Introduction to Capital Theory

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These notes are based on the wonderful book *Income, Wealth, and the Maximum Principle* by Martin L. Weitzman [16] and the paper by Robert Dorfman [3, 4].

1 Present discounted value

If you can invest one dollar at an annual rate of interest r, then in one year you will have $1 + r$ dollars. Equivalently, one dollar in one year is equivalent to $1/(1 + r)$ dollars today. If interest is compounded annually, one dollar today will be worth $(1 + r)^t$ dollars in t years. Thus one dollar in t years is worth $1/(1 + r)^t$ dollars today.

Interest is often compounded more frequently than annually. If interest is compounded n times annually, the annual interest rate is divided by n and credited at the end of each $1/n$th year, so the value in t years is $(1 + \frac{r}{n})^{nt}$. Now

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{nt} = e^{rt},$$

so with **continuous compounding** the **present discounted values** of a dollar at time t is just e^{-rt} today.

We shall measure time so that "now" is $t = 0$. Consider now a "flow" of income $f(t)$ at time t, for $t \geqslant 0$. Its present discounted value is

$$\int_0^\infty f(t)e^{-rt}\,dt.$$

See the Appendix for a generalization of this to time-varying interest rates.

2 A typical investment problem

A small firm has a **capital stock** that it uses to produce a *flow* of **income**. Let $f(K)$ denote the flow of gross income that the firm produces using capital level K, and assume that f is twice continuously differentiable, strictly concave, and strictly increasing with $f' > 0$ and $f'' < 0$. We assume that capital depreciates at the constant rate δ. Then the flow of **net income** is

$$F(K) = f(K) - \delta K.$$

We assume that depreciation is the only way for the firm to disinvest and reduce its capital stock. We also assume for simplicity that there is a maximal rate $\bar{I}$ of increase of the capital stock.

The firm can borrow and lend at the market interest rate $r > 0$, so what it cares about is the present discounted value of its net income. What the firm must choose is a time path, or **trajectory** of its **control variable** I over the time horizon $T = [0, \infty)$. (In what follows I shall use bold letters to denote trajectories.) The control influences the **state variable** K through the differential equation

$$\dot{\boldsymbol{K}}(t) = \boldsymbol{I}(t),$$

and it faces the constraints

$$K(0) = K_o,$$

and

$$-\delta \boldsymbol{K}(t) \leqslant \boldsymbol{I}(t) \leqslant \bar{I}.$$

Its goal is to maximize

$$\int_0^\infty \big[F\big(\boldsymbol{K}(t)\big) - \boldsymbol{I}(t)\big]\, e^{-rt}\, dt.$$

You might think that solving this requires a manager who is very far-sighted and can balance the trade-off between investing more now at the expense of current income to provide more income in the future,

> but in fact there are "prices" $\boldsymbol{p}$ that temporally decentralize this problem so that each instant t the manager chooses the level of investment $\boldsymbol{I}(t)$ to maximize a simple function of K, I, and p, called the **Hamiltonian**. The entire trade-off is summarized at each point in time by the value $\boldsymbol{p}(t)$.

3 A more general mathematical formulation

This is a special case of the following maximization problem,

$$\underset{\boldsymbol{I}}{\text{maximize}} \int_0^\infty G\big(\boldsymbol{K}(t), \boldsymbol{I}(t)\big) e^{-rt}\, dt$$

subject to the constraints (1)–(4).

$$\dot{\boldsymbol{K}}(t) = \boldsymbol{I}(t). \tag{1}$$

(Here I use the traditional dot notation to indicate derivatives with respect to time.)

$$\boldsymbol{K}(0) = K_o. \tag{2}$$

$$m\big(\boldsymbol{K}(t)\big) \leqslant \boldsymbol{I}(t) \leqslant M\big(\boldsymbol{K}(t)\big) \qquad t \geqslant 0, \tag{3}$$

where m and M are known functions that of course satisfy $m(K) \leqslant M(K)$ for all $K \geqslant 0$.[1] We shall require that m be convex and decreasing and M be concave and increasing. For many problems m is the zero function.

$$\boldsymbol{K}(t) \geqslant 0 \qquad t \geqslant 0. \tag{4}$$

The function G is the instantaneous **gain function**, and r is the **discount rate**. The arguments of G are the current levels of the **state variable** K, and the **control variable** I.

