the published display.) He continues:

> In words this means that if an individual selects batch one over batch two, he does not at the same time select two over one. The meaning of this is perfectly clear and will probably gain ready acquiescence. In any case the denial of this restriction would render invalid all of the former analysis of consumer's behaviour and the theory of index numbers as shown later.

There is an ambiguity here. The narrative indicates that $x \neq x'$, in which case $x' \otimes x$ can be interpreted in two ways, either as either $x\ S\ x'$ or as $(x \neq x' \ \ \&\ \ x\ V\ x')$.

10 The Weak Axiom of Revealed Preference in general

In the context of possibly non-singleton-valued choice correspondences, there is even more room to interpret Samuelson's original notion. I count four interpretations of the idea above:

$$\begin{aligned} x\ S\ y &\implies \neg\, y\ S\ x \\ x\ V\ y &\implies \neg\, y\ S\ x \\ x\ S\ y &\implies \neg\, y\ V\ x \\ (x\ V\ y \ \ \&\ \ x \neq y) &\implies \neg\, y\ V\ x \end{aligned}$$

Here is the interpretation that I think is most often used.

21 Definition (WARP: The Weak Axiom of Revealed Preference) *The choice c satisfies the Weak Axiom of Revealed Preference if the induced revealed preference relations V and S satisfy:*

$$(\forall x, y \in X\,)\ [\,x\ V\ y \implies \neg\, y\ S\ x\,].$$

That is, if x is directly revealed as good y, then y cannot be directly strict sense revealed preferred to x. Note that this is equivalent to the contrapositive form (interchanging the dummy variables x and y),

$$(\forall x, y \in X)\ [x\ S\ y \implies \neg\, y\ V\ x].$$

The second (contrapositive) version of WARP appears as condition C5 in Arrow [6].

It is also possible to express this idea directly in terms of the choice function without mentioning the revealed preference relation explicitly. The following axiom is what Mas-Colell [43, Definition 1.C.1, p. 10] refers to as the Weak Axiom of Revealed Preference, but Kreps [39, 2.9, p. 13] refers to it as Houthakker's Axiom, but he has since changed his terminology.[6]

22 Definition (Kreps's Choice Consistency Axiom) *The choice c satisfies Kreps's Axiom if for every $A, B \in \mathcal{B}$ and every $x, y \in X$,*

$$\Big(x, y \in A \cap B \ \ \&\ \ x \in c(A) \ \ \&\ \ y \in c(B)\Big) \implies x \in c(B).$$

[6]While Houthakker [31, 32] has made important contribution to revealed preference theory, he never formulates this axiom. Kreps reports via e-mail that he does not recall why he attributed this axiom to Houthakker, but perhaps it comes from a course he took from Bob Wilson in 1972.

The next axiom is the V-relation version of Richter's Congruence Axiom, so Sen [57] refers to it as the Weak Congruence Axiom.

23 Definition (Weak Congruence Axiom) *The choice c satisfies the Weak Congruence Axiom if for every $B \in \mathcal{B}$ and every $x, y \in X$,*

$$\Big(x, y \in B \;\;\&\;\; y \in c(B) \;\;\&\;\; x \; V \; y\Big) \implies x \in c(B).$$

24 Proposition *The following are equivalent.*

1. *The choice correspondence c satisfies WARP.*

2. *The choice correspondence c satisfies Kreps's Axiom.*

3. *The choice correspondence c satisfies the Weak Congruence Axiom.*

25 Exercise Prove Proposition 24. That is, all three of these axioms (WARP, Kreps, and WCA) characterize the same class of choice correspondences. □

The following interpretation is the one characterized by Richter [48] as Samuelson's.

26 Definition (SWARP: Samuelson's Weak Axiom of Revealed Preference)

$$x \; S \; y \implies \neg\, y \; S \; x. \tag{SWARP}$$

And finally we come to the last variation on Samuelson's theme:

$$\text{If } x \neq y, \text{ then } x \; V \; y \implies \neg\, y \; V \; x. \tag{SWARP$'$}$$

27 Exercise What is the connection between rationality and WARP? What is the connection between SWARP$'$, SWARP, and WARP? Hint:

- Rationality $\not\Longrightarrow$ WARP.
- Regular-rationality $\implies$ WARP.
- WARP $\not\Longrightarrow$ Rationality.
- [WARP & Decisiveness] $\implies$ Rationality.
- [WARP & Decisiveness & Saturated $\mathcal{B}$] $\implies$ Regular-rationality.
- SWARP$'$ $\implies$ WARP $\implies$ SWARP.
- SWARP $\not\Longrightarrow$ WARP $\not\Longrightarrow$ SWARP$'$ (even for decisive choice functions).
- If c is Univalent, then SWARP $\iff$ WARP $\iff$ SWARP$'$.

