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# Brief Notes on the Arrow–Debreu–McKenzie Model of an Economy

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## 1 Commodities

The first primitive concept is that of a **commodity**. A commodity is any good or service that may be produced, consumed, or traded. Commodities may distinguished by date, location, and state of the world. For mathematical simplicity we usually assume there is a finite number $\ell$ of commodities. The **commodity space** is thus $\boldsymbol{R}^\ell$.

## 2 Technology

The next concept is that of a **production unit** or **enterprise** which is characterized by its **technology set** $Y$. For $y$ belonging to $Y$, $y_k < 0$ indicates that commodity $k$ is used as an input and $y_k > 0$ indicates that it is an output.

In general there may be $n$ enterprises.

Conditions on production.

1. There is a possibility of inaction. That is, $0 \in Y_j$ for each $j$.
2. The aggregate production set $Y = \sum_{j=1}^n Y_j$ is closed. (Note that each $Y^j$ may be closed without $Y$ being closed.)
3. The aggregate production set $Y = \sum_{j=1}^n Y_j$ is convex.
4. Production is irreversible. That is, $Y \cap (-Y) \subset \{0\}$.
5. There is free disposability. That is, if $y \in Y$, then $\{y\} - \boldsymbol{R}^\ell_+ \subset Y$.[1]

## 3 Tastes

The next concept is that of an idealized **consumer** or **household**. A consumer is partially described by a consumption set $X$, which is a subset of the commodity space. Elements $x$ of $X$ are ordered lists of quantities of commodities consumed. If $x_k < 0$ it indicates that commodity $k$ is a labor service being supplied. The other part of the description of a consumer is the consumer's **preference relation** $\succcurlyeq$ on $X$, which is generally assumed

[1] This condition is usually written as $-\boldsymbol{R}^\ell_+ \subset Y$. My formulation makes it easier to construct economies satisfying free disposability and irreversibility, yet violating the possibility of inaction.

to be transitive, total, and reflexive. The relation $x \succcurlyeq y$ is read $x$ is at least as good as $y$. The **strict preference** relation $\succ$ is defined by

$$x \succ y \qquad \text{if} \qquad x \succcurlyeq y \text{ but not } y \succcurlyeq x,$$

and **indifference** $\sim$ is defined by

$$x \sim y \qquad \text{if} \qquad x \succcurlyeq y \text{ and } y \succcurlyeq x.$$

The set $\{y \in X : y \sim x\}$ is the **indifference class** of $x$ or the **indifference curve** through $x$. The set $\{y \in X : y \succcurlyeq x\}$ is the **upper contour set** at $x$, and $\{y \in X : y \succ x\}$ is the **strict upper contour set** at $x$. The relation $y \preccurlyeq x$ means $x \succcurlyeq y$, etc.

In general there may be $m$ consumers.

We may make use of the following assumptions.

Conditions on consumption sets.

1. Each $X_i$ is closed.
2. Each $X_i$ is convex.
3. Each $X_i$ is bounded below.

Conditions on preferences.

1. Each $\underset{i}{\succcurlyeq}$ is nonsatiated.
2. Each $\underset{i}{\succcurlyeq}$ is continuous.
3. Preferences are convex. That is, if $x \underset{i}{\succ} y$, then for every $\lambda \in (0,1)$, if $(1-\lambda)x + \lambda y \in X_i$,[2] then
$$(1-\lambda)x + \lambda y \underset{i}{\succ} y.$$

# 4 Resources

The third element in the description of an economy is the aggregate endowment $\omega \in \boldsymbol{R}^{\ell}$. We typically assume $\omega \geqq 0$, but that is mainly a definition of what it means to be a resource.

# 5 Allocations

An **economy** is thus summarized by a list

$$E = \big((X_i, \underset{i}{\succcurlyeq})_{i=1}^{m}, (Y_j)_{j=1}^{n}, \omega\big).$$

An **allocation** for the economy $E$ is a list

$$(x^1, \ldots, x^m, y^1, \ldots, y^n)$$

satisfying

$$x^i \in X_i \quad i = 1, \ldots, m$$

[2]The provision is explicit so that violations of condition 2 do not imply a violation of 3.

$$y^j \in Y_j \quad j = 1, \dots, n$$
$$\sum_{i=1}^m x^i = \omega + \sum_{j=1}^n y^j.$$

A natural question is whether allocations exist at all. Let $X = \sum_{i=1}^m X^i$. The question is whether $X \cap (Y + \omega) \neq \varnothing$. The typical way to guarantee this is to assume $0 \in Y^j$ (possibility of inaction) for each producer $j$ and that for each consumer $i$ there is some $\hat{x}^i \in X^i$ such that $\hat{x}^1 + \cdots + \hat{x}^m = \omega$, or $\hat{x}^1 + \cdots + \hat{x}^m \leqq \omega$ and assume additionally that $Y$ exhibits free disposability.

## 6 Efficiency

An allocation $(\bar{x}^1, \dots, \bar{x}^m, \bar{y}^1, \dots, \bar{y}^n)$ is said to be **inefficient**[3] if there is another allocation $(x^1, \dots, x^m, y^1, \dots, y^n)$ such that

$$x^i \succsim_i \bar{x}^i \quad \text{for all } i,$$

and

$$x^i \succ_i \bar{x}^i \quad \text{for at least one } i.$$

An allocation is **efficient**[4] if it is not inefficient.