1 Assumption *G is concave and continuously differentiable, and satisfies $G(0,0) = 0$, and $D_1 G \geqslant 0$ and $D_2 G < 0$ everywhere.*

Admissible controls

We restrict attention to control trajectories that are piecewise continuous. In other words, $\boldsymbol{I}$ is required to have at most finitely many discontinuities in any finite time interval.

[1] The general formulation of the Maximum Principle allows for additional constraints and a more general objective function. In particular G may have t as an argument, which allows for nondiscounting and for finite time horizons.

Steady states

A **steady state** is a pair $(\boldsymbol{K}, \boldsymbol{I})$ of trajectories satisfying

$$\boldsymbol{K}(t) = K, \quad \boldsymbol{I}(t) = 0 \quad \text{for all} \quad t \geqslant 0.$$

We shall refer to a steady state by the level K of the capital stock it maintains. A steady state may or may not be optimal. So fix $K > 0$ and let

$$\varphi_0 = \int_0^\infty G(K,0)e^{-rt}\,dt = G(K,0)/r,$$

the present value of the steady state K. A standard technique from the calculus of variations is to look at trajectory and consider a variation on it. The variation on $\boldsymbol{I} = 0$ that I want to consider is this. Invest at the rate ε/δ for a short time δ to increase the capital stock to $K + \varepsilon$. Intuitively, this incurs an "instantaneous" cost on the order of $D_2(K,0)\varepsilon$ now, but provides an increment in the present discounted value of the flow of $D_1G(K,0)\varepsilon/r$. Thus it will pay to adjust the capital stock up or down by ε unless $-D_1G(K,0)/D_2G(K,0) = r$. Weitzman defines the **stationary return on capital** by

$$R(K) = -\frac{D_1G(K,0)}{D_2G(K,0)}.$$

We now make some tedious approximation arguments to make this intuition rigorous.

So let $\varepsilon > 0$ and define the piecewise continuous trajectory

$$\boldsymbol{v}_{\varepsilon,\delta}(t) = \begin{cases} \frac{\varepsilon}{\delta} & t < \delta \\ 0 & t \geqslant \delta. \end{cases}$$

where $\delta > 0$. This leads to the following capital stock trajectory:

$$\boldsymbol{K}_{\varepsilon,\delta}(t) = \begin{cases} K + \frac{\varepsilon}{\delta}t & t < \delta \\ K + \varepsilon & t \geqslant \delta \end{cases},$$

and the resulting value is

$$\begin{aligned}
\varphi(\varepsilon,\delta) &= \int_0^\infty G\Big(\boldsymbol{K}_{\varepsilon,\delta}(t), \boldsymbol{v}_{\varepsilon,\delta}(t)\Big)e^{-rt}\,dt \\
&= \int_0^\delta G\Big(K + \frac{\varepsilon}{\delta}t, \frac{\varepsilon}{\delta}\Big)e^{-rt}\,dt + \int_\delta^\infty G(K+\varepsilon,0)e^{-rt}\,dt \\
&\leqslant \int_0^\delta G\Big(K, \frac{\varepsilon}{\delta}\Big)e^{-rt}\,dt + \int_\delta^\infty G(K+\varepsilon,0)e^{-rt}\,dt.
\end{aligned}$$

Since concave functions lie below their tangent lines (the supergradient inequality) we have

$$G\Big(K, \frac{\varepsilon}{\delta}\Big) \leqslant G(K,0) + D_2G(K,0)\frac{\varepsilon}{\delta}$$

and

$$G(K+\varepsilon,0) \leqslant G(K,0) + D_1G(K,0)\varepsilon,$$

so

$$\varphi(\varepsilon,\delta) \leqslant \int_0^\delta \Big(G(K,0) + D_2G(K,0)\frac{\varepsilon}{\delta}\Big)e^{-rt}\,dt + \int_0^\infty \Big(G(K,0) + D_1G(K,0)\varepsilon\Big)e^{-rt}\,dt = \bar\varphi(\varepsilon,\delta).$$

