

## Lecture 19: Significance Tests, I

### Relevant textbook passages:

Larsen–Marx [10]: Sections 7.2, 7.4, 7.5; 7.3, Appendices

### 19.1 A simple statistical problem

Consider the issue of bias of our coins from the grand flipping coin experiment. Over eight years we collected 212,480 tosses of which 106,048 were Tails. That is 49.91% Tails. Is this different enough from  $1/2$  to warrant concluding that  $p$ , the probability of Tails is *not*  $1/2$ ?

Let's think about why we care about the particular value  $p = 1/2$ . First  $p = 1/2$  seems to be the default assumption—it's hard to justify believing something else, so the *hypothesis* that  $p = 1/2$  seems to have a special place, and I shall refer to it as the **null hypothesis**. It would take some convincing evidence for me to *reject* the hypothesis, but can I quantify that?

Well, if in truth,  $p = 1/2$ , then if  $X$  is the number of Tails in  $n$  tosses, then the random variable

$$Z = \frac{X - n/2}{\sqrt{n/4}} \sim N(0, 1),$$

at least approximately. Now let me choose a probability  $\alpha$  that is low enough so that if I observe a value of  $Z$  that is sufficiently “unlikely,” then I am willing to reject my null hypothesis  $p = 1/2$ . That is, I conclude that the true value of  $p$  is “significantly” different from  $1/2$ . This probability  $\alpha$  is thus called the **level of significance**. For many people  $\alpha = 0.05$  seems like the right number. So let's provisionally adopt this. Now I know (and you have calculated on your homework) that for a standard Normal random variable  $Z$ ,

$$P(|Z| \geq 1.96) = 0.05.$$

This suggests the following *decision rule*:

$$\text{If } \left| \frac{x - n/2}{\sqrt{n/4}} \right| \geq 1.96, \text{ then reject the null hypothesis } p = 1/2.$$

Otherwise *accept* (fail to reject) the null hypothesis.

Note that by construction, even if in truth  $p = 1/2$ , then with probability  $\alpha = 0.05$ , I will still reject the null hypothesis.

On the other hand, if in truth the probability of tails is some other value  $p$ , then the random variable  $(X - np)/\sqrt{np(1-p)}$  is distributed approximately  $N(0, 1)$ , but we will still accept the null hypothesis whenever  $|(X - n/2)|/2\sqrt{n} < 1.96$ . What is the probability of this? It is small when  $p$  is near  $1/2$ , but gets larger as  $p$  becomes farther from  $1/2$ .

Why has  $\alpha = 0.05$  become so popular as the default for rejecting hypotheses? The answer goes back to Ronald A. Fisher who wrote:

It will illustrate the meaning of tests of significance if we consider for how many years the produce (i.e., results) should have been recorded in order to make the evidence convincing. First, if the experimenter could say that in twenty years experience with uniform treatment the difference in favour of the acre treated with

manure had never before touched 10 per cent, the evidence would have reached a point which may be called the verge of significance. This level, which we may call the 5 per cent. point, would be indicated, though very roughly, by the greatest chance deviation observed in twenty successive trials. ... If one in twenty does not seem high enough odds, we may, if we prefer it, draw the line at one in fifty (the 2 per cent. point), or one in a hundred (the 1 per cent. point).

—Ronald A. Fisher, The arrangement of field experiments, reprinted in *Break-throughs in Statistics*, Springer, 1992, pp. 82–91. Originally published in 1926.

## 19.2 Another simple statistical problem

Let  $X_1, \dots, X_n$  be an independent and identically distributed sample from a  $\text{Normal}(\mu, \sigma^2)$  distribution. What can we conclude about  $\mu$  and  $\sigma^2$ ? Let's simplify the problem by first assuming that we know that

$$\sigma^2 = 1.$$

(We'll come back to the general problem in a bit.) What can we say about  $\mu$ ?

We already know that given a sample  $x_1, \dots, x_n$ , the sample average

$$\bar{x} = \frac{x_1 + \dots + x_n}{n}$$

is the method of moments and the maximum likelihood estimator of  $\mu$ , that it is an unbiased and consistent estimator, and that the 95% confidence for  $\mu$  interval is given by

$$\left( \bar{x} - \frac{1.96}{\sqrt{n}\sigma}, \bar{x} + \frac{1.96}{\sqrt{n}\sigma} \right).$$

We also know that

$$\underbrace{\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}}_{=1} \sim N(0, 1).$$

Let's simply the question further by assuming that either  $\mu = 0$  or  $\mu = 1$ , and that it is our job to decide which.

Consider Figure 19.1, which shows the probability densities of the two normals,  $\text{Normal}(0, 1)$  and  $\text{Normal}(1, 1)$ . Let's make the hypothesis that  $\mu = 0$  our null hypothesis. It is intuitive that large values of  $\bar{x}$  make it “less likely” that  $\mu = 0$  is the parameter generating the data.

If in truth  $\mu = 0$ , then

$$P\left(\frac{\bar{X}}{1/\sqrt{n}} \geq 1.96\right) = 0.025$$

So setting a critical value  $c = 1.96$  and testing whether  $\sqrt{n}\bar{X} \geq c$  (a so-called one-sided test) gives a probability of 0.025 of rejecting the null hypothesis even when it is true.

Now suppose that in fact  $\mu = 1$ . Then  $\sqrt{n}(\bar{X} - 1) \sim N(0, 1)$ . Let's compute  $\text{Prob}(\sqrt{n}\bar{x} < c)$  in this case. This is the case where we decide  $\mu = 0$  when in truth  $\mu = 1$ . Now

$$\sqrt{n}\bar{x} < c \iff \sqrt{n}(\bar{x} - 1) < c - \sqrt{n}$$

so when  $\mu = 1$ ,

$$\text{Prob}(\sqrt{n}\bar{X} < c) = \text{Prob}(\sqrt{n}(\bar{X} - 1) < c - \sqrt{n}) = \Phi(c - \sqrt{n}).$$

This is the probability of deciding  $\mu = 0$ , when in truth  $\mu = 1$ . Note that this probability  $\rightarrow 0$  as  $n \rightarrow \infty$ .

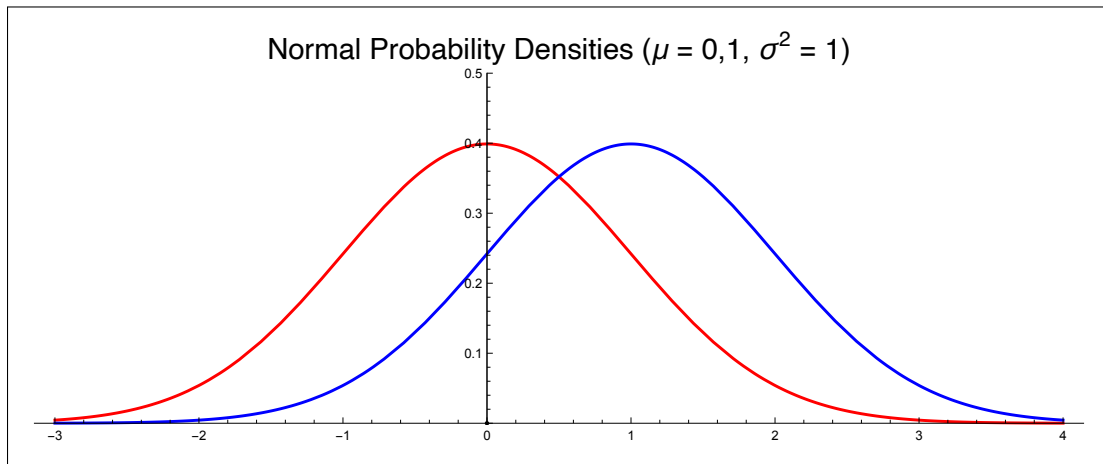


Figure 19.1. A simple hypothesis test.