## 7 Prices

A **price system** assigns to each commodity a price per unit of that commodity in some unit of account, usually monetary. Prices may be zero, or even negative, but most models deal with nonnegative prices. In our framework, prices subsume wages. That is, when the commodity is used as a factor of production, its wage is the same thing as its price.

Given a commodity vector $x$, the dot product $p \cdot x = \sum_{k=1}^{\ell} p_k x_k$ gives the value of the commodity vector.

If $y$ is a production plan, then because of the sign convention on inputs and outputs $p \cdot y$ is the profit (revenues minus costs) associated with the production plan.

## 8 Valuation equilibrium

A **valuation equilibrium** consists of an allocation together with a price system,

$$(\bar{x}^1, \dots, \bar{x}^m, \bar{y}^1, \dots, \bar{y}^n, \bar{p}),$$

with the following properties:

1. (Profit Maximization) For every firm $j$,

$$\bar{y}^j \in Y_j \quad \text{and} \quad \bar{p} \cdot \bar{y}^j \geqslant \bar{p} \cdot y^j \text{ for all } y^j \in Y^j.$$

[3] Or **Pareto dominated**

[4] Or **Pareto efficient** or **Pareto optimal**

2. (Preference Maximization) For every consumer $i$,

$$\big(x^i \in X_i \text{ and } p \cdot x^i \leqslant p \cdot \bar{x}^i\big) \implies \bar{x}^i \succcurlyeq_i x^i,$$

or equivalently

$$x^i \succ \bar{x}^i \implies p \cdot x^i > p \cdot \bar{x}^i.$$

3. (Market clearing) $(\bar{x}^1, \ldots, \bar{x}^m, \bar{y}^1, \ldots, \bar{y}^n)$ is an allocation, that is,

$$\sum_{i=1}^{m} \bar{x}^i = \sum_{i=1}^{m} \omega^i + \sum_{j=1}^{n} \bar{y}^j.$$

A valuation equilibrium is closely related to the concept of a Walrasian equilibrium below.

### 8.1 Valuation quasiequilibrium

Closely related is the concept a **Valuation quasiequilibrium**, in which the preference maximization property is replaced by an expenditure minimization property.

2′. (Expenditure minimization) For every consumer $i$,

$$p \cdot \bar{x}^i \leqslant p \cdot x^i \text{ for all } x^i \text{ satisfying } x^i \succcurlyeq \bar{x}^i,$$

or equivalently

$$x^i \succcurlyeq \bar{x}^i \implies p \cdot x^i \geqslant p \cdot \bar{x}^i.$$

## 9 Private property

In an economy with the social convention of **private property**, the aggregate endowment and all the enterprises are wholly owned by the consumers. To completely describe such an economy and its property system A **private ownership economy** $E$ is a list $\big((X_i, \succcurlyeq_i, \omega^i)_{i=1}^m, (Y_j)_{j=1}^n, (\theta_j^i)_{j=1,\ldots,n}^{i=1,\ldots,m}\big)$. Here $\omega^i$ is a list of consumer $i$'s **initial private endowment** of each commodity, so

$$\omega = \sum_{i=1}^{m} \omega^i,$$

and $\theta_j^i$ is the share of firm $j$ owned by consumer $i$. These shares are nonnegative and sum to unity:

$$\theta_j^i \geqslant 0, \text{ for all } i, j, \qquad \text{and} \qquad \sum_{i=1}^{m} \theta_j^i = 1 \text{ for all } j.$$

## 10 Walrasian equilibrium

The outcome of competitive markets in a private ownership economy is modeled as a **Walrasian equilibrium**, which is an allocation together with a price system that is characterized by three properties.

1. Each firm maximizes profits, taking prices as given.

2. Each consumer maximizes preferences subject to their budget constraint.

3. All markets clear.

Due to our sign conventions on inputs and outputs, the profit generated by the input-output plan $y$ at price vector $p$ is $p \cdot y$. So formally a **Walrasian equilibrium** is a list

$$(\bar{x}^1, \dots, \bar{x}^m, \bar{y}^1, \dots, \bar{y}^n, \bar{p}),$$

where

1. (Profit Maximization) For every firm $j$,

$$\bar{y}^j \in Y_j \quad \text{and} \quad \bar{p} \cdot \bar{y}^j \geqslant \bar{p} \cdot y^j \text{ for all } y^j \in Y^j.$$

2. (Preference Maximization) For every consumer $i$,

$$\bar{x}^i \in B_i = \{x^i \in X_i : \bar{p} \cdot x^i \leqslant \bar{p} \cdot \omega^i + \sum_{j=1}^n \theta^i_j \bar{p} \cdot \bar{y}^j\} \quad \text{and} \quad \bar{x}^i \underset{i}{\succcurlyeq} x^i \text{ for all } x^i \in B_i.$$

3. (Market clearing) $(\bar{x}^1, \dots, \bar{x}^m, \bar{y}^1, \dots, \bar{y}^n)$ is an allocation, that is,

$$\sum_{i=1}^m \bar{x}^i = \sum_{i=1}^m \omega^i + \sum_{j=1}^n \bar{y}^j.$$

### 10.1 Walrasian quasiequilibrium

A closely related concept is that of a **Walrasian quasiequilibrium**, in which the preference maximization property is replaced by an expenditure minimization property.

2′. (Expenditure minimization) For every consumer $i$,

$$p \cdot \bar{x}^i \leqslant p \cdot x^i \text{ for all } x^i \text{ satisfying } x^i \succcurlyeq \bar{x}^i.$$

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