□

10.1 Strict revealed preference revisited

Earlier we introduced the strict sense revealed preference relation S, defined, as you should recall, by

$$x \; S \; y \text{ if } (\, \exists B \in \mathcal{B}\,) \, [\, x \in c(B) \;\; \& \;\; y \in B \;\; \& \;\; y \notin c(B)\,].$$

There is another way to define a strict revealed preference in terms of the asymmetric part $\widehat{V}$ of the direct revealed preference relation V, which is defined by

$$x \; \widehat{V} \; y \text{ if } x \; V \; y \;\; \& \;\; \neg \, y \; V \; x.$$

If the choice function satisfies WARP, then the two are equivalent (cf. [43, Exercise 1.C.3]):

28 Proposition *For any choice function,*

$$x \; \widehat{V} \; y \implies x \; S \; y.$$

If the choice function satisfies WARP, then

$$x \; \widehat{V} \; y \iff x \; S \; y.$$

29 Exercise Prove Proposition 28. □

11 When every finite subset is a budget

We now turn to the special case where the set $\mathcal{B}$ of budgets includes all nonempty finite subsets of X, which I previously called the saturated case. This is the case considered by Kreps [39, Ch. 2], Arrow [6], and Sen [56]. The next result is Kreps's Prop. 2.14 and Mas-Colell's [43] Proposition 1.D.2. To prove these results we do not need that every nonempty finite subset belongs to $\mathcal{B}$, it suffices to require that every two-element and every three-element set belongs to $\mathcal{B}$.[7]

30 Theorem *Let c be a decisive choice and assume that $\mathcal{B}$ contains every nonempty finite subset of X. Then c is regular-rational if and only if c satisfies the Weak Axiom of Revealed Preference.*

31 Exercise Prove Theorem 30. Hint: Is the revealed preference relation V a regular preference? □

[7] Strictly speaking we also need one-element subsets to belong to $\mathcal{B}$, but we can omit this requirement if we are willing to modify the definition of V so that by definition $x \; V \; x$ for every x.

12 Other revealed preference axioms

Houthakker [31] introduced the Strong Axiom of Revealed Preference in the context of competitive budgets. Ville [67, 68] used an "infinitesimal" version of it.

32 Strong Axiom of Revealed Preference

$$(\forall x, y \in X)\ [\,x\ H\ y \implies \neg\, y\ H\ x\,],$$

where H is the transitive closure of S.

33 Proposition (Richter) *If c is univalent, then c satisfies the W-axiom if and only if it satisfies the Strong Axiom of Revealed Preference.*

Most of the revealed preference axioms discussed so far do not make full use of the information encoded in the choice function. For example, the statements $x\ V\ y$ and $y\ V\ x$ ignore the relation, if any, between the sets out of which x and y are chosen. Sen's [56] $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ axioms deal with nested budget sets, and refine Kreps's Axiom.

34 Definition (Sen's $\boldsymbol{\alpha}$) *A choice function c satisfies Sen's $\boldsymbol{\alpha}$ if for every pair $A, B \in \mathcal{B}$ with*

$$A \subset B,$$
$$\big(x \in c(B)\ \ \&\ \ x \in A\big) \implies x \in c(A).$$

35 Definition (Sen's $\boldsymbol{\beta}$) *A choice function c satisfies Sen's $\boldsymbol{\beta}$ if for every pair $A, B \in \mathcal{B}$ with*

$$A \subset B,$$
$$\big(x, y \in c(A)\ \ \&\ \ y \in c(B)\big) \implies x \in c(B).$$

Sen's $\boldsymbol{\alpha}$ axiom was proposed earlier by Nash [44] under the name "Independence of Irrelevant Alternatives" in his axiomatization of the Nash bargaining solution, and also by Chernoff [13] as a rationality criterion for statistical decision procedures. It is also Axiom C3 in Arrow [6].

36 Exercise True or False? A choice function satisfies Kreps's Axiom if and only if it satisfies both Sen's $\boldsymbol{\alpha}$ and Sen's $\boldsymbol{\beta}$. □

37 Exercise

i. Prove that every rational choice function satisfies Sen's $\boldsymbol{\alpha}$.

ii. Do Decisiveness, saturated budgets, and Sen's $\boldsymbol{\alpha}$ together imply rationality?

iii. Exhibit a rational choice function that does not satisfy Sen's $\boldsymbol{\beta}$.

□

Wilson [69] provides an analysis of other variations on revealed preference axioms.

13 Path independence

An alternative to rationality is the notion of path independence as formulated by Plott [45].