### 19.3 Hypothesis testing in the abstract

A typical **data model** or **probability model** is a density/likelihood function  $f(x; \theta)$  depending on a datum  $x$  from some set  $\mathcal{X}$ , and parameterized by  $\theta$  belonging to some set  $\Theta$ . Having estimated parameters is usually not enough. We know that the estimates are random, and that if we are very unlucky our estimates can be very misleading. Usually, we want to *test* some *hypothesis* about the probability model.

- There are two kinds of such hypotheses.
- The first kind is a **hypothesis about the parameter**. A typical such hypothesis is of the form:  $\theta \in A$ , for some subset  $A$  of the parameter space  $\Theta$ . For instance, we may be interested in whether the mean  $\mu$  of a normal is greater than zero,  $\mu \in [0, \infty)$ . These tests are usually referred to as **significance tests**.
- The second kind of hypothesis is whether our model is **misspecified**. Misspecification occurs when the data generating process is governed by a different functional form  $g(x; \psi)$  for some possibly alternate parameter set. (Remember that we could take as our parameter space the set of *all* probability measures on  $\mathcal{X}$ . But this space is so large we usually call this a nonparametric model.) An example of this might be whether the data come from *any* normal distribution, or whether an entirely different family, such as the Cauchy distribution, is a “better” model. These tests are frequently usually to as **specification tests**.

We’ll start with the first kind of testing, parametric hypothesis testing in the context of a fixed data model  $f$ .

#### A shipload of definitions

- The **data model**:  $f: \mathcal{X} \times \Theta \rightarrow \mathbf{R}$ . Typically,  $\mathcal{X}$  and  $\Theta$  are subsets of some finite dimensional Euclidean spaces.  $X$  is a random variable (or random vector) with either a density or a mass function given by  $f(x; \theta)$ . For convenience, I will typically refer to  $f$  as a pdf. Recall that the likelihood function  $L(\theta; x)$  is just  $f(x; \theta)$  so I may also refer to  $f$  as the likelihood function. Finally, when it is convenient, I may also write  $f_\theta(x)$  for  $f(x; \theta)$ .

Larsen–  
 Marx [10]:  
 Section 6.2

- The **Null Hypothesis**: Denoted  $H_0$ , it takes the form of statement  $\theta \in \Theta_0 \subset \Theta$ . It could be as simple as  $\theta = 0$  (more properly  $\theta \in \{0\}$ ). The next section elaborates on the role of the null hypothesis.

Often the null hypothesis is something the researcher hopes to prove false. For instance, if you want to show that a drug improves the cure rate for a disease, the null hypothesis is probably going to be that the difference in the cure rate (over the control group) is zero. If the drug is actually useful, then you should reject the null hypothesis.

- **Alternative Hypothesis**:  $H_1$  is the hypothesis that  $\theta \in \Theta_1 = \Theta \setminus \Theta_0$ . The two sets  $\Theta_0$  and  $\Theta_1$  partition the parameter space  $\Theta$ .

E.g.,  $\Theta = [0, \infty)$ ,  $\Theta_0 = \{0\}$ , and  $\Theta_1 = (0, \infty)$ . The null hypothesis is  $\theta = 0$  and the alternative is  $\theta > 0$ .

- A **simple hypothesis** is that  $\Theta_i$  has just one point, and a **composite hypothesis** is that  $\Theta_i$  has more than one point.

Often a null hypothesis is simple and the alternative is composite, but that needn't be.

The point of a **test** is to, based on the datum (vector)  $x$ , either **reject** the null hypothesis in favor of the alternative, or to **fail to reject** the null hypothesis. (It is considered a faux pas to say that you accept the null hypothesis.)

How do you decide which hypothesis gets to be the null hypothesis, and which gets to be the alternative?

## 19.4 Choosing the null hypothesis

Brad Efron [3, pp. 556–557] gives a nice discussion of the role of the null and alternative hypotheses in scientific investigation. In the quotation below, the case he is referring to is known as **Bode's Law**, namely that the distance from the sun of the  $n^{\text{th}}$  planet is of the form the  $d_n = a + b2^n$ . (Bode did not have a large sample of solar systems.) Efron is critiquing the analysis of Good [6] who was in turn critiquing Bode. In the quote. Model B is Bode's Law, model  $\bar{B}$  is Good's alternative hypothesis, and C is Efron's alternative. Note this is concerned with model specification rather than parametric hypotheses, but the comments on the role of the null hypothesis are still relevant.

The most interesting feature of this problem is the light it casts on the role of the null hypothesis in hypothesis testing. These terms are used here in Fisher's sense in which the null hypothesis is by design a hypothesis of uninteresting structure compared to that which we are considering as an alternative, though it may contain its own interesting features. [...]

However it is not necessary to believe in the null hypothesis in order to use it as a test against the alternative of interest. Very often, perhaps most of the time, we do not believe in the validity of the Fisherian null hypothesis, whether or not the test based on it accepts or rejects in the usual sense. [...]

The null hypothesis in the context of this discussion plays the role of devil's advocate, a competitor that an alternative of interest to us must soundly discredit in order to show its strength. [...]

The conclusions of a significance test are bound to be less than completely satisfying given the indirect route of the argument. In the case at hand for instance, accepting C doesn't mean we believe C is true (Figure B mildly discourages such a belief). All we can say is that a statistical model that is relatively uninteresting compared to Bode's law would often yield data as "simple" as that actually observed,

and this undermines the necessity of our belief in the law’s validity. Conversely even if we had decisively rejected  $C$  we still might fear that we had overlooked some other reasonable null hypothesis which would do better.

One should not be dismayed by the limitations of the Fisherian significance test since it is designed only to give us some direction toward the correct answer in situations like the present one where there is little data to work with. As more data accumulate in any given problem, significance testing becomes superfluous.

[...]

By definition “estimation” refers to situations where we believe we know all the possible relevant statistical models and we are simply trying to choose the correct one. Estimation is an inherently more satisfying operation than significance testing, but demands more data or more theoretical knowledge from the statistician.

## 19.5 The abstract mechanics of a statistical test

We can reduce the previous discussion to the following definition of a test. The definition may seem overly abstract, but it the ideas will become familiar with some examples.

**19.5.1 Definition** A **statistical test** of the null hypothesis  $H_0: \theta \in \Theta_0$  versus the alternative  $H_1: \theta \in \Theta_1$  is characterized by a pair  $(T, C)$ , where  $T$  is a **test statistic** taking values in a set  $\mathcal{X}$  and  $C$  is a subset of  $\mathcal{X}$ , called the **critical region**. The outcome of the test is to **reject** the null hypothesis  $H_0$  in favor of  $H_1$  if  $T \in C$ , and to **fail to reject** or **accept**  $H_0$  if  $T \notin C$ .

This requires a fair amount of elaboration:

- **Test statistic:**  $T$  is a function of the data, and also of the null hypothesis.

E.g.,  $T = (\bar{x} - \theta_0)/(s/\sqrt{n})$ .

When we discussed estimators, I said that a statistic must be a function of the data, and not of the unknown parameters.

If we have a simple null hypothesis, say  $\theta = \theta_0$ , the test statistic is allowed to depend on  $\theta_0$  because in its role as the null hypothesis,  $\theta_0$  is not unknown to us—we know quite well what its value is because we picked it.

Even if the null hypothesis is composite, but say of the form  $\theta < \theta_0$ , we can allow the test statistic depend on the boundary point  $\theta_0$  (or any other feature that characterizes the set  $\Theta_0$ ).