If $\varphi(\varepsilon,\delta) > \varphi_0$, then the steady state K cannot be optimal and $\bar\varphi(\varepsilon,\delta) > \varphi_0$. That is,

$$\begin{aligned}
0 < \bar\varphi(\varepsilon,\delta) - \varphi_0 &= \int_0^\delta \Big(D_2G(K,0)\frac{\varepsilon}{\delta}\Big)e^{-rt}\,dt + \int_\delta^\infty D_1G(K,0)\varepsilon e^{-rt}\,dt \\
&\leqslant \int_0^\delta \Big(D_2G(K,0)\frac{\varepsilon}{\delta}\Big)e^{-r\delta}\,dt + \int_0^\infty D_1G(K,0)\varepsilon e^{-rt}\,dt
\end{aligned},$$

(remember $D_2G < 0$)

$$= D_2G(K,0)\varepsilon e^{-r\delta} + \frac{D_1G(K,0)\varepsilon}{r}$$

which for $\varepsilon > 0$ implies (given that $D_2 G < 0$)

$$-\frac{D_1 G(K,0)}{D_2 G(K,0)} > re^{-r\delta}.$$

Now if the steady state K is optimal, this variation cannot be an improvement so the reverse inequality above most hold. Letting $\delta \downarrow 0$ we see that i K is an optinmal steady state, then

$$R(K) = -\frac{D_1 G(K,0)}{D_2 G(K,0)} \leqslant r.$$

We now have to consider the variations with $\varepsilon < 0$. The above argument shows that for this case, if K is an optimal steady state, then

$$R(K) = -\frac{D_1 G(K,0)}{D_2 G(K,0)} \geqslant r.$$

So if a steady state $K > 0$ is optimal, then

$$R(K) = -\frac{D_1 G(K,0)}{D_2 G(K,0)} = r \tag{5}$$

must necessarily hold.

Related functions

We now define three related functions to the problem above. The first is the **value function** V. It is the maximized value of the objective function as a function of the initial capital stock. That is,

$$V(K) = \max_{\boldsymbol{I}} \int_0^\infty G\big(\boldsymbol{K}(t), \boldsymbol{I}(t)\big) e^{-rt}\, dt$$

where the maximum is taken with respect to trajectories satisfying the constraints (1)–(4) with $K_o = K$. This of course assumes that a maximum exists for $K_o = K$. Also, if we want to index the problem by K, we really ought to index the optimal trajectories by K, but we shan't. The thing to note about the value function is that if satisfies **Bellman's Principle of Optimality**, which states that if $\boldsymbol{I}^*, \boldsymbol{K}^*$ are optimal trajectories starting at $\boldsymbol{K}(0) = K_o$, then for any time $t \geqslant 0$,

$$V(K_o) = \int_0^t G\big(\boldsymbol{K}^*(s), \boldsymbol{I}^*(s)\big) e^{-rs}\, ds + e^{-rt} V\big(\boldsymbol{K}^*(t)\big). \tag{6}$$

What this says is that when the capital stock reaches $\boldsymbol{K}^*(t)$ at time t, the optimal continuation is the same as if we reset the clock to zero, and then followed the optimal trajectory for $K_o = \boldsymbol{K}^*(t)$. This implies that if an optimal trajectory $\boldsymbol{K}^*$ exists starting at K_o, then an optimal trajectory exists for every starting value $\boldsymbol{K}^*(t)$. In particular, V is defined for every $\boldsymbol{K}^*(t)$. Since $\boldsymbol{K}^*$ has a derivative (namely $\boldsymbol{I}$), it is continuous, so its range is an interval. Thus V must be defined on some interval (perhaps degenerate).