Let's say that a family $\mathcal{E}$ of nonempty subsets of X **covers** B if each $E \in \mathcal{E}$ is a subset of B and $\bigcup \mathcal{E} = B$. (Note that we do not require that the sets in the cover $\mathcal{E}$ be disjoint.) A choice c satisfies **path independence** if for every pair $\mathcal{E}$ and $\mathcal{F}$ of families of subsets that cover B, we have

$$c\Big(\bigcup_{E\in\mathcal{E}} c(E)\Big) = c\Big(\bigcup_{F\in\mathcal{F}} c(F)\Big) = c(B).$$

Clearly, regular-rationality implies path independence.

38 Exercise Prove that if decisive c is regular-rational, then it is path independent. What if c is not decisive? □

Path independence does not imply rationality.

39 Example (Path independence does not imply rationality)

$$c(\{x,y\}) = \{x,y\} \quad c(\{x,z\}) = \{x,z\} \quad c(\{y,z\}) = \{y,z\} \quad c(\{x,y,z\}) = \{x,y\}$$

□

40 Exercise Explain why the example above is an example of what it claims to be. □

[*** More to come ***]

14 Utility and revealed preference

A **utility** for a regular preference relation $\succcurlyeq$ on X is a function $u\colon X \to \boldsymbol{R}$ satisfying

$$x \succcurlyeq y \quad \iff \quad u(x) \geqslant u(y).$$

In the neoclassical case, we say that u is **monotone** if $x \gg y \implies u(x) > u(y)$.

It is well known that not every regular preference has a utility.

41 Example (A preference with no utility) The **lexicographic preference** on the plane is given by $(x, y) \succcurlyeq (x', y')$ if $\Big[x > x' \text{ or } (x = x' \text{ and } y \geqslant y')\Big]$. To see that no utility exists for this preference relation, let $x > x'$. Then any utility u would imply the existence of rational numbers q_x and $q_{x'}$ satisfying

$$u\Big((x, 1)\Big) > q_x > u\Big((x, 0)\Big) > u\Big((x', 1)\Big) > q_{x'} > u\Big((x', 0)\Big).$$

This defines a one-to-one correspondence $x \longleftrightarrow q_x$ between the reals and a subset of the rational numbers. But Cantor proved long ago via his famous "diagonal procedure" that no such correspondence can exist (see, for instance, the *Hitchhiker's Guide* [4, p. 11]). □

As an aside, Debreu [15, §4.6] proves that a continuous utility exists for a continuous regular preference on any connected subset of $\boldsymbol{R}^{\mathrm{n}}$. (A regular preference $\succcurlyeq$ on X is *continuous* if $\{(x, y) \in X \times X : x \succcurlyeq y\}$ is a closed subset of $X \times X$.)

But even though the lexicographic preference has no utility, the demand function it generates can also be generated by the continuous utility function $u(x, y) = x$.

42 Exercise Prove the assertion in the paragraph above. Hint: Find the demand generated by the lexicographic preference. □

This raises a natural question, if a demand is regular-rational, can it also be generated by a utility function? Following Richter, let us say that a choice c is **represented** by the utility u if for every budget $B \in \mathcal{B}$,

$$c(B) = \Big\{x \in B : (\forall y \in B)\ [\, u(x) \geqslant u(y)\,]\Big\}.$$

In this case we say that the choice is **representable**. In the competitive budget case, we say that c is **monotonely representable on its range** if it is representable by a utility that is monotone nondecreasing on the range of c.

Unfortunately, not every regular-rational choice is representable. The next example is based on Richter [48, Example 3, p. 47].

43 Example (A regular-rational demand that is not representable) Define the preference relation $\succcurlyeq$ on $\boldsymbol{R}^2_+$ by

$$(x, y) \succcurlyeq (x', y') \text{ if } \Big[(x + y > x' + y') \text{ or } (x + y = x' + y' \ \ \& \ \ y \geqslant y')\Big].$$

That is, the preference first looks at the sum, and then the y-coordinate. This is a regular preference relation. It is easy to see (draw a picture) that the demand c at price vector (p_x, p_y) and income m is

$$c\Big(\beta(p,w)\Big) = \begin{cases} (0, w/p_y) & \text{if } p_x \geqslant p_y \\ (w/p_x, 0) & \text{if } p_x < p_y. \end{cases}$$

But no utility can generate this demand. To see this, assume that c is represented by the utility u, fix $w = 1$ and consider two positive numbers $p > p'$. When $p_x = p_y = p$, then $(0, 1/p)$ is chosen over $(1/p, 0)$, see Figure 1. When $p_x = p_y = p'$, then $(0, 1/p')$ is chosen over $(1/p', 0)$. And when $p_x = p'$ and $p_y = p$, then $(1/p', 0)$ is chosen over $(0, 1/p)$. Therefore there must be some rational numbers $q_p, q_{p'}$ satisfying

