Supplement 4: Review of Integration

Relevant textbook passages:

Pitman [7]:

Larsen–Marx [6]:

This supplement provides a handy reference for many of the results that you learned in high school or Ma 1 on the Riemann integral. They are presented with references, but usually without proof. At the moment it is incomplete.

S4.1 The Classical Fundamental Theorems of Calculus

This section is adapted from my on-line notes.

We start with a review of the Fundamental Theorems of Calculus, as presented in Apostol [3]. The notion of integration employed is the Riemann integral.

S4.1.1 Definition *An* ***indefinite integral*** *F of f over the interval I is any function F such that for some a in I,*

$$F(x) = \int_a^x f(s)\,ds \quad \text{for all } x \text{ in } I.$$

Different values of a give rise to different indefinite integrals of f.

An antiderivative is distinct from the concept of an indefinite integral.

S4.1.2 Definition *A function P is a* ***primitive*** *or* ***antiderivative*** *of a function f on an interval I if*

$$P'(x) = f(x) \quad \text{for every } x \text{ in } I.$$

Leibniz' notation for this is $\int f(x)\,dx = P(x) + C$. Note that if P is an antiderivative of f, then so is $P + C$ for any constant function C.

Despite the similarity in notation, the statement that P is an antiderivative of f is a statement about the derivative of P, namely that $P'(x) = f(x)$ for all x in I; whereas the statement that F is an indefinite integral of f is a statement about the integral of f, namely that there exists some a in I with $\int_a^x f(s)\,ds = F(x)$ for all x in I. Nonetheless there is a close connection between the concepts, which justifies the similar notation. The connection is laid out in the two Fundamental Theorems of Calculus.

S4.1.3 Theorem (First Fundamental Theorem of Calculus [3, Theorem 5.1, p. 202]) *Let f be integrable on $[a, x]$ for each x in $[a, b]$. Let $a \leqslant c \leqslant b$, and let F be the indefinite integral of f defined by*

$$F(x) = \int_c^x f(s)\,ds.$$

Then F is differentiable at every x in (a, b) where f is continuous, and at such points $F'(x) = f(x)$.

Therefore an indefinite integral of a continuous function f is also an antiderivative of f.

S4.1.4 Theorem (Second Fundamental Theorem of Calculus [3, Theorem 5.3, p. 205]) *Let f be continuous on (a,b) and let P be any antiderivative of f on (a,b). The for each x and c in (a,b), we have*

$$P(x) = P(c) + \int_c^x f(s)\,ds.$$

That is, an antiderivative of a continuous function f is also an indefinite integral of f.

S4.2 Some definite integrals

Since the derivative of the arctan function is given by

$$\frac{d}{dx}\arctan(x) = \frac{1}{x^2+1},$$

(see, e.g., Apostol [3, Eqn. 6.48, p. 255]) the Second Fundamental Theorem of Calculus S4.1.4 gives the following.

S4.2.1 Fact *The indefinite integral*

$$\int \frac{1}{x^2+1} = \arctan(x),$$

so

$$\int_0^\infty \frac{1}{x^2+1} = \arctan(x)\Big|_0^\infty = \frac{\pi}{2}.$$

S4.3 Multiple integrals and Fubini's Theorem

See Apostol [4, Chapter 11] for a proper definition of multiple integrals. There are some computational tricks for multiple integrals that allow them to be computed as iterated one-dimensional integrals. Let $Q = [a,b] \times [c,d]$ be a rectangle in the plane, consider

$$\iint_Q f(x,y)\,dx\,dy.$$

Put

$$g(y) = \int_a^b f(x,y)\,dx$$

Is

$$\iint_Q f(x,y)\,dx\,dy = \int_c^d g(y)\,dy\ ?$$

There are two closely related theorems that give an affirmative answer. They are often stated for Lebesgue integrals (see below) instead of Riemann integrals, but I'll quote two results for Riemann integrals. The first appears as Theorem 11.6 in Apostol [4, p. 363], and I'll call it Fubini's Theorem, (cf. Aliprantis–Burkinshaw [2, Theorem 26.6, p. 212]).

S4.3.1 Theorem (Fubini's Theorem for continuous functions) *Let f be continuous on the rectangle $Q = [a,b] \times [c,d]$. Then f is integrable and*

$$\iint_Q f = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx.$$

The latter integrals in the theorem are referred to **iterated integrals**. Also, if $f(x,y) = g(x)h(y)$, then we have the corollary.