- **Critical region:**  $C$ . If the value of  $T$  belongs to  $C$ , the null hypothesis is **rejected** in favor of the alternative hypothesis. Otherwise we **fail to reject** the null hypothesis.

The critical region is often either an interval (possibly infinite) or the complement of an interval. The endpoint(s) define the **critical value(s)** of the test. For example, if  $C = [t^*, \infty)$ , then  $t^*$  is the critical value.

- Suppose the test statistic  $T$  has the value  $t$  and critical region is of the form  $[c^*, \infty)$ . The probability

$$P_{\theta_0}(T \geq t)$$

is called the **p-value** of  $t$  under the null hypothesis  $\theta_0$ .

An equivalent description of the test is to reject  $H_0$  whenever the  $p$ -value of the statistic is less than  $\alpha$ .

- If critical region is of the form  $(-\infty, c^*] \cup [c^*, \infty)$ , the **p-value** of  $t$  is the probability

$$P_{\theta_0}(|T| \geq |t|).$$

- A generalization of the critical region is the **critical function**  $\phi$ . This allows for randomizing when deciding whether or not to reject the null hypothesis. The value  $\phi(t)$  is the probability of rejecting the null hypothesis when  $T(x) = t$ . If the test uses a critical region  $C$ , then  $\phi$  is the indicator function  $\mathbf{1}_C$  of the critical region. When would you want to randomize? When the test statistic is discrete, and I largely ignore the issue, but read on to the next point.

- **Significance level:**  $\alpha$ . This is, roughly speaking, the probability  $P(T \in C)$  when  $H_0$  is true. That is, it is the probability of rejecting the null hypothesis when it is true. This is also called the **size** of the test. Note: people often misspeak and use  $1 - \alpha$  as the significance level. Or they might err and say that the test is “significant at the  $1 - \alpha$  confidence level.” Usually you know what they mean, and it is boorish to correct them—except in this class.

When the null hypothesis is simple, the meaning of  $P(T \in C)$  is clear. The probability  $P$  is actually  $P_{\theta_0}$ , the probability whose pdf is  $f(\cdot; \theta_0)$ . For a composite null hypothesis, we define the size  $\alpha$  to be

$$\alpha = \sup\{P_{\theta}(T \in C) : \theta \in \Theta_0\}.$$

That is, for a composite null hypothesis, the probability of rejecting the hypothesis for any parameter in the null hypothesis  $\Theta_0$  is no greater than  $\alpha$ . The actual probability will depend on which  $\theta \in \Theta_0$  is the “true” parameter.

Statisticians are inordinately fond of  $\alpha = 0.05$  and  $\alpha = 0.01$ . This goes back to Ronald A. Fisher’s [4, 5] dicta in 1925. It also goes back to the pre-statistical software era in which you had to look up critical values in a table in a book, which listed only a few values of  $\alpha$ . When I was a freshman, one of the first books that everyone bought was the Chemical Rubber Company’s *CRC Standard Math Tables* [16]. You can see a typical page in Figure 19.2.

If the test statistic  $T$  is a discrete random variable, for instance, a count of successes, it may be impossible to find a critical region of size exactly  $\alpha$  due to the “lumpiness” of  $T$ . This is when you might want to randomize for a particular value  $t$  of  $T$ , in order to get the probability of rejection exactly equal to  $\alpha$ . This is when to use a critical function instead of a critical region. When the null hypothesis is simple,  $\theta = \theta_0$ , then

$$E_{\theta_0} \phi(T) = \alpha.$$

- **The significance level and critical region can only be computed if we know the distribution of  $T$  given the parameters  $\theta$ .** This is why so much effort has been devoted to figuring out the distribution of various test statistics.

## 19.6 Statistics means never having to say you’re certain

- A **Type I error** occurs when  $H_0$  is rejected when in fact it is true. (False rejection.) This can only happen if we reject the hypothesis.
- A **Type II error** occurs when  $H_0$  fails to be rejected when in fact it is false. (False acceptance.) This can only happen if we fail to reject the hypothesis.
- **We may never know if an error of either type has occurred!**
- The probability of committing a Type I error is  $\alpha$ , the significance level of the test. *This is a choice made by the experimenter (we hope) before the experiment is conducted.*

Larsen–  
 Marx [10]:  
 § 6.4

PERCENTAGE POINTS, STUDENT'S *t*-DISTRIBUTION

This table gives values of *t* such that

$$F(t) = \int_{-\infty}^t \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi} \Gamma\left(\frac{n}{2}\right)} \left(1 + \frac{x^2}{n}\right)^{-\frac{n+1}{2}} dx$$

for *n*, the number of degrees of freedom, equal to 1, 2, . . . , 30, 40, 60, 120, ∞; and for *F*(*t*) = 0.60, 0.75, 0.90, 0.95, 0.975, 0.99, 0.995, and 0.9995. The *t*-distribution is symmetrical, so that *F*(-*t*) = 1 - *F*(*t*)

<i>n</i> \ <i>F</i>	.60	.75	.90	.95	.975	.99	.995	.9995
1	.325	1.000	3.078	6.314	12.706	31.821	63.657	636.619
2	.289	.816	1.886	2.920	4.303	6.965	9.925	31.598
3	.277	.765	1.638	2.353	3.182	4.541	5.841	12.924
4	.271	.741	1.533	2.132	2.776	3.747	4.604	8.610
5	.267	.727	1.476	2.015	2.571	3.365	4.032	6.869
6	.265	.718	1.440	1.943	2.447	3.143	3.707	5.959
7	.263	.711	1.415	1.895	2.365	2.998	3.499	5.408
8	.262	.706	1.397	1.860	2.306	2.896	3.355	5.041
9	.261	.703	1.383	1.833	2.262	2.821	3.250	4.781
10	.260	.700	1.372	1.812	2.228	2.764	3.169	4.587
11	.260	.697	1.363	1.796	2.201	2.718	3.106	4.437
12	.259	.695	1.356	1.782	2.179	2.681	3.055	4.318
13	.259	.694	1.350	1.771	2.160	2.650	3.012	4.221
14	.258	.692	1.345	1.761	2.145	2.624	2.977	4.140
15	.258	.691	1.341	1.753	2.131	2.602	2.947	4.073
16	.258	.690	1.337	1.746	2.120	2.583	2.921	4.015
17	.257	.689	1.333	1.740	2.110	2.567	2.898	3.965
18	.257	.688	1.330	1.734	2.101	2.552	2.878	3.922
19	.257	.688	1.328	1.729	2.093	2.539	2.861	3.883
20	.257	.687	1.325	1.725	2.086	2.528	2.845	3.850
21	.257	.686	1.323	1.721	2.080	2.518	2.831	3.819
22	.256	.686	1.321	1.717	2.074	2.508	2.819	3.792
23	.256	.685	1.319	1.714	2.069	2.500	2.807	3.767
24	.256	.685	1.318	1.711	2.064	2.492	2.797	3.745
25	.256	.684	1.316	1.708	2.060	2.485	2.787	3.725
26	.256	.684	1.315	1.706	2.056	2.479	2.779	3.707
27	.256	.684	1.314	1.703	2.052	2.473	2.771	3.690
28	.256	.683	1.313	1.701	2.048	2.467	2.763	3.674
29	.256	.683	1.311	1.699	2.045	2.462	2.756	3.659
30	.256	.683	1.310	1.697	2.042	2.457	2.750	3.646
40	.255	.681	1.303	1.684	2.021	2.423	2.704	3.551
60	.254	.679	1.296	1.671	2.000	2.390	2.660	3.460
120	.254	.677	1.289	1.658	1.980	2.358	2.617	3.373
∞	.253	.674	1.282	1.645	1.960	2.326	2.576	3.291

\* This table is abridged from the "Statistical Tables" of R. A. Fisher and Frank Yates published by Oliver & Boyd, Ltd., Edinburgh and London, 1938. It is here published with the kind permission of the authors and their publishers.