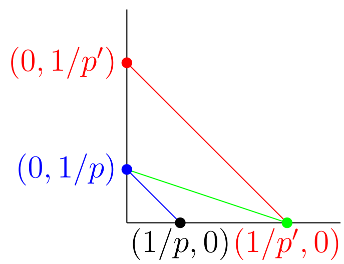

Figure 1.

$$u(0, 1/p') > q_{p'} > u(1/p', 0) > u(0, 1/p) > q_p > u(1/p, 0).$$

In particular, $q_p \neq q_{p'}$. This defines a one-to-one correspondence $p \longleftrightarrow q_p$ between the positive reals and a subset of the rational numbers, which is impossible. □

But we do have some positive results for demand functions (choices on the neoclassical budget space).

44 Theorem (Richter [48, Theorem 12, p. 50]) *Let x be a demand function with convex range. If x satisfies the Strong Axiom of Revealed Preference, then x is monotonely representable on its range.*

45 Theorem (Richter [48, Theorem 14, p. 51]) *Let x be a demand correspondence with convex range. Assume that $x(p, w)$ is closed for each (p, w). If x is regular-rational, then it is representable. If x satisfies budget exhaustion, then x is monotonely representable on its range.*

Note that while convexity of the range is sufficient, it is hardly necessary. Here is an example.

46 Example Here is an example of a demand generated by a monotone upper semicontinuous utility that does not have a convex range. Define the utility u on $\boldsymbol{R}^2_+$ by

$$u(x,y) = \begin{cases} y & y < 1 \\ \\ 1 + x & y \geqslant 1. \end{cases}$$

The range of the demand generated by this utility is the union of the line segment from $(0,0)$ to $(0,1)$ with the half-line $\{(\lambda, 1) : \lambda \geqslant 0\}$. See Figure 2, where the range is shown in red. □

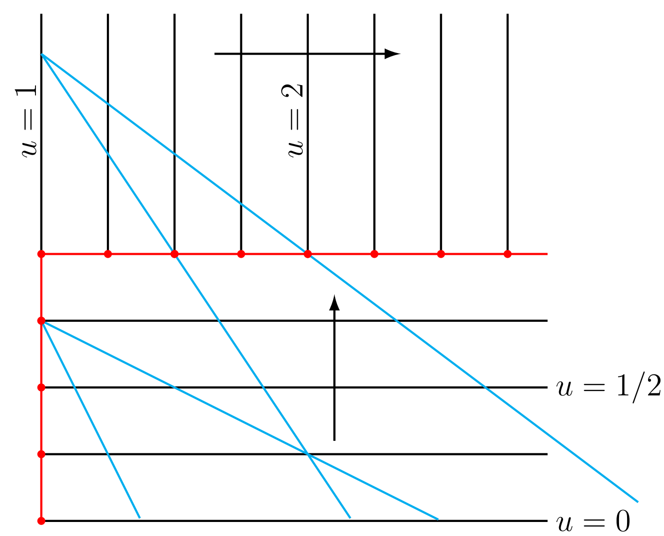

Figure 2. Preferences and demand for Example 46.

15 Other revealed preference axioms and competitive budgets

For competitive budgets $\beta(p,w)$, if the demand correspondence x satisfies budget exhaustion,

This section is very tentative. I need a better way to organize it.

$$x \in x(p,w) \implies p \cdot x = w,$$

then the income w is redundant, and we are often presented with a datum as a price-quantity pair, (p,x). The interpretation is that $x \in c\Big(\beta(p, p \cdot x)\Big)$. When the demand correspondence does not satisfy budget exhaustion, this way of presenting data is not so useful. So for this section, *assume every demand correspondence satisfies budget exhaustion.*

Samuelson [51, 52, 53] phrased the Weak Axiom of Revealed Preference for competitive budgets (while implicitly assuming univalence and budget exhaustion) this way.

47 Samuelson's Weak Axiom of Revealed Preference *Let x be a demand function that satisfies budget exhaustion. Let $x^0 = x(p^0, w^0)$ and $x^1 = x(p^1, w^1)$. Assume $x^0 \neq x^1$. Then*

$$p^0 \cdot x^1 \leqslant p^0 \cdot x^0 \quad \implies \quad p^1 \cdot x^0 > p^1 \cdot x^1.$$

This is just the Weak Axiom of Revealed Preference in disguise.

There are other notions of revealed preference that can be used with competitive budgets and budget exhaustion. In particular, there is another notion of strict revealed preference that was championed by Varian [65].

