Caltech Department of Mathematics

Ma 3/103 KC Border
Introduction to Probability and Statistics Winter 2020

Answers for HW 7

Sample answer for Exercise 1: Heare are some selected pieces of a solution

1.

> The likelihood function is
>
> $$L(\theta; \boldsymbol{x}) = \begin{cases} \left(\frac{1}{\theta}\right)^n & \text{if } 0 \leqslant x_i \leqslant \theta,\ i = 1, \dots, n, \\ 0 & \text{otherwise.} \end{cases}$$
>
> This likelihood is maximized by making θ as small as possible, subject to $\theta \geqslant x_i$ for all i, so the maximum likelihood estimator of θ is
>
> $$\hat{\theta}_{\text{MLE}}(\boldsymbol{x}) = \max_i x_i.$$

2. We know that the data must satisfy $\hat{\theta}_{\text{MLE}} = \max_i x_i \leqslant \theta$. Therefore a confidence interval for the parameter θ should look like $[\hat{\theta}_{\text{MLE}}, b]$, which is a lot different from the confidence interval for the mean of a Normal distribution, which is centered on the estimate. The question is, how is b determined?

   If we knew the parameter θ, the cumulative distribution function G_θ of $\hat{\theta}_{\text{MLE}}$ is given by

   $$G_\theta(t) = \left(\frac{t}{\theta}\right)^n$$

   and the density g_θ of the MLE is thus

   $$g_\theta(t) = \frac{n}{\theta^n} t^{n-1}.$$

   The density is strictly increasing in t (for $t \in [0, \theta]$), and the probability of any interval $[a, b]$ is just $\int_a^b g_\theta(t)\,dt$. So for a given probability $1 - \alpha$, xthe interval is shortest when g_θ is greatest, which means an interval of the form

   $$[a, \theta],$$

   and its probability is just $1 - G_\theta(a)$.

We want to choose a so that

$$P_\theta\Big(\hat{\theta}_{\text{MLE}} \in [a,\theta]\Big) = 1-\alpha,$$

so

$$\alpha = G_\theta(a) = \left(\frac{a}{\theta}\right)^n.$$

In other words,

$$\boxed{a = \sqrt[n]{\alpha}\theta.}$$

So, if we know θ, the shortest interval I with $P\Big(\hat{\theta}_{\text{MLE}} \in I\Big)$ is

$$[\sqrt[n]{\alpha}\theta,\ \theta]$$

3. The statement

$$\sqrt[n]{\alpha}\theta \leqslant \hat{\theta}_{\text{MLE}} \leqslant \theta$$

is equivalent to the statement

$$\boxed{\hat{\theta}_{\text{MLE}} \leqslant \theta \leqslant \hat{\theta}_{\text{MLE}}/\sqrt[n]{\alpha},}$$

so the $1-\alpha$ confidence interval for θ is

$$\left[\hat{\theta}_{\text{MLE}},\ \hat{\theta}_{\text{MLE}}/\sqrt[n]{\alpha}\right].$$

4.

Here is a table of right-hand endpoint coefficients $\alpha^{-1/n}$:

		$1-\alpha$				
		0.995	0.99	0.975	0.95	0.9
	1	200.0000	100.0000	40.0000	20.0000	10.0000
	2	14.1421	10.0000	6.3246	4.4721	3.1623
	4	3.7606	3.1623	2.5149	2.1147	1.7783
	8	1.9392	1.7783	1.5858	1.4542	1.3335
	16	1.3926	1.3335	1.2593	1.2059	1.1548
n	32	1.1801	1.1548	1.1222	1.0981	1.0746
	64	1.0863	1.0746	1.0593	1.0479	1.0366
	128	1.0423	1.0366	1.0292	1.0237	1.0182
	256	1.0209	1.0182	1.0145	1.0118	1.0090
	512	1.0104	1.0090	1.0072	1.0059	1.0045
	1024	1.0052	1.0045	1.0036	1.0029	1.0023

5.

> The method of moments estimator satisfies $\bar{x} = \theta/2$, where $\bar{x}$ is the sample average, $(x_1 + \cdots + x_n)/n$, or
>
> $$\hat{\theta}_{\text{MoM}} = 2\bar{x}.$$

6.

> No. The Method of Moments estimator could easily fall below the lower bound of the confidence interval. For instance,if the sample size is four, and the values drawn are $1, 2, 3, 10$, then the MOM is $2\bar{x} = 8$, which is strictly less than the MLE of 10, and so it falls outside the confidnece 95% interval, which is $[10, 21.47]$ according to Part 4.

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Sample answer for Exercise 2: The significance level α of the test it the probability that the null hypothesis is rejected when it is indeed true. For this test, it is just the probability of two or three white balls out of three, when the urn has 5 white and 5 red chips. These are governed by the hypergeometric probabilities (see Pitman, p. 125 or L&M pp. 110–111):

$$P(2 \text{ white, } 1 \text{ red}) = \frac{\binom{5}{2}\binom{5}{1}}{\binom{10}{3}} = \frac{5}{12} \qquad P(3 \text{ white, } 0 \text{ red}) = \frac{\binom{5}{3}\binom{5}{0}}{\binom{10}{3}} = \frac{1}{12}$$

so

$$P(2 \text{ or } 3 \text{ white}) = \frac{1}{2}.$$

> So the significance level is $\alpha = 0.5$.

The probability of a type II error, that is accepting the null hypothesis that there are 5 white and 5 red balls, when in fact there are k white balls, is just the provability of 0 or 1 white balls in the sample. So when k balls are white

$$P(1 \text{ white, } 2 \text{ red}) = \frac{\binom{k}{1}\binom{10-k}{2}}{\binom{10}{3}} \qquad P(0 \text{ white, } 3 \text{ red}) = \frac{\binom{k}{0}\binom{10-k}{3}}{\binom{10}{3}}$$

These probabilities are given in the following table.

	Number of white balls in total					
Number of white balls in sample	5	6	7	8	9	10
0	$\frac{1}{12}$	$\frac{1}{30}$	$\frac{1}{120}$	0	0	0
1	$\frac{5}{12}$	$\frac{3}{10}$	$\frac{7}{40}$	$\frac{1}{15}$	0	0
2	$\frac{5}{12}$	$\frac{1}{2}$	$\frac{21}{40}$	$\frac{7}{15}$	$\frac{3}{10}$	0
3	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{7}{24}$	$\frac{7}{15}$	$\frac{7}{10}$	1

So the probability of a Type II error with k white balls is given in the last column

$$\begin{array}{lccccc} & P(0\text{ white}) & + & P(1\text{ white}) & = & P(\text{Type II error}) \\ k=5 & \frac{1}{12} & + & \frac{5}{12} & = & \frac{1}{2} \\ k=6 & \frac{1}{30} & + & \frac{3}{10} & = & \frac{1}{3} \\ k=7 & \frac{1}{120} & + & \frac{7}{40} & = & \frac{11}{60} \\ k=8 & 0 & + & \frac{1}{15} & = & \frac{1}{15} \\ k=9 & 0 & + & 0 & = & 0 \\ k=10 & 0 & + & 0 & = & 0 \end{array}$$

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Sample answer for Exercise 3:

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Sample answer for Exercise 4:

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Sample answer for Exercise 5:

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Sample answer for Exercise 6:

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Sample answer for Exercise 7:

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