Figure 19.2. A typical statistical table from the *CRC Standard Math Tables* [16].

- Once we have chosen the critical region  $C$  and the associated level of significance, with talent and cleverness we can often calculate the probability of a Type II error. This probability is frequently referred to as  $\beta(\theta)$ . It depends on the actual parameter value, which of course we do not know.
- The complement of a type II error is to properly *reject  $H_0$  when it is indeed false*. The probability of this is  $1 - \beta(\theta)$ , and is called the **power** of the test. A graph of the power vs.  $\theta$  is called the **power curve** of the test. The function  $\beta(\theta)$ , the probability of a Type II error, is sometimes called the **operating characteristic** of the test [1, 7]. It is simply the probability of “accepting” the null hypothesis, as a function of the (unknown) parameter value.
- **Warning:** The classic bible of hypothesis testing by E. L. Lehmann [11, p. 61] uses  $\beta$  to denote the power. Thus his  $\beta$  is Larsen and Marx’s  $1 - \beta$  and vice versa. A brief inquiry shows that my colleague Bob Sherman and references [7, 8, 12] agree with Larsen and Marx.
- If we have two tests with the same significance level  $\alpha$ , if one is always more powerful than the other, then it is a (statistically) better test. (One way to increase the power is to get more data, which may be expensive.)
- The power of a test is influenced by the shape of the critical region. For instance, why do we usually take the critical region to be the regions where the density of  $T$  is smallest? Because it makes the test more powerful. This is the essence of the **Neyman–Pearson Lemma**.
- Every test is characterized by its pair  $(T, C)$  of test statistic and  $T$  and critical region  $C$ . Let  $\beta_{T,C}(\theta)$  denote the test’s probability of Type II error (false acceptance of the null) when the parameter is  $\theta$ ,

$$\beta_{T,C}(\theta) = P_{\theta}(T \notin C).$$

**19.6.1 Definition**    *If a test  $(T^*, C^*)$  has size (significance level)  $\alpha$ , and if for every  $\theta \in \Theta_1$ ,*

$$1 - \beta_{T^*, C^*}(\theta) \geq \max\{1 - \beta_{T,C}(\theta) : \text{test } (T, C) \text{ has size } \alpha\},$$

*then we say that  $(T^*, C^*)$  is a **uniformly most powerful test (UMP)**.*

- UMPs are good to have, but they don’t always exist. We shall discuss situations where UMPs do exist, and what they look like, next time.
- When a UMP does not exist, many practitioners argue that a test should be selected to minimize a weighted average of  $\alpha$  and  $\beta$ , where the weights reflect the tester’s concerns about the two types of errors.

## 19.7 ★ A more nuanced discussion of $p$ -values



Kuffner and Walker [9] argue that the definition that I (and your textbook [10]) give you is misleading. We started with the choice of  $\alpha$ , the size or significance level, and then chose a critical region  $C$ . They argue that we should choose a critical region  $C_{\alpha}$  for each  $\alpha \in (0, 1)$ . Presumably we should do this in a way to minimize the probability of Type II error for each  $\alpha$ . Then we can define the  $p$ -value of the realization of  $T = T(X)$  to be

$$p(X) = \inf\{\alpha : T(X) \in C_{\alpha}\}. \tag{1}$$

This scales  $T$  so that its units becomes commensurable with the size  $\alpha$  of the test. (By the way, equation (1) is the definition of a “valid  $p$ -value” that is used by Casella and Berger [2, p. 397].)



Elaborate on this.

This definition actually agrees with what I said earlier, but Kuffner and Walker argue that while the distinction is subtle, it is important that with their way of defining the  $p$ -value, it is not defined to be a probability. They emphasize that hypothesis testing, is a special case of statistical decision theory which requires that a **decision rule** be used. A decision rule is a function from the data  $X$  to the set of possible decisions, which in this case is just {reject, fail to reject}. Unless  $\alpha$  is chosen in advance, the decision rule is undefined.

It is the fact that a decision rule is being employed is what enable us to give a frequency interpretation to the Type 1 and Type 2 errors.

On the other hand, as Casella and Berger [2, p. 397] argue, “The advantage to reporting a test result as a  $p$ -value is that each reader can choose the  $\alpha$  that he or she considers appropriate and the compare the reported  $[p(X)]$  to  $\alpha$  and know whether these data lead to acceptance or rejection of  $H_0$ . Furthermore, the smaller the  $p$ -value, the stronger the evidence for rejecting  $H_0$ . Hence a  $p$ -value reports the results of a test on a more continuous scale, rather than a dichotomous decision ‘Accept  $H_0$ ’ or ‘Reject  $H_0$ .’”

### 19.7.1 The distribution of $p$ -values

Suppose we are testing a null hypothesis  $H_0: \theta = \theta_0$  at the  $\alpha$  level of significance with the test statistic  $T$  and critical region  $T \geq t^*$ , where

$$P_{\theta_0}(T > t^*) = \alpha.$$

Assume that the null hypothesis is true. Define the function  $\text{pv}(x)$  by

$$\text{pv}(x) = P_{\theta_0}(T > x).$$

This is just the survival function of the random variable  $T$  under the null hypothesis. But in the testing situation it has a different interpretation.

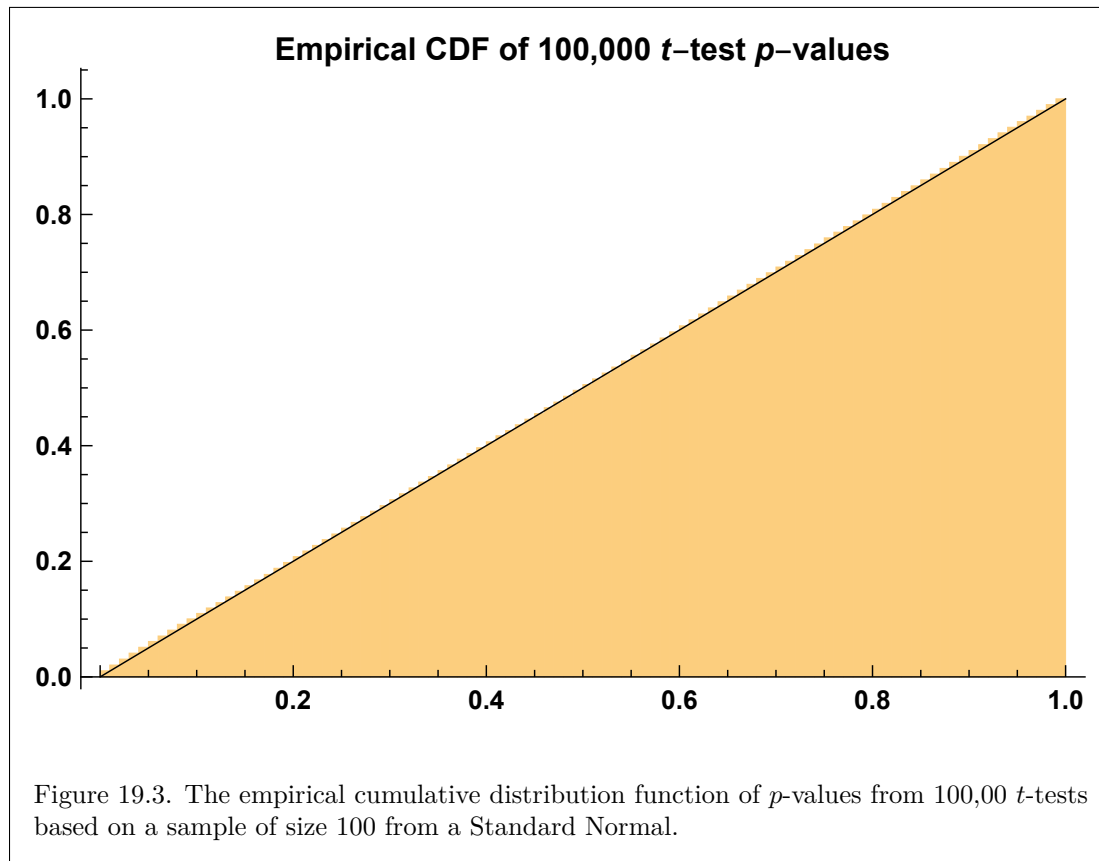
$\text{pv}(T)$  is the  $p$ -value of the outcome of the experiment’s test.