48 Definition (Budget-sense strict revealed preference) *Let x be a demand correspondence and let $x^0 \in x(p^0, w^0)$. Then x^0 is* ***(direct) strictly revealed preferred to x^1 in the budget sense****, written*

$$x^0 \; A \; x^1 \text{ if } p^0 \cdot x^1 < p^0 \cdot x^0.$$

That is, $x^0 \; A \; x^1$ if there is some budget at which x^0 is demanded and x^1 is strictly less expensive than x^0.

The next condition is called the **Weak Weak Axiom of Revealed Preference** by Kihlstrom, Mas-Colell, and Sonnenschein [36].

49 Axiom (Weak Weak Axiom of Revealed Preference (WWA)) *Let x be a demand function that satisfies budget exhaustion. Then*

$$x \; A \; y \quad \implies \quad \neg \, y \; S \; x,$$

or equivalently, taking the contrapositive and interchanging the dummy variables x and y,

$$x \; S \; y \quad \implies \quad \neg \, y \; A \; x,$$

The next condition is called the **Generalized Axiom of Revealed Preference** by Varian [63, 64]

50 Axiom (Generalized Axiom of Revealed Preference (GARP)) *Let x be a demand function that satisfies budget exhaustion. Then*

$$x \; W \; y \quad \implies \quad \neg \, y \; A \; x.$$

This axiom is explored by Varian [63], and is related to results of Afriat [1, 3]. (In fact, the A relation is named in honor of Afriat.)

[*** More to come ***]

16 Motivated choice and intransitivity

The definition of rational choice we have been using is that a choice is rational if the choice set from each budget is the set of greatest elements in the budget for some binary relation. For a regular preference $\succcurlyeq$, the set of greatest and maximal elements is the same.

51 Proposition *If $\succcurlyeq$ is a regular preference on a set B, then the set of $\succcurlyeq$-greatest elements of B coincides with the set of $\succ$-maximal elements. That is,*

$$(\forall y \in B)\ [x \succcurlyeq y] \quad \iff \quad (\forall y \in B)\ [y \neq x \implies \neg\, y \succ x].$$

For a binary relation that fails to be complete or transitive the above result does not hold. Since completeness and transitivity of indifference are less plausible than transitivity of strict preference, it may be worth exploring another way that binary relations define choice functions.

52 Definition (Motivated choice) *A binary relation $\succ$ on X **motivates** the choice c over the budget space $(X, \mathcal{B})$ if for every $B \in \mathcal{B}$, the choice set $c(B)$ is the set of $\succ$-maximal elements of B, that is,*

$$c(B) = \{x \in B : (\forall y \in B)\ [y \neq x \implies \neg\, y \succ x]\}.$$

*In this case we say that c is a **motivated choice**.*

The main references here are Shafer [58], Kihlstrom, Mas-Colell, and Sonnenschein [36], Suzumura [59], Clark [14], Richter and Kim [38], and Kim [37].

[*** More to come ***]

17 Desultory methodological musings

Theorem 8 says that any choice function is rational if and only if it satisfies the V-axiom. Theorem 17 says that any decisive choice function is regular-rational if and only if it satisfies the W-axiom. Theorem 30 says that if the budget space is saturated, then a decisive choice function is regular-rational if and only if it satisfies the Weak Axiom of Revealed Preference. Since rationality does not imply WARP, but for decisive choice functions WARP implies rationality (Exercise 27), and since in general WARP does not imply the W-axiom, which is the "better" or more important result? The answer to this question depends on what you think is the point of the theory. Need to mention this earlier.

Varian [63, 64] and Echenique, Golovin, and Wierman [18], and probably Richter [47, 48] would argue that they want to test the hypothesis that choice functions are rational by applying the results to observable data. Varian [63, p. 945] is quite explicit:

> The economic theory of consumer demand is extremely simple. The basic behavioral hypothesis is that the consumer chooses a bundle of goods that is preferred to all other bundles that he can afford. Applied demand analysis typically addresses three sorts of issues concerning this behavioral hypothesis.
>
> (i) Consistency. When is observed behavior consistent with the preference maximization model?
>
> (ii) Recoverability. How can we recover preferences given observations on consumer behavior?
>
> (iii) Extrapolation. Given consumer behavior for some price configurations how can we forecast behavior for other price configurations?
>
> [...] I will show how one can directly and simply test a finite body of data for consistency with preference maximization, recover the underlying preferences in a variety of formats, and use them to extrapolate demand behavior to new price configurations.

Given this point of view, theorems that apply only when the budget space is saturated are useless. We only get observations on competitive budgets, not pairs or triples, and only finitely many of them.

