Caltech Department of Mathematics

Answers for Assignment 3

Sample answer for Exercise 1: To be supplied at a later date. ◇

Sample answer for Exercise 2:

1. (15 pts) Let E_i be the even that urn i contains at least one ball. Then

$$P(E_i) = 1 - P(E_i^c),$$

and $P(E_i^c)$ is the probability that no ball is in urn i. There are m urns, so the probability that any given ball does not hit urn i is just $(m-1)/m$. In n independent trials the probability that none of the n balls hit urn i is

$$P(E_i^c) = \left(\frac{m-1}{m}\right)^n,$$

so

$$P(E_i) = 1 - \left(\frac{m-1}{m}\right)^n.$$

Let X denote the number of urns with at least one ball. Then

$$\boxed{\boldsymbol{E}\, X = \sum_{i=1}^{m} P(E_i) = m\left(1 - \left(\frac{m-1}{m}\right)^n\right).}$$

2. (15 pts) Let E_i be the event that card i is a Club, and let X be the random variable that tells how many Clubs are in the hand. Then by the argument above,

$$\boldsymbol{E}\, X = \sum_{i=1}^{7} P(E_i).$$

By symmetry each E_i has probability $1/4$, so

$$\boxed{\boldsymbol{E}\, X = 1\tfrac{3}{4}.}$$

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Sample answer for Exercise 3: Start by constructing the sample space for the experiment. It is a countably infinite set of finite sequences from $\{1, \dots, 6\}$. Let S_n be the event that experiment stops on roll n. That is,

$$S_1 = \{6\} \quad \text{and} \quad S_n = \{1, \dots, 5\}^{n-1} \times \{6\}, \quad n > 1.$$

Then the sample space S for the experiment is the countably infinite set

$$S = \bigcup_{n=1}^{\infty} S_n,$$

As an aside, $P(S_n) = (5/6)^{n-1}(1/6)$.

The random variable W of interest is the waiting time to the first 6, and is the number of rolls the experiment lasts. That is, the event $(W = n)$ is just S_n.

Let E be the event in S that all the rolls are even. The conditional expectation of W given E is

$$\boldsymbol{E}(W \mid E) = \sum_{n=1}^{\infty} nP\left(W = n \mid E\right) = \sum_{n=1}^{\infty} nP(S_n \mid E).$$

So we need to compute the conditional probabilities $P(S_n \mid E)$. Start by letting

$$E_n = E \cap S_n,$$

and observe that

$$E_1 = \{6\} \quad \text{and} \quad E_n = \{2, 4\}^{n-1} \times \{6\}, \quad n > 1.$$

So the probability of the event E_n is

$$P(E_n) = (1/3)^{n-1}(1/6), \tag{1}$$

which is not quite geometric. Since $E = \bigcup_{n=1}^{\infty} E_n$, and the E_n are pairwise disjoint, we have

$$P(E) = \sum_{n=1}^{\infty} P(E_n) = \sum_{n=1}^{\infty} (1/3)^{n-1}(1/6) = \frac{1}{4}. \tag{2}$$

Now from (1) and (2),

$$P(S_n \mid E) = \frac{P(S_n \cap E)}{P(E)} = \frac{P(E_n)}{P(E)} = (2/3)(1/3)^{n-1}. \tag{3}$$

Note that this is a geometric probability with parameter $p = 2/3$. So we know that

$$\boldsymbol{E}(W \mid E) = \sum_{n=1}^{\infty} nP(S_n \mid E) = \sum_{n=1}^{\infty} n(2/3)(1/3)^{n-1} = \frac{3}{2}.$$

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