Now  $\text{pv}(T)$  as a function of  $T$  is itself a random variable. What is its distribution? Assuming  $T$  has an absolutely continuous cumulative distribution function  $F_T$ , then  $\text{pv}(x) = 1 - F_T(x)$ . But recall from Proposition 12.1.2 that  $F_T(T)$  is uniformly distributed on  $[0, 1]$  for any continuous random variable  $T$ . Therefore,

if the null hypothesis is true, the  $p$ -value  $\text{pv}(T)$  of the outcome of the experiment’s test is a Uniform $[0, 1]$  random variable.

For the sake of illustration, I conducted a Monte Carlo simulation of 100,00 experiments where a sample of size 100 was drawn from a Standard Normal distribution. For each sample, I tested the hypothesis that  $\mu = 0$  using a two-sided  $t$ -test. I recorded the resulting  $p$ -value of the  $t$ -statistic. Figure 19.3 shows the empirical cumulative distribution function of these  $p$ -values. Notice how well they fit the Uniform distribution.

All this means that even if the null hypothesis is true, the resulting data will randomly give evidence of varying “strength” (in the sense of Casella and Berger) to reject the null hypothesis. The probability of rejecting the null hypothesis  $H_0$  at the  $\alpha$  level of significance is still  $\alpha$ . So why can’t we just forgo the experiment, and decide whether to reject  $H_0$  by simply drawing a value from any uniform distribution on  $[0, 1]$ ? We could, but that we be stupid. The test would still be test at the  $\alpha$ -level of significance, but if we draw from a uniform distribution that is independent of the experiment, we lose control of the Type II error. The power curve would be flat and always take the value  $\alpha$ .



## 19.8 The role of hypothesis testing

Jerzy Neyman and Egon S. Pearson [13] offer this on why we need to understand the nature and uses hypothesis tests. (Emphasis mine.)

One of the most common as well as most important problems which arise in the interpretation of statistical results, is that of deciding whether or not a particular sample may be judged as likely to have been randomly drawn from a certain population, whose form may be either completely or only partially specified. We may term Hypothesis A the hypothesis that the population from which the sample  $\Sigma$  has been randomly drawn is that specified, namely  $\Pi$ . In general the method of procedure is to apply certain tests or criteria, the results of which will enable the investigator to decide with a greater or less degree of confidence whether to accept or reject Hypothesis A, or, as is often the case, will show him that further data are required before a decision can be reached. At first sight the problem may be thought to be a simple one, but upon fuller examination one is forced to the conclusion that **in many cases there is probably no single “best” method of solution. The sum total of the reasons which will weigh with the investigator in accepting or rejecting the hypothesis can very rarely be expressed in numerical terms.** All that is possible for him is to balance the results of a mathematical summary, formed upon certain assumptions, against other less precise impressions based upon *à priori* or *à posteriori* considerations. **The tests themselves give no final verdict, but as tools help the worker who is using them to form his final decision;** one man may prefer to use one method, a second another, and yet in the long run there may be little to choose between the value of their conclusions.

What is of chief importance in order that sound judgment may be formed is that the method adopted, its scope and its limitations, should be clearly understood, and it is because we believe this often not to be the case that is has seemed worth while to us to discuss the principles involved in some detail and to illustrate their application to certain important sampling tests.

## 19.9 ★ Likelihood Ratio Tests for simple hypotheses

**19.9.1 Example (Likelihood ratio test for the mean of a Normal( $\mu, 1$ ))** Let us return to the simple example of Section 19.2 to make sense out of all this jargon and abstraction. We know that the sample average  $\bar{X}$  is either a Normal  $N(0, 1/n)$  random variable or a Normal  $N(1, 1/n)$  random variable. On the basis of the datum  $\bar{x}$  we have to decide which.

The parameter set  $\Theta$  for the mean  $\mu$  has only two points,  $\Theta = \{0, 1\}$ . To avoid needless confusion, let me say that the null hypothesis  $H_0$ , and the alternative  $H_1$  are

$$H_0: \mu = 0, \quad H_1: \mu = \mu_1 = 1.$$

Let me also write  $f_0(\bar{x})$  for the pdf of  $\bar{X}$  under the null hypothesis, and  $f_1(\bar{x})$  for the pdf under the alternative.

The same intuition that motivated our maximum likelihood estimation suggest that perhaps we ought to use a test like the following:

Let the test statistic  $T(\bar{x})$  just be  $\bar{x}$  itself.

Define the **likelihood ratio**

$$\lambda(x) = \frac{f_1(\bar{x})}{f_0(\bar{x})}$$

Change this for next year!

**Warning:** I am writing the ratio the way that Lehmann [11, p. 64] writes it. Larsen–Marx [10, p. 380] and Casella and Berger [2, p. 375] invert this ratio. It’s merely a convention, but you have to know which one is being used or your inequalities will be reversed. I took an informal survey of my colleagues, to ask how they write likelihood ratios, and most of them agreed with Lehmann. However, [Wikipedia](#) agrees with Larsen and Marx.

A **likelihood ratio test** takes the form:

- Choose a cutoff  $c > 0$ ,
- and reject  $H_0$  if  $\lambda(\bar{x}) \geq c$ ,
- otherwise fail to reject  $H_0$ , or accept  $H_0$  over  $H_1$ .

Note that if you invert the likelihood ratio the way that Larsen and Marx do, you want to reject  $H_0$  if  $\lambda \leq 1/c$ .

So there is a one-parameter family of likelihood ratio tests parametrized by  $k$ . It is this freedom that lets you use the Likelihood Principle, but still accommodate cost/benefit considerations.

**An extended normal example**

For our normal case with known variance  $\sigma^2 = 1$ , the sample average  $\bar{X}$  for a sample of size  $n$  has variance  $1/n$ , so the likelihood ratio for  $0 < \mu_1$  is

$$\lambda(\bar{x}) = \frac{e^{-\frac{1}{2n}(\bar{x}-\mu_1)^2}}{e^{-\frac{1}{2n}(\bar{x}-0)^2}} = e^{\frac{1}{2n}(2\bar{x}-\mu_1)},$$

which is increasing in  $\bar{x}$ . In this case, the likelihood ratio test is equivalent to

- Choose a cutoff  $c$ ,
- and reject  $H_0$  in favor of  $H_1$  if  $\bar{x} \geq c$ ,
- otherwise (when  $\bar{x} < c$ ) fail to reject (accept)  $H_0$  over  $H_1$ .