But there is another goal of the theory. Arrow, Plott, and Sen are motivated at least in part by problems of group decision making. In particular they seek to find (but do not truly succeed) procedures that aggregate individuals' preferences into a group choice. They want the choices made to exhibit "reasonable" consistency or "rationality" properties. With this point of view revealed preference axioms are consistency criteria, and it is desirable to know how they relate to rationality. While a group may never have to enumerate the choices that they would make over all budgets, the consistency conditions can have their reasonableness validated by asking them to apply to saturated budget spaces. Sen [57, p. 312] argues as follows.

> [...] In particular the following two questions are relevant.
>
> (1) Are the rationality axioms to be used only after establishing them to be true?
>
> (2) Are there reasons to expect that some of the rationality axioms will tend to be satisfied in choices over "budget sets" but not for other choices?
>
> [...] There are an infinite (and uncountable) number of budget sets even for the two-commodity case and choices only over a few will be observed. What is then the status of an axiom that is used in an exercise having been seen not to be violated over a certain *proper* subset [...]? Clearly it is still an assumption rather than an established fact. [...] But then the question arises: why assume the axioms to be true only for "budget sets" [...]

Another argument in favor of using finite sets even for demand theory is given by Arrow [6, p. 122] and appeals to simplicity.

> It is the suggestion of this paper that the demand function point of view would be greatly simplified if the range over which the choice functions are considered to be determined is broadened to include all finite sets. Indeed, as Georgescu-Roegen has remarked, the intuitive justification of such assumptions as the Weak Axiom of Revealed Preference has no relation to the special form of the budget constraint sets but is based rather on implicit consideration of two element sets [...].

Note that this quote refers to "intuitive justification" of an axiom. Ordinarily, science should not use principles because of their intuitive appeal, but since we are discussing human behavior and economists are ourselves human, this may be excusable. It is also quite common. Recall Samuelson's [51] remark cited above, "The meaning of [the Weak Axiom] is perfectly clear and will probably gain ready acquiescence."

Appendices

A Binary relations

N. Bourbaki [11, Section I.1.1, p. 16; Section II.3.1, p. 75] and K. J. Devlin [16, p. viii] are rather insistent that a **binary relation** R between members of a nonempty set X and members of a nonempty set Y defines a *statement*, $x\ R\ y$, about ordered pairs (x, y) in $X \times Y$, and is not itself a *subset* of $X \times Y$. Nonetheless a relation is completely characterized by its **graph**, the subset of $X \times Y$ for which the statement is true. That is,

$$\text{gr } R = \{(x, y) \in X \times Y : x\ R\ y\}.$$

Indeed, many authors do define a relation to be its graph, and they would argue that only a pedant would insist on such a distinction. (To which I say, "sticks and stones ...") When $X = Y$, we say we have a **binary relation on X**. We also write $x\ R\ y\ R\ z$ to mean $x\ R\ y\ \ \&\ \ y\ R\ z$.

The binary relation S **extends** the binary relation R if $x\ R\ y \implies x\ S\ y$. In terms of graphs, gr $S \supset$ gr R.

Given a binary relation R on X, we define its **asymmetric part** $\widehat{R}$ by

$$x\ \widehat{R}\ y \iff (x\ R\ y\ \ \&\ \ \neg\, y\ R\ x).$$

We define the **symmetric part** $\widetilde{R}$ by

$$x\ \widetilde{R}\ y \iff (x\ R\ y\ \ \&\ \ y\ R\ x).$$

In terms of their graphs (subsets of $X \times X$) we have gr $R=$ gr $\widetilde{R} \bigcup$ gr $\widehat{R}$ and gr $\widetilde{R} \bigcap$ gr $\widehat{R} = \varnothing$. When R is thought of as a preference relation "as good as," the symmetric part is an indifference relation and the asymmetric part is a strict preference relation.

The following definitions describe various kinds of binary relations. *Not all authors use the same terminology as I.* Each of these definitions should be interpreted as if prefaced by the appropriate universal quantifiers "for every x, y, z," etc. The symbol $\neg$ indicates negation.