The next function we define is the **Hamiltonian** (more precisely, the **current value Hamiltonian**) for the problem,

$$H(K, I, p) = G(K, I) + pI.$$

It is the sum of the gain function and a **multiplier** or **costate** variable p multiplying the function that defines $\dot{\boldsymbol{K}}$. Why we do this will become apparent later. Closely related is the **maximized Hamiltonian** $\tilde{H}$, defined by

$$\tilde{H}(K, p) = \max_{I : m(K) \leqslant I \leqslant M(K)} H(K, I, p).$$

Theorem

Assumptions

Here are the assumptions that Weitzman uses. They are satisfied for many economic problems. He notes that there are weaker assumptions under which the theorem remains true, but they are less easy to understand and the proofs are less intuitive.

1. G is concave and continuously differentiable, $G(0,0) = 0$, and $D_1 G \geqslant 0$ and $D_2 G < 0$ everywhere.

2. The functions m and M are twice continuously differentiable, m is convex and increasing, and M is concave and nondecreasing (so for $K > 0$, $m'(K) \leqslant 0$, $m''(K) \geqslant 0$, $M'(K) \geqslant 0$, $M''(K) \leqslant 0$). In addition, for $K \geqslant 0$, we assume $m(K) \leqslant 0 \leqslant M(K)$. To make sure that K never becomes negative, we also assume $m(0) = 0$. Even if we do not allow capital to be consumed, it may still depreciate, in which case we generally take $m(K) = -\delta K$.

3. An optimal trajectory exists for $K_o = 0$.

4. **Accessibility Hypothesis:** Define $R(K) = -D_1 G(K,0)/D_2 G(K,0)$. If there exists $\hat{K}$ satisfying $R(\hat{K}) = r$ (that is, $\hat{K}$ is a candidate for an optimal steady state), then $R'(\hat{K}) < 0$ (which implies $\hat{K}$ is locally unique) and $m(\hat{K}) < 0 < M(\hat{K})$ (which implies that $\hat{K}$ is accessible from both sides). Note that this rules out $m(K) = 0$ for all K if such a $\hat{K}$ exists.

The (One-Dimensional) Maximum Principle

Under the assumptions above, the pair of trajectories $(\boldsymbol{K}^*, \boldsymbol{I}^*)$ is optimal (within the class of piecewise continuous trajectories) if and only if there exists a trajectory $\boldsymbol{p}^*$ of the costate variable such that for all $t \geqslant 0$,

$$\boldsymbol{p}^*(t) \geqslant 0; \tag{7}$$

at each time t, $\boldsymbol{I}(t)$ is chosen to

$$\underset{I}{\text{maximize}}\, H\big(\boldsymbol{K}(t), I, \boldsymbol{p}(t)\big) \quad \textit{subject to} \quad m\big(\boldsymbol{K}(t)\big) \leqslant I \leqslant M\big(\boldsymbol{K}(t)\big),$$

that is,

$$H\big(\boldsymbol{K}^*(t), \boldsymbol{I}^*(t), \boldsymbol{p}^*(t)\big) = \tilde{H}\big(\boldsymbol{K}^*(t), \boldsymbol{p}^*(t)\big); \tag{8}$$

the trajectory $\boldsymbol{p}^*$ satisfies

$$\dot{\boldsymbol{p}}^*(t) = -D_1 \tilde{H}\big(\boldsymbol{K}^*(t), \boldsymbol{p}^*(t)\big) + r\boldsymbol{p}^*(t); \tag{9}$$

*and the following **transversality condition** holds,*

$$\lim_{t \to \infty} \boldsymbol{p}^*(t)\boldsymbol{K}^*(t)e^{-rt} = 0. \tag{10}$$

Moreover, the value function V is concave, continuously differentiable, nondecreasing, nonnegative, and its derivative is the costate variable. That is, for all $t \geqslant 0$,

$$\boldsymbol{p}^*(t) = V'\big(\boldsymbol{K}^*(t)\big).$$

For a proof see Chapter 3 of Weitzman [16]. Let me just say here that the proof proceeds by defining the **wealth function**

$$W(t) = V\big(\boldsymbol{K}^*(t)\big),$$

and using Bellman's optimality principle

$$V(K_o) = \int_0^t G\big(\boldsymbol{K}^*(s), \boldsymbol{I}^*(s)\big)e^{-rs}\,ds + e^{-rt}\underbrace{V\big(\boldsymbol{K}^*(t)\big)}_{W(t)}.$$