S4.3.2 Corollary *Let $f(x,y) = g(x)h(y)$ be continuous on the rectangle $[a,b] \times [c,d]$. Then*

$$\iint_Q f = \left(\int_a^b g(x)\,dx \right) \cdot \left(\int_c^d h(y)\,dy \right).$$

The next theorem appears as Theorem 11.5 in Apostol [4, p. 358], and I'll call it Tonelli's Theorem, (cf. Aliprantis–Burkinshaw [2, Theorem 26.7, p. 213]).

S4.3.3 Theorem (Tonelli's Theorem) *Let f be bounded on the rectangle $Q = [a,b] \times [c,d]$ and assume that f is integrable. For each $y \in [c,d]$ assume that $\int_a^b f(x,y)\,dx$ exists, and denote this value by $h(y)$. If $\int_c^d h(y)\,dy$ exists, then it is equal to*

$$\iint_Q f = \int_c^d \underbrace{\left[\int_a^b f(x,y)\,dx \right]}_{=h(y)} dy.$$

By induction we can reduce an integral over $\boldsymbol{R}^{\mathrm{n}}$ to an iterated sequence of 1-dimensional integrals.

S4.4 Change of variables (substitution)

Apostol [3, § 5.7, pp. 212–217] has a nice exposition of the technique of integration by substitution, also known as change of variables. The main result is the next theorem.

S4.4.1 Theorem (Substitution Theorem, [3, Theorem 5,4, p. 215]) *Let g be continuously differentiable on an open interval $I \subset \boldsymbol{R}$ and let f be a continuous function on the range of g. Then for each $x, a \in I$, we have*

$$\int_a^x f\big[g(t)\big] g'(t)\,dt = \int_{g(a)}^{g(x)} f(u)\,du.$$

When applying the above theorem, we say we are making the **substitution** or making the **change of variable**

$$u = g(t).$$

The next theorem is for multiple integrals. It may be found in Hogg and Craig [5, G4.3 and 4.5] or Apostol [4, Eqn. 11.32, p. 394]. The function H plays a role similar to g in the above theorem.

S4.4.2 Theorem (Change of Variables # 2) *Let $U \subset \boldsymbol{R}^{\mathrm{n}}$, $H \colon U \to X \subset \boldsymbol{R}^{\mathrm{n}}$ be one-to-one and continuously differentiable. Define*

$$J_H(u) = \det \left[\tfrac{\partial H^j}{\partial u_i}(u) \right]$$

and assume that $J_H(u) \neq 0$ for all $u \in U$. Let $f \colon X \to \boldsymbol{R}$ be integrable. Then

$$\int_X f(x)\,dx_1, \ldots, dx_n = \int_U f\big(H(u)\big)\ |J_H(u)|\ du_1, \ldots, du_n.$$

S4.5 Integration by parts

The Fundamental Theorems enable us to prove the following result, which appears in Apostol [3, Section 5.9, pp. 217–218].

S4.5.1 Theorem (Integration by Parts) *Suppose f and g are continuously differentiable on the open interval I. Let $a < b$ belong to I. Then*

$$\int_a^b f(x)g'(x)\,dx + \int_a^b f'(x)g(x)\,dx = f(b)g(b) - f(a)g(a).$$

This result is usually written less symmetrically as

$$\int_a^b f(x)g'(x)\,dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)\,dx,$$

where the integral on the left is one that you already know how to evaluate. Sometimes one uses the language of change of variables: Letting $u = f(x)$ and $v = g(x)$, write $du = f'(x)\,dx$, $dv = g'(x)\,dx$, and

$$\int u\,dv = uv - \int v\,du.$$

S4.6 ⋆ A nasty, brutish, and short intro to the Lebesgue integral

You can probably get by without this material, but maybe you're curious. There are many more thorough expositions. I learned from Royden [9], but I also like Aliprantis and Burkinshaw [2], Aliprantis and Border [1], and Rosentrater [8]. Tao [10] is a highly recommended text by an award-winning teacher and researcher.

The Riemann approach and the Lebesgue approach to integration agree on how to compute the integral of a step function (piecewise constant function) on an interval $[a, b]$. For a partition $a = x_0 < x_1 < \cdots < x_n = b$ of $[a, b]$, if the step function φ takes on the value y_k on the interval $I_k = [x_{k-1}, x_k)$, then the integral of φ is just

$$\int \varphi = \sum_{k=1}^n y_k \lambda(I_k),$$

where λ denotes the length of the interval.