A binary relation R on a set X is:

- **reflexive** if $x \; R \; x$.
- **irreflexive** if $\neg \, x \; R \; x$.
- **symmetric** if $x \; R \; y \implies y \; R \; x$. Note that symmetry does not imply reflexivity.
- **asymmetric** if $x \; R \; y \implies \neg \, y \; R \; x$. An asymmetric relation is irreflexive.
- **antisymmetric** if $(x \; R \; y \;\; \& \;\; y \; R \; x) \implies x = y$. An antisymmetric relation may or may not be reflexive.
- **transitive** if $(x \; R \; y \;\; \& \;\; y \; R \; z) \implies x \; R \; z$.
- **quasitransitive** if its asymmetric part $\widehat{R}$ is transitive.
- **acyclic** if its asymmetric part $\widehat{R}$ has no cycles. (A cycle is a finite set $x_1, \ldots, x_n = x_1$ satisfying $x_1 \; \widehat{R} \; x_2, \ldots, x_{n-1} \; \widehat{R} \; x_n = x_1$.)
- **negatively transitive** if $(\neg \, x \; R \; y \;\; \& \;\; \neg \, y \; R \; z) \implies \neg \, x \; R \; z$.
- **complete**, or **connected**, if either $x \; R \; y$ or $y \; R \; x$ or both. Note that a complete relation is reflexive.
- **total**, or **weakly connected**, if $x \neq y$ implies either $x \; R \; y$ or $y \; R \; x$ or both. Note that a total relation may or may not be reflexive. Some authors call a total relation complete.
- a **partial order** if it is a reflexive, transitive, antisymmetric relation. Some authors (notably Kelley [35]) do not require a partial order to be reflexive.
- a **linear order** if it is a total, transitive, antisymmetric relation; a total partial order, if you will. It obeys the following **trichotomy law**: For every pair x, y exactly one of $x \; R \; y$, $y \; R \; x$, or $x = y$ holds.
- an **equivalence relation** if it is reflexive, symmetric, and transitive.
- a **preorder**, or **quasiorder**, if it is reflexive and transitive. An antisymmetric preorder is a partial order.

The **transitive closure** of the binary relation R on X is the binary relation T on X defined by

$$x \; T \; y \text{ if there exists a finite sequence } z_1, \dots, z_n \text{ such that } x = z_1 \; R \cdots R \; z_n = y.$$

Clearly T is transitive, extends R, and its graph is the smallest of any transitive relation that extends S.

A.1 Equivalence relations

Equivalence relations are among the most important. As defined above, an **equivalence relation** is a reflexive, symmetric, and transitive relation, often denoted $\sim$. Equality is an equivalence relation. Given any function f with domain X, we can define an equivalence relation $\sim$ on X by $x \sim y$ if and only if $f(x) = f(y)$.

Given an equivalence relation $\sim$ on X, for any $x \in X$, the **equivalence class** of x is defined to be $\{y \in X : y \sim x\}$. It is often denoted $[x]$. It is easy to see that if $x \sim y$, then $[x] = [y]$. Also if $[x] \bigcap [y] \neq \varnothing$, then $x \sim y$ and $[x] = [y]$. Since $\sim$ is reflexive, for each x we have $x \in [x]$. Thus the collection of equivalence classes of $\sim$ form a **partition** of X.

The symmetric part of a reflexive transitive relation is an equivalence relation. Thus the indifference relation derived from a regular preference is an equivalence relation. The equivalence classes are often called **indifference curves**.

A.2 Orders and such

A **partial order** (or partial ordering, or simply **order**) is a reflexive, transitive, and antisymmetric binary relation. It is traditional to use the symbol $\geqslant$ to denote a partial order. A set X equipped with a partial order is a **partially ordered set**, sometimes called a **poset**. A **total order** or **linear order** $\geqslant$ is a partial order with the property that if $x \neq y$, then either $x \geqslant y$ or $y \geqslant x$. That is, a total order is a partial order that is total. A **chain** in a partially ordered set is a subset on which the order is total. That is, any two distinct elements of a chain are ranked by the partial order. In a partially ordered set the notation $x > y$ means $x \geq y$ and $x \neq y$.

Let X be a partially ordered set. An **upper bound** for a set $A \subset X$ is an element $x \in X$ satisfying $x \geqslant y$ for all $y \in A$. An element x is a **maximal element** of X if there is no y in X for which $y > x$. Similarly, a **lower bound** for A is an $x \in X$ satisfying $y \geqslant x$ for all $y \in A$. **Minimal elements** are defined analogously. A **greatest element** of A is an $x \in A$ satisfying $x \geqslant y$ for all $y \in A$. **Least elements** are defined in the obvious fashion. Clearly every greatest element is maximal, and if $\geqslant$ is complete, then every maximal element is greatest.

B Zorn's Lemma

There are a number of propositions that are equivalent to the Axiom of Choice in Zermelo–Frankel Set Theory. One of the most useful of these is Zorn's Lemma, due to M. Zorn [70]. That is, Zorn's Lemma is a theorem if the Axiom of Choice is assumed, but if Zorn's Lemma is taken as an axiom, then the Axiom of Choice becomes a theorem. For a thorough discussion of Zorn's Lemma and its equivalent formulations see Rubin and Rubin [50]. In addition, Halmos [26] and Kelley [35, Chapter 0] have extended discussions of the Axiom of Choice.