to write

$$W(t) = e^{rt}\big[W(0) - \int_0^t G\big(\boldsymbol{K}^*(s), \boldsymbol{I}^*(s)\big)e^{-rs}\,ds\big],$$

which proves that W is differentiable. Since V is concave (this is easy to show), it has left- and right-hand derivatives. The chain rule for left- and right-hand derivatives (there is one) thus implies that V is differentiable and

$$V'\big(\boldsymbol{K}^*(t)\big) = \frac{\dot{W}(t)}{\boldsymbol{I}^*(t)},$$

provided $\boldsymbol{I}^*(t) \neq 0$. (The case $\boldsymbol{I}^*(t) = 0$ requires a bit more work.) This now allows us to define $\boldsymbol{p}^*(t)$ to be $V'\big(\boldsymbol{K}^*(t)\big)$, and the remaining properties follow by more or less standard methods. Since a differentiable concave function is continuously differentiable, we conclude that $\boldsymbol{p}^*(t)$ is continuous.

Commentary

On $\boldsymbol{p}^*$ and the Hamiltonian

According to the theorem, the costate variable $\boldsymbol{p}^*$ is the derivative of the value function, that is, it is the marginal value of a unit of capital to the firm, or the **shadow price** of investment. It is precisely the value of investment. The Hamiltonian is the sum

$$G(K, I) + pI,$$

the sum of the net income plus the value of investment. The fact that this is maximized at each point in time says that the firm should choose its investment to maximize the sum of its dividends G plus retained earnings $\boldsymbol{p}^*I$, where the retained capital I is valued at its true marginal value $\boldsymbol{p}^* = V'(\boldsymbol{K}^*)$.

On $\dot{\boldsymbol{p}}^*$

By the Envelope Theorem, *if $\boldsymbol{I}^*(t)$ is an interior maximizer of the Hamiltonian*, the derivative of the maximized Hamiltonian with respect to K or p is the partial derivative of the Hamiltonian. That is,

$$D_1\tilde{H}\big(\boldsymbol{K}^*(t), \boldsymbol{p}^*(t)\big) = D_1H\big(\boldsymbol{K}^*(t), \boldsymbol{I}^*(t), \boldsymbol{p}^*(t)\big) = D_1G\big(\boldsymbol{K}^*(t), \boldsymbol{I}^*(t)\big)$$

and

$$D_2\tilde{H}\big(\boldsymbol{K}^*(t), \boldsymbol{p}^*(t)\big) = D_3H\big(\boldsymbol{K}^*(t), \boldsymbol{I}^*(t), \boldsymbol{p}^*(t)\big) = \boldsymbol{I}^*(t).$$

In this case, (9) can be rewritten as

$$\dot{\boldsymbol{p}}^*(t) = -D_1G\big(\boldsymbol{K}^*(t), \boldsymbol{I}^*(t)\big) + r\boldsymbol{p}^*(t). \tag{11}$$

This can be interpreted as a **no-arbitrage condition**. At time t I can buy Δ units of capital at a price $\boldsymbol{p}(t)$ and use it earn an incremental flow of income at the rate $D_1G \cdot \Delta$ for a length of time ε, and then resell it time $t + \varepsilon$ at a price $\boldsymbol{p}(t + \varepsilon)$. The gain from this is

$$\Delta\big[\boldsymbol{p}(t + \varepsilon) - \boldsymbol{p}(t) + \varepsilon D_1G\big].$$

Or I could take $\boldsymbol{p}(t)\Delta$ and lend it at the interest rate r for a period of length ε and earn $\boldsymbol{p}(t)\Delta\varepsilon r$. Absence of arbitrage implies that these two strategies must have the same return, or

$$\boldsymbol{p}(t+\varepsilon) - \boldsymbol{p}(t) + \varepsilon D_1 G = \boldsymbol{p}(t)\varepsilon r.$$

Dividing by ε and taking the limit as $\varepsilon \to 0$ implies (11).