In order to keep the significance level the same as the sample size varies, because  $\bar{X}$  has standard deviation  $1/\sqrt{n}$ , we have to scale the cutoff point for  $\bar{x}$  by  $1/\sqrt{n}$ . Under the null hypothesis,

$$Z = \bar{X}/(1/\sqrt{n}) = \sqrt{n}\bar{X} \sim N(0, 1),$$

so

$$P_0(\bar{X} \geq c/\sqrt{n}) = P_0\left(\frac{\bar{X}}{1/\sqrt{n}} \geq c\right) = 1 - \Phi(c).$$

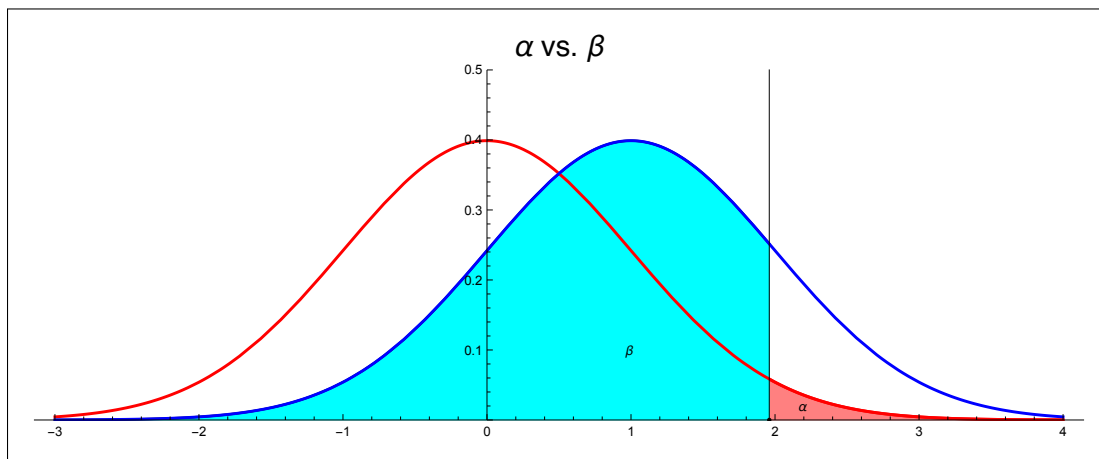
Setting  $c = 1.64485$  and using the rule “reject the null if  $\bar{x} \geq c/\sqrt{n}$ ” yields  $\alpha = 0.05$  independent of  $n$ . (This is a one-sided test.)

The probability of a Type II error (False Acceptance) is the probability that  $\bar{X} < c/\sqrt{n}$  when the mean  $\mu$  is  $\mu_1 = 1$ . Now under the alternative  $H_1$ , the random variable  $(\bar{X} - \mu_1)/(1/\sqrt{n}) \sim N(0, 1)$ , so

$$\beta(\mu_1) = P_1(\bar{X} < c/\sqrt{n}) = P_1\left(\bar{X} - \mu_1 < \frac{c}{\sqrt{n}} - \mu_1\right) = P_1\left(\frac{\bar{X} - \mu_1}{1/\sqrt{n}} < c - \sqrt{n}\mu_1\right) = \Phi(c - \sqrt{n}\mu_1).$$

So the power is

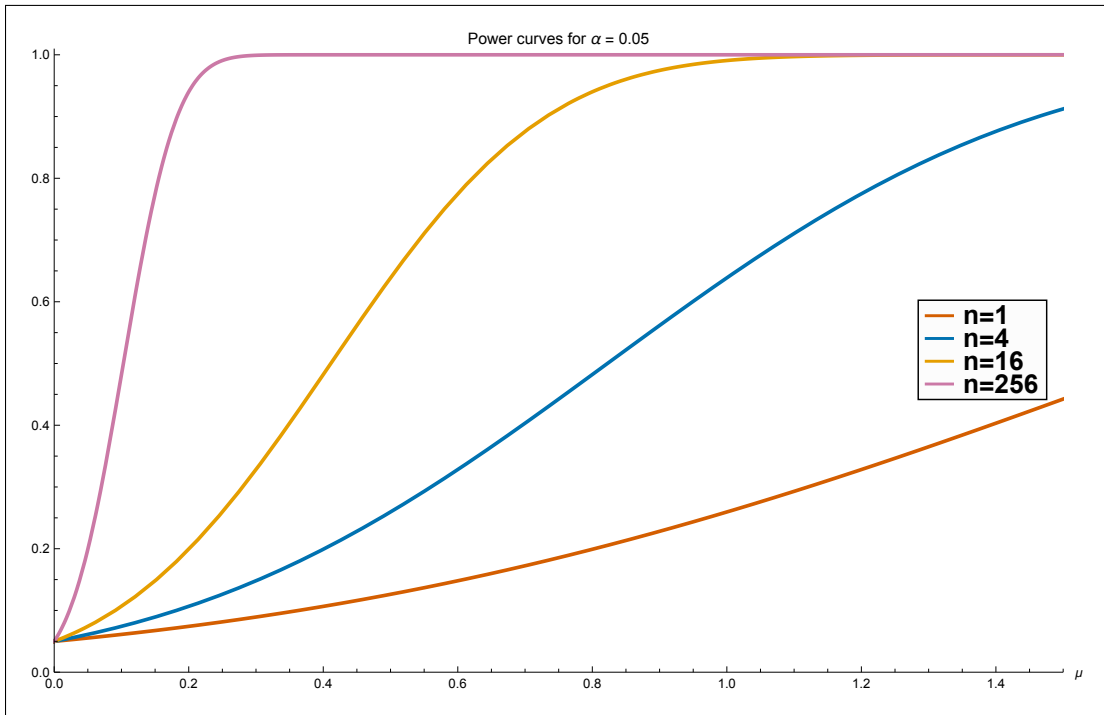
$$1 - \Phi(c - \sqrt{n}\mu_1).$$



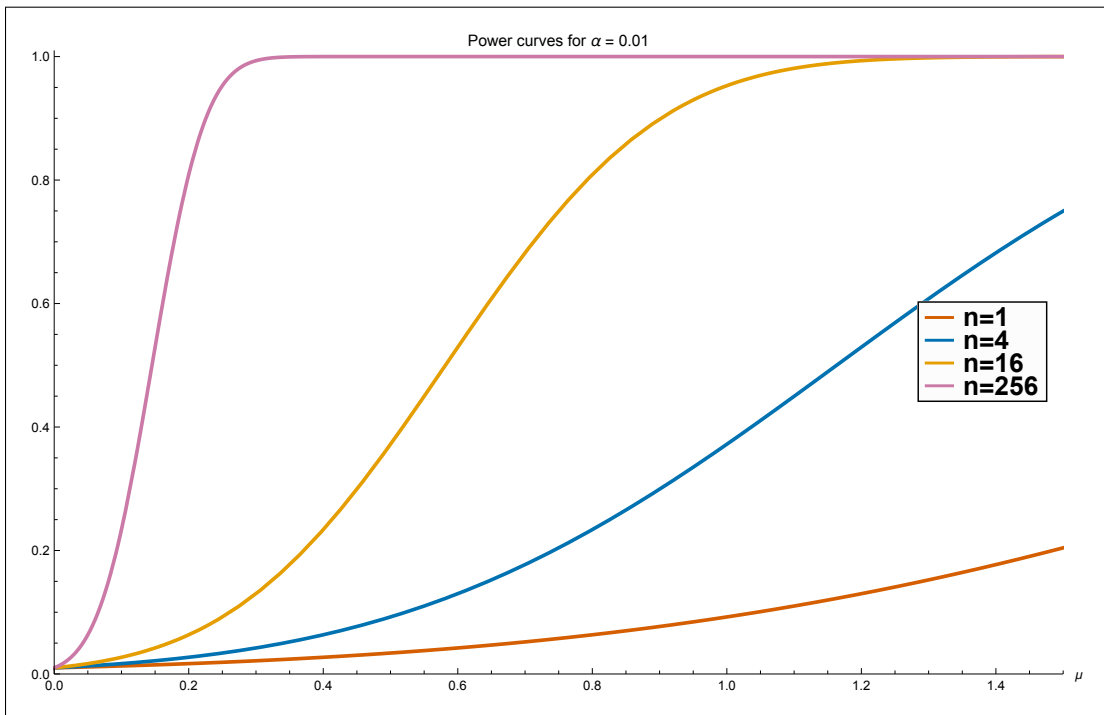
More generally if the alternative hypothesis is  $\mu > 0$ , the **power function** is given by

$$1 - \beta(\mu) = 1 - \Phi(c - \sqrt{n}\mu).$$

Here is the graph of the power function for various  $n$ .