For more general functions f, the Riemann approach approximates f by step functions on finer and finer partitions of $[a, b]$ that agree with f at some point in each subinterval. If there is some number I_f such that for every $\varepsilon > 0$, for sufficiently fine partition, such an approximating step function φ has $|I_f - \int \varphi| < \varepsilon$, the that I is defined to be the **Riemann integral** of f, denoted $\int_a^b f(x)\,dx$.

The Lebesgue approach also approximates f by a simple function (not necessarily a step function), but the approximation is gotten by partitioning the range of f, not the domain. For a *bounded* nonnegative function f with range in $[a, b]$ take a partition of the *range* of the into finitely many subintervals with endpoints $a = y_0 < y_1 < \cdots < y_n = b$. Define

$$E_k = f^{-1}\big[[y_{k-1}, y_k)\big] = \{x : y_k \leqslant f(x) < y_{k+1}\}.$$

Now approximate f from below by the simple function φ defined by

$$\varphi(x) = y_k \text{ whenever } x \in E_k.$$

The define

$$\int \varphi = \sum_{i=1}^n y_k \lambda(E_k),$$

where λ denotes the generalization of length known as **Lebesgue measure**. It is a countably additive set nonnegative function on the σ-algebra generated by the intervals (known as the **Borel σ-algebra**) that agrees with length for intervals. (It is nontrivial to prove that such a thing exists.) Provided f is **measurable** in the sense that the inverse image of every interval belongs to the Borel σ-algebra, then the supremum of the $\int \varphi$ exists as the partition of the range becomes finer and finer. The supremum is defined to be the **Lebesgue integral** of f, denoted $\int_{[a,b]} f \, d\lambda$ or $\int_{[a,b]} f(x) \, d\lambda(x)$ or $\int_{[a,b]} f(x) \, \lambda(dx)$.

This may seem like a lot of trouble to go through, so there should be some good reason for defining a whole new approach to integration. One reason is that every function with a Riemann integral has a Lebesgue integral and they agree, but there are many functions which have a well defined Lebesgue integral, but for which the Riemann integral does not exist. (Also you have probably forgotten how complicated the definition of the Riemann integral really is.) Another reason is that Lebesgue integration generalizes in a straightforward way to integrals on abstract probability spaces. Indeed the best way to define the expectation of a random variable is as a Lebesgue integral.

S4.6.1 Theorem (Cf. [2, Theorem 23.6, p. 183]) *If $f : [a,b] \to \boldsymbol{R}$ is Riemann integrable, then it is Lebesgue integrable and the integrals coincide.*

The next theorem may be found in Aliprantis–Burkinshaw [2, Theorem 23.7, p. 184].

S4.6.2 Theorem (Lebesgue–Vitali) *A bounded function $f : [a,b] \to \boldsymbol{R}$ is Riemann integrable if and only if its set of discontinuities has Lebesgue measure zero.*

S4.6.3 Lemma *Every countable subset of $\boldsymbol{R}$ has Lebesgue measure zero.*

Proof: Let $\{x_1, x_2, \dots\}$ be a countable set. Each point x_i is a degenerate interval $[x_i, x_i]$, so $\lambda(x_i) = 0$. Since λ is countably additive,

$$\lambda\big(\{x_1, x_2, \dots\}\big) = \sum_i \lambda(x_i) = 0.$$

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S4.6.4 Example Let $\mathbb{Q}$ denote the rationals and $\mathbb{J}$ denote the irrationals in $[0,1]$. The indicator function $\mathbf{1}_{\mathbb{J}}$ of the irrationals in $[0,1]$, that is,

$$\mathbf{1}_{\mathbb{J}}(x) = \begin{cases} 1, & \text{if } x \text{ is irrational, and } 0 < x < 1, \\ 0, & \text{otherwise,} \end{cases}$$

has no Riemann integral, but its Lebesgue integral is one.

Since $\mathbf{1}_{\mathbb{J}}$ is simple, its Lebesgue integral is just $1 \cdot \lambda(\mathbb{J})$. Now $[0,1]$ is the disjoint union $\mathbb{Q} \cup \mathbb{J}$. Thus $\lambda(\mathbb{Q}) + \lambda(\mathbb{J}) = \lambda\big([0,1]\big) = 1$. But $\mathbb{Q}$ is countable, so we have $\lambda(\mathbb{Q}) = 0$ (Lemma S4.6.3), and so $\lambda(\mathbb{J}) = 1$.

To see that $\mathbf{1}_{\mathbb{J}}$ has no Riemann integral, just observe that any subinterval of positive length contains both an irrational and a rational number, so we can pick step functions φ that are identically zero or identically one for our approximations, so a limit does not exist. □

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