Stationary optima

A stationary optimum need not exist, but suppose $\hat{K} > 0$ is a stationary optimum. That is, if $K_0 = \hat{K}$, then $\boldsymbol{K}^*(t) = \hat{K}$ for all $t \geqslant 0$. Then $\boldsymbol{I}^*(t) = 0$ for all t. If this is an interior maximizer of the Hamiltonian, then the first order condition implies

$$D_2 G(\hat{K}, 0) + \boldsymbol{p}^*(t) = 0,$$

so that $\boldsymbol{p}^*$ must also be constant. Then (9) and (11) imply

$$-D_1 G\big(\boldsymbol{K}^*(t), \boldsymbol{I}^*(t)\big) + r\boldsymbol{p}^*(t) = 0,$$

or in terms of the stationary return on capital function R,

$$R(\hat{K}) = r.$$

The transversality condition

To get a feel for the necessity of the transversality condition, first observe that it is satisfied in every stationary optimum, as $\boldsymbol{p}^*$ and $\boldsymbol{K}^*$ are constant and $e^{-rt} \to 0$. For the general case, we use the concavity of V. Since concave function lie below their tangent lines $V(0) \leqslant V(K) + V'(K)(0-K)$ for all K. In particular, for $K = \boldsymbol{K}^*(t)$, using the fact that $\boldsymbol{p}^*(t) = V'\big(\boldsymbol{K}^*(t)\big)$, we can rearrange this to get

$$V\big(\boldsymbol{K}^*(t)\big) - V(0) \geqslant \boldsymbol{p}^*(t)\boldsymbol{K}^*(t) \geqslant 0$$

for all $t > 0$. Thus

$$e^{-rt}\left(V\big(\boldsymbol{K}^*(t)\big) - V(0)\right) \geqslant e^{-rt}\boldsymbol{p}^*(t)\boldsymbol{K}^*(t) \geqslant 0. \tag{12}$$

By the Principle of Optimality (6)

$$\begin{aligned} e^{-rt}V\big(\boldsymbol{K}^*(t)\big) &= V(K_o) - \int_0^t G\big(\boldsymbol{K}^*(s), \boldsymbol{I}^*(s)\big)e^{-rs}\,ds \\ &= \int_t^\infty G\big(\boldsymbol{K}^*(s), \boldsymbol{I}^*(s)\big)e^{-rs}\,ds \end{aligned}$$

Since the integral is convergent, we have

$$\lim_{t\to\infty} e^{-rt}V\big(\boldsymbol{K}^*(t)\big) = \lim_{t\to\infty}\int_t^\infty G\big(\boldsymbol{K}^*(s), \boldsymbol{I}^*(s)\big)e^{-rs}\,ds = 0.$$

Thus (12) implies

$$\lim_{t\to\infty} e^{-rt}\boldsymbol{p}^*(t)\boldsymbol{K}^*(t) = 0.$$

The economics of the transversality condition

The transversality condition also has an economic interpretation as another no-arbitrage condition. Suppose it failed—that is, suppose that

$$\limsup_{t\to\infty} e^{-rt}\boldsymbol{p}^*(t)\boldsymbol{K}^*(t) = A > 0.$$

Suppose I adopt the strategy of running the firm until time T, then selling it and investing the proceeds at the interest rate r. The present value of this is

$$\int_0^T G\big(\boldsymbol{K}^*(t), \boldsymbol{I}^*(t)\big)e^{-rt}\,dt + \boldsymbol{p}^*(T)\boldsymbol{K}^*(T)e^{-rT}.$$

By choosing T large enough I can make this arbitrarily close to

$$\int_0^\infty G\big(\boldsymbol{K}^*(t), \boldsymbol{I}^*(t)\big)e^{-rt}\,dt + A,$$

creating an arbitrage profit of just less than A. In order for this not to be profitable, the transversality condition must hold.