53 Zorn's Lemma *If every chain in a partially ordered set X has an upper bound, then X has a maximal element.*

C Extension of preorders

It is always possible to extend any binary relation R on a set X to the total relation S defined by $x\ S\ y$ for all x, y. (For this appendix, S is just a binary relation, not the Samuelson revealed preference relation.) But this is not very interesting since it destroys any asymmetry present in R. Let us say that the binary relation S on a set X is a **compatible extension** of the relation R if S extends R and preserves the asymmetry of R. That is, $x\ R\ y \implies x\ S\ y$, and $x\ \widehat{R}\ y$ implies $x\ \widehat{S}\ y$, where as above, the $\widehat{\ }$ indicates the asymmetric part of a relation. The following theorem is due to Szpilrajn [60]. (Szpilrajn proved that every partial order has a compatible extension to a linear order, but the proof is the same.)

54 Szpilrajn's Theorem on total extension of preorders *Any preorder has a compatible extension to a total preorder.*

Proof: Let R be a preorder on X. That is, R is a reflexive and transitive binary relation on the set X. For this proof we identify a relation with its graph, so an extension of a relation can be thought of as a superset of the relation. Let us now commit the sin of identifying a relation with its graph, and let $\mathcal{S}$ be the collection of all reflexive and transitive compatible extensions of R, partially ordered by inclusion of graphs as subsets of $X \times X$, and let $\mathcal{C}$ be a nonempty chain in $\mathcal{S}$. (The collection $\mathcal{S}$ is nonempty since R itself belongs to $\mathcal{S}$.) We claim that the binary relation $T = \bigcup\{S : S \in \mathcal{C}\}$ is an upper bound for $\mathcal{C}$ in $\mathcal{S}$. Clearly $x\ R\ y \implies x\ T\ y$, and T is reflexive. To see that T is transitive, suppose $x\ T\ y$ and $y\ T\ z$. Then $x\ S_1\ y$ and $y\ S_2\ z$ for some $S_1, S_2 \in \mathcal{C}$. Since $\mathcal{C}$ is chain, $\operatorname{gr} S_1 \subset \operatorname{gr} S_2$ or $\operatorname{gr} S_2 \subset \operatorname{gr} S_1$. Either way $x\ S_i\ y$ and $y\ S_i\ z$ for some i. Since S_i is transitive, $x\ S_i\ z$, so $x\ T\ z$.

Suppose now that $x\ \widehat{R}\ y$, that is, $x\ R\ y$ and $\neg\, y\ R\ x$. By compatibility $\neg\, y\ S\ x$ for any S in $\mathcal{S}$, it follows that $\neg\, y\ T\ x$. Thus T is a reflexive and transitive compatible extension of R, and T is also an upper bound for $\mathcal{C}$ in $\mathcal{S}$. Therefore by Zorn's Lemma 53, the collection $\mathcal{S}$ of compatible extensions of R has a maximal element.

We now show that any maximal element of $\mathcal{S}$ must be a total relation. So fix S in $\mathcal{S}$, and suppose that S is not total. Then there is a pair $\{x, y\}$ of distinct elements such that neither $x \ S \ y$ nor $y \ S \ x$. Define the relation T by $\operatorname{gr} T = \operatorname{gr} S \cup \Big\{(x, y)\Big\}$, and let W be the transitive closure of T. Clearly W is transitive, and extends R, since S does. We now verify that W is a compatible extension of S. Suppose, by way of contradiction, that $u \ S \ v$, $\neg v \ S \ u$, but $v \ W \ u$ for some u, v. By the definition of W as the transitive closure of T, there exists a finite sequence $v = u_0, u_1, \dots, u_n = u$ of elements of X with $v = u_0 \ T \ u_1 \cdots u_{n-1} \ T \ u_n = u$. Since T differs from S only in that its graph contains the ordered pair (x, y), and S is irreflexive and transitive-, it follows that for some i, $x = u_i$ and $y = u_{i+1}$. (To see this, suppose $v = u_0 \ S \ u_1 \cdots u_{n-1} \ S \ u_n = u$, so $v \ S \ u$. But by hypothesis, $\neg v \ S \ u$, a contradiction.) We can find such a sequence in which x occurs once, so $y = u_{i+1} \ T \ u_{i+2} \cdots u_{n-1} \ T \ u_n = u \ S \ v = u_0 \ T \ u_1 \cdots u_{i-1} \ T \ u_i = x$. In each of these links we may replace T by S, and conclude that $y \ S \ x$, a contradiction. Therefore W is a compatible extension of R, and since it strictly includes S, we see that S cannot be maximal in $\mathcal{S}$. Therefore any maximal element must be total. ∎

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