Caltech Department of Mathematics

Ma 3/103 KC Border
Introduction to Probability and Statistics Winter 2020

Assignment 7: Exercises on Hypothesis Testing

Due Tuesday, March 3 by 4:00 p.m.
in the dropbox in the lobby of Linde Hall.

Instructions:

When asked for a probability or an expectation, give both a formula and an explanation for why you used that formula, and also give a numerical value when available.

When asked to plot something, use informative labels (even if handwritten), so the TA knows what you are plotting, attach a copy of the plot, and, if appropriate, the commands that produced it.

Exercise 1 (Confidence Intervals) (47 pts)

For this question, the calculations are trivial. What matters is your reason for doing them. You *must* explain and defend your reasoning.

Consider the continuous "German tank problem." Here the probability model is

$$f(x;\theta) = \begin{cases} \frac{1}{\theta} & 0 \leqslant x \leqslant \theta \\ 0 & \text{otherwise} \end{cases}$$

where $\theta > 0$ is the unknown parameter. The data comprise a sample of n independent draws $\boldsymbol{x} = (x_1, \dots, x_n)$ from this distribution.

1. (5 pts) Write down the likelihood function. What is the maximum likelihood estimator $\hat{\theta}$ of θ, written as a function of the datum $\boldsymbol{x}$?

Okay, here comes the real question: How should you construct a $1-\alpha$ confidence interval for θ? It helps to break down your answer into steps:

2. (10 pts) If you knew θ, what is the smallest interval $[a,b]$ (the endpoints may depend on θ) such that
$$P_\theta\Big(\hat{\theta}(\boldsymbol{x}) \in [a,b]\Big) = 1-\alpha?$$

3. (10 pts) Find an interval $[\bar{a}(\hat{\theta}), \bar{b}(\hat{\theta})]$ such that
$$\Big(\hat{\theta} \in [a,b]\Big) \iff \Big(\theta \in [\bar{a}(\hat{\theta}), \bar{b}(\hat{\theta})]\Big).$$

4. (4 pts) Compute $\bar{a}(\hat{\theta}), \bar{b}(\hat{\theta})$ for $n = 5$, and $\alpha = 0.10, 0.05, 0.01$.
5. (10 pts) Derive the method of moments estimator of θ.
6. (4 pts) Must the method of moments estimate fall in the confidence interval? (Why or why not?) □

The following problems are taken from Larsen and Marx [2]. I have slightly modified the text of some of the questions. These problems are not intellectually challenging, but they are representative of the sort of routine hypothesis testing that every researcher needs to carry out in order to publish their work. (It's kind of like practicing scales on a musical instrument. It's not fun [for most people], but you want to be able to do it without using your higher brain functions. Legend has it that Charlie Parker spent eight to sixteen hours a day for five years practicing scales in a woodshed. It's unlikely we would have bebop if he hadn't.)

Exercise 2 (20 pts) Problem 6.4.12, p. 378.

An urn contains ten chips. An unknown number of chips are white, the rest are red. We wish to test

$$H_0\colon \text{exactly half the chips are white}$$
$$\text{against}$$
$$H_1\colon \text{more than half the chips are white.}$$

We will draw, without replacement, three chips and reject H_0 if two or more are white. Find α, the significance level of this test. Also, find β, the probability of a type II error, when the urn contains k white chips, for $k = 6, 7, \dots, 10$. □

Exercise 3 (10 pts) Problem 6.3.2, p. 366.

Efforts to find a genetic explanation for why certain people are right-handed and others left-handed have been largely unsuccessful. Reliable data are difficult to find because of environmental factors that also influence a a child's "handedness." To avoid that complication, researchers often study the analogous problem of "pawedness" in animals, where both genotypes and the environment can be partially controlled.

In one such experiment [1], mice were put into a cage having a feeding tube that was equally accessible from the right or the left. Each mouse was carefully watched over a number of feedings. If it used its right paw more than half the time, to activate the tube, it was defined to be "right-pawed." Observations of this sort showed that 67% of mice belonging to strain A/J are right-pawed. A similar protocol was followed on a sample of thirty-five mice belonging to strain A/HeJ. Of those thirty-five, a total of eighteen were eventually classified as right-pawed.

• Test whether the proportion of right-pawed mice found in the A/HeJ sample was significantly different from what was known about the A/J strain. Use a two-sided alternative and let 0.05 be the probability associated with the critical region. □

Exercise 4 (15 pts) Problem 6.4.4, p. 377.

Plot a power curve for the $\alpha = 0.05$ test of $H_0\colon \mu = 60$ versus $H_1\colon \mu \neq 60$ if the data consist of a random sample of size 16 from a normal distribution having $\sigma = 4$. □

Exercise 5 (15 pts) Problem 6.4.8, p. 378.

Will $n = 45$ be a sufficiently large sample to test $H_0\colon \mu = 10$ versus $H_1\colon \mu \neq 10$ at the $\alpha = 0.05$ level of significance if the experimenter wants the Type II error probability to be no greater than 0.20 when $\mu = 12$? Assume that $\sigma = 4$. □

Exercise 6 (15 pts) Problem 6.4.10, p. 378.

Suppose a sample of size 1 is taken from the pdf $f_Y(y) = (1/\lambda)e^{-y/\lambda}$, $y > 0$, for the purpose of testing $H_0\colon \lambda = 1$ versus $H_1\colon \lambda > 1$. The null hypothesis will be rejected if $y \geqslant 3.20$.

1. Calculate the probability of committing a Type I error.
2. Calculate the probability of committing a Type II error when $\lambda = 4/3$.
3. Draw a diagram that shows the α and β and calculated in parts (1) and (2) as areas. □

Exercise 7 (10 pts) Problem 7.4.6, p. 399.

Let $\bar{Y}$ and S denote the the sample mean and sample standard deviation, respectively, based on a set of $n = 20$ measurements taken from a normal distribution with $\mu = 90.6$ Find the function $k(S)$ for which

$$P\Big(90.6 - k(S) \leqslant \bar{Y} \leqslant 90.6 + k(S)\Big) = 0.99$$

□

Exercise 8 (10 pts) How much total time did you spend on the preceding exercises? **Please put the answer to this exercise on the *front page* of your answers and identify it as such.** □

Exercise 9 (Optional Exercise) (50 pts) **No collaboration is allowed on optional exercises**.

A sample of size n is taken form a uniform distribution on $[a, b]$, where a and b are unknown. What is the maximum likelihood estimate of the length $b - a$ of the support of the uniform distribution? Construct a 95% confidence interval for the length. □

References

[1] R. L. Collins. 1968. On the inheritance of handedness. *Journal of Heredity* 59(1):9–12. http://jhered.oxfordjournals.org/content/59/1/9.full.pdf+html

[2] R. J. Larsen and M. L. Marx. 2012. *An introduction to mathematical statistics and its applications*, fifth ed. Boston: Prentice Hall.