The present value Hamiltonian

The Hamiltonian H I defined is called the **current value Hamiltonian**. Some authors prefer to work with the **present value Hamiltonian**

$$\begin{aligned}\hat{H}(K, I, t, q) &= e^{-rt}G(K, I) + qI \\ &= e^{-rt}H(K, I, e^{rt}q).\end{aligned}$$

So in other words

$$H(K, I, p) = e^{rt}\hat{H}(K, I, t, e^{-rt}p).$$

Thus if $\boldsymbol{I}^*(t)$ maximizes $H(K, I, p)$ it will also maximize $\hat{H}(K, I, t, e^{-rt}p)$. Thus the **maximized present value Hamiltonian** $\tilde{\hat{H}}$ satisfies

$$\tilde{\hat{H}}(K, t, q) = \tilde{H}(K, e^{rt}q).$$

Then defining $\boldsymbol{q}^*(t) = e^{-rt}\boldsymbol{p}^*(t)$, the transversality condition becomes

$$\boldsymbol{q}^*(t)\boldsymbol{K}^*(t) \to 0 \text{ as } t \to \infty.$$

Also

$$\begin{aligned}\dot{q}^*(t) &= -re^{-rt}\boldsymbol{p}^*(t) + e^{-rt}\dot{\boldsymbol{p}}^*(t) \\ &= -re^{-rt}\boldsymbol{p}^*(t) + e^{-rt}\left(-D_1\tilde{H}\big(\boldsymbol{K}^*(t), \boldsymbol{p}^*(t)\big) + r\boldsymbol{p}^*(t)\right) \\ &= -D_1\tilde{\hat{H}}\big(\boldsymbol{K}^*(t), e^{-rt}\boldsymbol{p}^*(t)\big) \\ &= -D_1\tilde{\hat{H}}\big(\boldsymbol{K}^*(t), \boldsymbol{q}^*(t)\big)\end{aligned}$$

The Wealth and Income Version of the Maximum Principle

This statement is sometimes called the Hamilton–Jacobi formulation, or Jacobi's integral form of Hamilton's equations of motion.

Under the assumptions here, the feasible trajectories $(\boldsymbol{K}^*, \boldsymbol{I}^*)$ *are optimal if and only if there exists a continuous nonnegative price trajectory* $\boldsymbol{p}^*$ *satisfying for all* $t \geqslant 0$,

$$\begin{aligned} rV\big(\boldsymbol{K}^*(t)\big) &= G\big(\boldsymbol{K}^*(t), \boldsymbol{I}^*(t)\big) + \boldsymbol{p}^*(t)\boldsymbol{I}^*(t) \\ &= \tilde{H}\big(\boldsymbol{K}^*(t), \boldsymbol{p}^*(t)\big). \end{aligned} \tag{13}$$

Let's interpret this in economic terms. On the left-hand side, $V\big(\boldsymbol{K}^*(t)\big)$ is the value of the optimal time-t capital stock, in other words it is **market value of the firm's equity shares** (wealth). So $rV\big(\boldsymbol{K}^*(t)\big)$ is flow of interest that this equity would generate if invested at the market rate of interest (income). It is equated to the right-hand side, which consists of two parts: $G\big(\boldsymbol{K}^*(t), \boldsymbol{I}^*(t)\big)$, the instantaneous net income, that is, the **dividends paid out**; plus $\boldsymbol{p}^*(t)\boldsymbol{I}^*(t)$, the value of the optimal time-t investment at prices $\boldsymbol{p}^*(t)$, which is the firm's **internal shadow price of capital**.

4 Appendix: The economics of first-order linear differential equations

This really has nothing to do with the maximum principle, but it's fun. The following theorem is a standard statement of the solution to a first order linear differential equation. I took it from Apostol [1, Theorems 8.2 and 8.3, pp. 309–310].

2 Theorem (First order linear differential equation) *Assume* P, Q *are continuous on the open interval* I. *Let* $a \in I$, $b \in \boldsymbol{R}$.

Then there is one and only one function $y = f(x)$ *that satisfies the initial value problem*

$$y' + P(x)y = Q(x) \tag{14}$$

with $f(a) = b$. *It is given by*

$$f(x) = be^{-A(x)} + e^{-A(x)} \int_a^x Q(t)e^{A(t)}\,dt$$

where

$$A(x) = \int_a^x P(t)\,dt.$$

The theorem appears a bit mysterious in this form, but I can give it an economic interpretation that makes it obvious (at least to me). The first thing we will do is change the variable on which y depends from x to time, t.