What if I want a smaller value of  $\alpha$ ? To get  $\alpha = 0.01$ , I need to set  $c = 2.32635$ . Here are the new power curves:



□

Can we get more power? No, this is the best we can do. This was first proven by Jerzy Neyman and Egon Pearson [14].<sup>1</sup>

<sup>1</sup>Egon Pearson is the son of Karl Pearson, who originated the chi-square test, among other things.

## 19.10 Neyman–Pearson Fundamental Lemma

**19.10.1 Neyman–Pearson Fundamental Lemma** *For testing a simple null versus a simple alternative, a likelihood ratio test maximizes the power, given the size.*

Here is a reasonably convincing proof for the case of absolutely continuous test statistics, but if you want to dot all your *i*'s and cross all your *t*'s, you need to use critical functions, not critical regions. See Lehmann [11, pp. 65–67] for a more complete proof.

*Proof:* Pick a point  $c$  in the critical region, and a point  $d$  in the non-critical region and imagine swapping tiny intervals about them:

$$\begin{aligned}\Delta\alpha &\approx -f_0(c)\delta + f_0(d)\varepsilon, \\ \Delta\beta &\approx f_1(c)\delta - f_1(d)\varepsilon,\end{aligned}$$

where  $\delta$  and  $\varepsilon$  are the widths of the intervals around  $c$  and  $d$ . Then

$$\Delta\alpha = 0 \implies \varepsilon = \frac{f_0(c)}{f_0(d)}\delta.$$

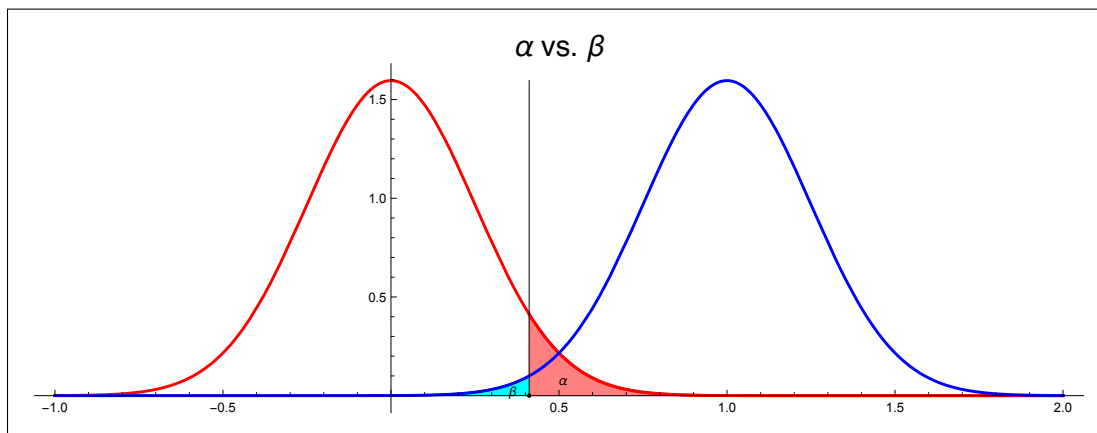
So

$$\begin{aligned}\Delta\beta &= \left[ f_1(c) - f_1(d) \frac{f_0(c)}{f_0(d)} \right] \delta \\ &= \left[ \frac{f_1(c)}{f_0(c)} - \frac{f_1(d)}{f_0(d)} \right] f_0(c) \delta \geq 0.\end{aligned}$$

$(\geq k^*) \quad (\leq k^*)$

That is, any small change to the critical region that keeps the significance (size)  $\alpha$  constant (or reduces it) must increase  $\beta$ , the probability of Type II error, reducing the power. ■

So what can we do to get a more powerful test? Increasing the sample size reduces the standard deviation, so the power increases. The next chart shows the effect of reducing the standard deviation by a factor of 4 (increasing the sample size by a factor of 16) for our toy problem. Of course, increasing the sample size can be expensive, especially if your experiment involves medical testing, colliding large hadrons, or mapping the climate on assorted planets.



## 19.11 ★ The monotone likelihood ratio property

For composite hypotheses, likelihood ratio tests work best when the data model  $f(x; \theta)$  satisfies an additional property, known as the monotone likelihood ratio property. In this case, the

Neyman–Pearson Fundamental Lemma generalizes, and likelihood ratio tests are UMP, they are characterized by critical values, and the notion of a  $p$ -value applies.

When  $\Theta \subset \mathbf{R}$ , the probability model  $f(x; \theta)$  satisfies the **Monotone Likelihood Ratio Property (MLRP) with respect to the statistic  $T$**  if for every pair  $\theta, \theta'$ , with

$$\theta < \theta'$$

the likelihood ratio

$$\frac{f(x; \theta')}{f(x; \theta)} \text{ is nondecreasing in } T(x).$$

We then say the likelihood ratio is monotone in  $T(x)$ .

An equivalent way to rewrite the MLRP is:

$$(\theta < \theta' \ \& \ T(x) < T(x')) \implies \frac{f(x; \theta')}{f(x; \theta)} \leq \frac{f(x'; \theta')}{f(x'; \theta)}.$$

**19.11.1 Example (MLRP and the Normal Family)** Consider the Normal family  $N(\mu, 1)$ . Here the one-dimensional parameter is  $\mu$  and the parameter space is  $\Theta = \mathbf{R}$ . If we take a sample of  $n$  independent observations  $x_1, \dots, x_n$ , the sample average  $\bar{x}$  is a sufficient statistic for  $\mu$ , and the sample average has a Normal distribution with mean  $\mu$  and variance  $1/n$ . Its density is

$$f(\bar{x}; \mu) = \frac{1}{\sqrt{2\pi}} e^{-(\bar{x}-\mu)^2/(2/n)}.$$

So for  $\mu < \mu'$ , the likelihood ratio is

$$\frac{e^{-(\bar{x}-\mu')^2/(2/n)}}{e^{-(\bar{x}-\mu)^2/(2/n)}} = e^{\frac{n}{2}((\bar{x}-\mu)^2 - (\bar{x}-\mu')^2)} = e^{\frac{n}{2}(\mu^2 - \mu'^2 + 2(\mu' - \mu)\bar{x})}$$

which is a strictly increasing function of  $\bar{x}$ . This remains true for each  $\sigma^2 \neq 1$ . Thus the Normal family has the MLRP for  $\mu$  with respect to  $\bar{x}$  for fixed  $\sigma^2$ .  $\square$

**19.11.2 Example (MLRP and the Poisson distribution)** The Poisson( $\mu$ ) pmf is

$$f(k; \mu) = e^{-\mu} \frac{\mu^k}{k!} \quad (k = 0, 1, 2, \dots).$$

The sample mean  $\bar{x} = (k_1 + \dots + k_n)/n$  of an independent sample of size  $n$  has pmf

$$f(\bar{x}; \mu) = \frac{1}{k_1! \dots k_n!} e^{-n\mu} \mu^{n\bar{x}}$$

For  $\mu < \mu'$ , the likelihood ratio is

$$\lambda(\bar{x}) = \frac{\frac{1}{k_1! \dots k_n!} e^{-n\mu'} \mu'^{n\bar{x}}}{\frac{1}{k_1! \dots k_n!} e^{-n\mu} \mu^{n\bar{x}}} = e^{-n(\mu' - \mu)} \left( \frac{\mu'}{\mu} \right)^{n\bar{x}}, \quad (>1)$$

which is a strictly increasing function of  $\bar{x}$ .  $\square$

### 19.12★ UMP Tests for MLRP Densities

When  $\theta$  is a one-dimensional parameter, we say that the null hypothesis is **one-sided** if it is of the form

$$H_0: \theta \leq \bar{\theta} \quad \text{so} \quad H_1: \theta > \bar{\theta},$$

(or if we reverse the sense of the inequalities).

