Ec 181
Convex Analysis and Economic Theory

KC Border
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Topic 29: The Simplex Method

In 1945, the economist George J. Stigler [17] undertook to find the minimum cost diet meeting various nutritional constraints. His results were obtained by examining a promising but limited subset of diets. He wrote (p. 310) that "the procedure is experimental because there does not appear to be any direct method of finding the minimum of a linear function subject to linear conditions."[1]

Just two years later, the simplex algorithm was invented to solve just such problems. According to George B. Dantzig [4, p. 24], the widely acknowledged originator of the algorithm,

> During the summer of 1947, Leonid Hurwicz, well-known econometrician associated with the Cowles Commission, worked with the author on techniques for solving linear programming problems. This effort and some suggestions of T. C. Koopmans resulted in the "Simplex Method."

The Simplex Method is an algorithmic method for solving linear programs. It is no longer the fastest method, but it is easy to understand and program, and if you use rational arithmetic, it is exact. It is also closely related to the polyhedral structure of the feasible set. These notes borrow extensively from the lucid expositions by David Gale [8], Joel Franklin [7], and George Dantzig [4].

[1] Just for the fun of it, here is Table 2 from Stigler [17, p. 311] outlining the minimum cost diet in August 1939 and in August 1944. Bon appétit!

Minimum Cost Annual Diets

	August 1939		August 1944	
Commodity	Quantity	Cost	Quantity	Cost
wheat flour	370 lb.	$13.33	585 lb.	$34.58
evaporated milk	57 cans	3.84	—	—
cabbage	111 lb.	4.11	107 lb.	5.23
spinach	23 lb.	1.85	13 lb.	1.56
dried navy beans	285 lb.	16.80	—	—
pancake flour	—	—	134 lb.	18.08
pork liver	—	—	45 lb.	5.48
Total cost		$39.93		$59.88

29.1 The simplex method

We are now ready to apply the replacement operation to linear programming. Dantzig [4] draws a distinction between the **simplex *method*** and the **simplex *algorithm***. The simplex method consists of two **phases**, each of which uses the simplex algorithm. The simplex algorithm is a rule for choosing pivots for successive replacement operations until a stopping condition is reached.

For concreteness, consider the following linear programming problem. Let A be an $m \times n$ matrix, let q belong to $\boldsymbol{R}^{\mathrm{m}}$, and p belong to $\boldsymbol{R}^{\mathrm{n}}$. To use the simplex algorithm we need to write the primal program in canonical equality form:

$$\underset{x \in \boldsymbol{R}^{\mathrm{n}}}{\text{maximize}} \; p \cdot x$$

subject to

$$\begin{aligned} Ax &= q \\ x &\geqq 0 \end{aligned}$$

This program has n variables $x_1, \ldots, x_n$, m equality constraints $A_i \cdot x_i = q_i$, and n nonnegativity constraints on the x_i's.

The dual of the is program is:

$$\underset{y \in \boldsymbol{R}^{\mathrm{m}}}{\text{minimize}} \; q \cdot y$$

subject to

$$A'y \geqq p$$

Notice that there are no sign constraints on y.

A vector x is **feasible** for the primal if $Ax = q$ and $x \geqq 0$, and it is **optimal** if it is feasible and attains the maximum. **Phase 1** of the simplex method uses the simplex algorithm to find a feasible vector, or else proves that none exists. **Phase 2** starts with a feasible vector, and uses the simplex algorithm to find an optimal vector. Paradoxically, Phase 1 uses Phase 2, so we start with that. Phase 1 is covered in Section 29.4.

29.1.1 The simplex *tableau* and Phase 2

The matrix A is $m \times n$, so each column is a vector in $\boldsymbol{R}^{\mathrm{m}}$. The linear span of the columns is called the **column space** of A. The dimension of the column space is called the **rank** of A. For the time being, assume:

29.1.1 Assumption (Rank Assumption) *The column space of the $m \times n$ matrix A has dimension m.*

Under the Rank Assumption, every basis for the column space of A has m elements, so we must have $n \geqslant m$, that is, at least as many variables as constraints. If our original problem involves only inequality constraints, we can convert it into a problem with equality constraints by adding "slack" variables for each constraint, which will guarantee that the Rank Assumption holds. Nevertheless, the Rank Assumption is only used to simplify the analysis and is not crucial. Also, if the primal has more constraints than variables, then the dual will have more variables than constraints.

Elaborate on this point.

In fact we can find a basis (usually more than one) consisting only of columns of A. By Proposition 28.6.2 if there is an optimal solution, then there is an optimal solution that depends only on an independent subset of the columns.