Interpret $y(t)$ as the value of a savings account at time t. At each point of time it earns an instantaneous rate of return $r(t)$. Moreover, we add a "flow" of additional savings to the account at the rate $s(t)$. Thus the rate of change of the value of the account is

$$y'(t) = r(t)y(t) + s(t). \tag{15}$$

Moreover, let's rewrite the initial condition as $y(t_0) = y_0$. This yields the following translation.

3 Theorem (First order linear differential equation) *Assume* r, s *are continuous on the open interval* I. *Let* $t_0 \in I$, $y_0 \in \boldsymbol{R}$.

Then there is one and only one function y *that satisfies the initial value problem*

$$y' = r(t)y + s(t)$$

with $y(t_0) = y_0$*. It is given by*

$$y(t) = [y_0 + S(t)]\, e^{\overline{r}(t)(t-t_0)}$$

where

$$\overline{r}(t) = \frac{1}{t-t_0} \int_{t_0}^{t} r(\tau)\, d\tau$$

and

$$S(t) = \int_{t_0}^{t} s(\tau) e^{-\overline{r}(\tau)(\tau-t_0)}\, d\tau.$$

But this version is obviously true!

Proof: We rely on the following well-known (easily proved) result:

$$\lim_{n\to\infty} \big(1 + (r/n)\big)^{nt} = e^{rt}.$$

That is, the result of compounding interest on a dollar continuously over t periods is e^{rt} dollars.

Case 1: $s = 0$. If the **instantaneous rate of return** at time t is $r(t)$, the **average rate of return** $\overline{r}(t)$ over the interval $[t_0, t]$ is just

$$\overline{r}(t) = \frac{1}{t-t_0} \int_{t_0}^{t} r(\tau)\, d\tau.$$

Now if we add nothing to the initial investment over time, that is, if $s(t) = 0$ for all t, then I claim that the value of the account at time t is given by

$$y(t) = y_0 e^{\overline{r}(t)(t-t_0)}. \tag{16}$$

That is, earning the varying rate of return r over the interval $[t_0, t]$ is the same as earning the average rate of return $\overline{r}$ over the interval. We can verify this by showing that y given by (16) solves (15).

$$\begin{aligned}
\frac{dy}{dt} &= \frac{d}{dt} y_0 e^{\overline{r}(t)(t-t_0)} \\
&= y_0 e^{\overline{r}(t)(t-t_0)} \frac{d}{dt}\left(\overline{r}(t)(t-t_0)\right) \\
&= y_0 e^{\overline{r}(t)(t-t_0)} \frac{d}{dt} \int_{t_0}^{t} r(\tau)\, d\tau \\
&= y_0 e^{\overline{r}(t)(t-t_0)} r(t) \\
&= r(t) y(t),
\end{aligned}$$

which is (15) with $s = 0$.

Case 2: General s. But in general, the additional savings $s(t)$ is not zero. In order to deal with the general case, we use the incredibly useful notion of **present value**. If you invest \$1 at time t_0 it will be worth $\$e^{\overline{r}(t)(t-t_0)}$ at time t, so the value at time t_0, that is,

the present value of \$1 at time t is $e^{-\overline{r}(t)(t-t_0)}$.

For if you invest $e^{-\overline{r}(t)(t-t_0)}$ at t_0, you will have $e^{-\overline{r}(t)(t-t_0)} e^{\overline{r}(t)(t-t_0)} = 1$ dollar at time t.

The present value of the flow $s(t)$ is $s(t)e^{-\overline{r}(t)(t-t_0)}$. The present value $S(t)$ of all the additional savings up to time t is thus

$$S(t) = \int_{t_0}^{t} s(\tau) e^{-\overline{r}(\tau)(\tau-t_0)}\, d\tau.$$

But at time t this present value will be worth an additional

$$S(t)e^{\overline{r}(t)(t-t_0)}.$$

Thus the total value of the savings account at time t is given by

$$y(t) = (y_0 + S(t))\, e^{\overline{r}(t)(t-t_0)}.$$

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