When the density has the MLRP for the statistic  $T$ , then a Uniformly Most Powerful Test exists. (See Definition 19.6.1.) The next result may be found in Lehmann [11, Theorem 2, p. 68].

**19.12.1 Theorem** *Let  $\Theta$  be one-dimensional, and assume the probability model  $f(x; \theta)$  has monotone likelihood ratio in  $T(x)$ . For testing the null hypothesis  $H_0: \theta \leq \theta_0$  against the alternative  $H_1: \theta > \theta_0$ , there is a UMP test with a critical region of the form  $[c^*, \infty)$ . That is, there is a critical value  $c^*$ , so that the test*

$$\text{rejects } H_0 \text{ if } T(x) \geq c^*.$$

The size of the test, by definition

$$\alpha = \sup\{P_{\theta_0}(T(X) \geq c^*) : \theta \leq \theta_0\}$$

is achieved for  $\theta = \theta_0$ , that is,

$$P_{\theta_0}(T(X) \geq c^*) = \alpha.$$

In addition, the power function (Larsen and Marx's  $1 - \beta(\theta)$ ) is strictly increasing in  $\theta$  (up to the point where it becomes 1, and then it is constant).

*Sketch of proof:*

- If we test the simple null  $\theta = \theta_0$  against a simple alternative  $\theta = \theta'$ , where  $\theta' > \theta$  the Neyman–Pearson Lemma tells us the most powerful test is a likelihood ratio test.

- Because the likelihood ratio is monotone in  $T$ , the test takes the form:

$$\text{Reject } \theta = \theta_0 \text{ against the alternative } \theta = \theta' \text{ if } T \geq c^*$$

for some critical value  $c^*$ .

- Find  $c^*$  to give you the desired level of significance  $\alpha$ .

- Now observe that because of the MLRP the same test specified by  $c^*$  is also a likelihood ratio test of the null  $\theta = \theta_0$  against the simple alternative  $\theta = \theta''$  for any  $\theta'' > \theta_0$ , and it also has significance level  $\alpha$ . So by the Neyman–Pearson Lemma, it is the most powerful such test.

- This means that the test with critical value  $c^*$  is Uniformly Most Powerful for testing the simple null  $\theta = \theta_0$  against the *composite* hypothesis  $\theta > \theta_0$ .

- The MLRP also implies the test is UMP for the composite null. For  $\theta < \theta_0$ , we have  $P_{\theta}(T \geq c^*) \leq P_{\theta_0}(T \geq c^*)$ .

■

The details are spelled out in [11, p. 69], but you can probably work them out yourself.

In this setting, there is another quantity of interest.

- Given such a test we reject  $H_0$  if  $T \geq c^*$ , where  $c^*$  is chosen so that

$$P_{\theta_0}(T \geq c^*) = \alpha.$$

Suppose the test statistic  $T$  has the value  $t$ . The probability

$$P_{\theta_0}(T \geq t)$$

is called the **p-value** of  $t$ .

An equivalent description of the test is to reject  $H_0$  whenever the  $p$ -value of the statistic is less than  $\alpha$ .



### 19.13 Likelihood Ratio Tests for composite hypotheses without MLRP

Larsen–Marx [10]:  
Section 6.5

Likelihood ratio tests can also be used with composite hypotheses even in the absence of the MLRP. For testing the Null Hypothesis  $H_0: \theta \in \Theta_0$  versus the Alternative Hypothesis  $H_1: \theta \in \Theta_1$ , let  $\hat{\theta}_0$  be the maximum likelihood estimator of  $\theta$  over  $\Theta_0$  and let  $\hat{\theta}_1$  be the maximum likelihood estimator of  $\theta$  over  $\Theta_1$ . That is,

$$L(\hat{\theta}_0(\mathbf{x}); \mathbf{x}) = \max\{L(\theta; \mathbf{x}) : \theta \in \Theta_0\}$$

and

$$L(\hat{\theta}_1(\mathbf{x}); \mathbf{x}) = \max\{L(\theta; \mathbf{x}) : \theta \in \Theta_1\}$$

Then the likelihood ratio

$$\lambda(\mathbf{x}) = \frac{L(\hat{\theta}_1(\mathbf{x}); \mathbf{x})}{L(\hat{\theta}_0(\mathbf{x}); \mathbf{x})} = \frac{\max\{L(\theta; \mathbf{x}) : \theta \in \Theta_1\}}{\max\{L(\theta; \mathbf{x}) : \theta \in \Theta_0\}}$$

may serve as a test of the null hypothesis  $H_0: \theta \in \Theta_0$  versus the alternative  $H_1: \theta \in \Theta_1$ .

Now  $\lambda(\mathbf{x})$  depends on the sample  $\mathbf{x}$ , and so is a random variable, which L&M call  $\Lambda$ ,

$$\Lambda = \lambda(\mathbf{x}).$$

According to Larsen–Marx [10, Definition 6.5.1, p. 381] you should choose a critical value  $\lambda^*$  so that the null hypothesis is rejected if

$$\lambda(\mathbf{x}) \geq \lambda^* \geq 0.$$

**Warning:** Larsen–Marx [10] invert the definition of  $\lambda$ , so the above inequality is reversed in their treatment.

They assert the significance level of such a test is given by the  $\alpha$  such that

$$P(\Lambda \geq \lambda^* \mid H_0 \text{ is true}) = \alpha. \tag{*}$$



**19.13.1 Remark** When  $\Theta_0$  consists of just a single parameter value  $\theta_0$ , I know how to make sense of L&M’s statement (\*): Compute the probability using  $P = P_{\theta_0}$ . When  $\Theta_0$  has more than one possible parameter, then (\*) is ambiguous at best, and meaningless at worst. Here is what they mean. For each  $\theta$ , there is a density  $f(\mathbf{x}; \theta)$  of  $\mathbf{x}$ . Then for each  $\theta$ ,

$$P_\theta(\lambda(\mathbf{x}) \geq \hat{\lambda}) = \int \mathbf{1}_{\lambda(\mathbf{x}) \geq \hat{\lambda}}(\mathbf{x}) f(\mathbf{x}; \theta) d\mathbf{x} = \alpha. \tag{**}$$

defines a critical value  $\hat{\lambda}(\theta)$  that makes (\*\*) true. The critical value that should be used for the test is most stringent one,

$$\lambda^* = \max\{\hat{\lambda}(\theta) : \theta \in \Theta_0\}.$$

That way, for every  $\theta \in \Theta_0$  the probability of a Type I error is *no more* than  $\alpha$ . Remember the role of the null hypothesis is the hypothesis that you want to thoroughly discredit it before you are willing to give it up.

Larsen and Marx called this a **generalized likelihood ratio test**, but other statisticians may drop the “generalized.” Again, the usefulness of such test is not because we attach magical properties to the likelihood function, but because test constructed in this way usually have desirable properties.

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