Assume that we have somehow found (in Phase 1) a feasible solution $x = (x_1, \dots, x_n) \geqq 0$ of $Ax = q$ that depends on a basis $\{A^{c_1}, \dots, A^{c_m}\}$ of m columns of A. That is,

$$q = \sum_{j=1}^{n} x_j A^j = \sum_{i=1}^{m} x_{c_i} A^{c_i},$$

where

$$x_{c_i} \geqslant 0, \; i = 1, \dots, m, \quad \text{and} \quad x_j = 0 \text{ for } j \notin \{c_1, \dots, c_m\}.$$

Under the Rank Assumption, in fact, $x_{c_i} > 0$, $i = 1, \dots, m$. Also the Rank Assumption, every column A^j is a unique linear combination of the basis columns $\{A^{c_1}, \dots, A^{c_m}\}$, say

$$A^j = \sum_{i=1}^{m} t_{i,j} A^{c_i}, \quad j = 1, \dots, n.$$

Given this uniqueness, the basis determines x, and so determines $p \cdot x$. Thus:

> The linear programming problem can be thought of as finding the optimal basis of the columns of A. The simplex algorithm is a rule for replacing columns in the basis, one at a time, until the optimal basis is found.

29.1.2 Replacement operations on the simplex *tableau*

The idea behind the simplex algorithm is to choose the replacement pivot so that at each stage $p \cdot x$ increases, or at least does not decrease. In order to do this, we must examine how the *tableau* changes when we change the basis.

Start with the following *tableau*.

	A^1	$\cdots$	A^ℓ	$\cdots$	A^n	q
A^{c_1}	$t_{1,1}$	$\cdots$	$t_{1,\ell}$	$\cdots$	$t_{1,n}$	x_{c_1}
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
A^{c_k}	$t_{k,1}$	$\cdots$	$t_{k,\ell}$	$\cdots$	$t_{k,n}$	x_{c_k}
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
A^{c_m}	$t_{m,1}$	$\cdots$	$t_{m,\ell}$	$\cdots$	$t_{m,n}$	x_{c_m}

Bear with me while we see what happens when we pivot on $t_{k,\ell}$ in order to replace A^{c_k} by A^ℓ. This replacement will yield the new *tableau*

	A^1	$\cdots$	$A^{\ell-1}$	A^ℓ	$A^{\ell+1}$	$\cdots$	A^n	q
A^{c_1}	$t'_{1,1}$	$\cdots$	$t'_{1,\ell-1}$	0	$t'_{1,\ell+1}$	$\cdots$	$t'_{1,n}$	x'_{c_1}
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\cdots$
$A^{c_{k-1}}$	$t'_{k-1,1}$	$\cdots$	$t'_{k-1,\ell-1}$	0	$t'_{k-1,\ell+1}$	$\cdots$	$t'_{k-1,n}$	$x'_{c_{k-1}}$
$A^\ell = A^{c'_k}$	$t'_{k,1}$	$\cdots$	$t'_{k,\ell-1}$	1	$t'_{k,\ell+1}$	$\cdots$	$t'_{k,n}$	$x'_\ell = x'_{c'_k}$
$A^{c_{k+1}}$	$t'_{k+1,1}$	$\cdots$	$t'_{k+1,\ell-1}$	0	$t'_{k+1,\ell+1}$	$\cdots$	$t'_{k+1,n}$	$x'_{c_{k+1}}$
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\cdots$
A^{c_m}	$t'_{m,1}$	$\cdots$	$t'_{m,\ell-1}$	0	$t'_{m,\ell+1}$	$\cdots$	$t'_{m,n}$	x'_{c_m}

The new ℓ^{th} column is the k^{th} unit coordinate vector e^k. The new k^{th} row has

$$t'_{k,j} = \frac{t_{k,j}}{t_{k,\ell}}, \quad j = 1, \ldots, n \tag{1}$$

(which implies $t'_{k,\ell} = 1$) and

$$x'_{c_k} = \frac{x_{c_k}}{t_{k,\ell}}. \tag{2}$$

As for the remainder of the *tableau*, the new i^{th} row for $i \neq k$ has

$$t'_{i,j} = t_{i,j} - \frac{t_{k,j}}{t_{k,\ell}} t_{i,\ell}, \quad j = 1, \ldots, n \tag{3}$$

(note that this implies $t'_{i,\ell} = 0$) and

$$x'_{c_i} = x_{c_i} - \frac{x_{c_k}}{t_{k,\ell}} t_{i,\ell}. \tag{4}$$

We can now compute what happens to $p \cdot x$ when A^{c_k} is replaced by A^ℓ. Initially

$$p \cdot x = \sum_{i=1}^m p_{c_i} x_{c_i}.$$

After the replacement,

$$\begin{aligned}
p \cdot x' &= \sum_{i=1}^{m} p_{c_i} x'_{c_i} \\
&= p_\ell \underbrace{\frac{x_{c_k}}{t_{k,\ell}}}_{x'_\ell = x'_{c_k}} + \sum_{\substack{i=1\\ i\neq k}}^{m} p_{c_i} \underbrace{\left(x_{c_i} - \frac{x_{c_k}}{t_{k,\ell}} t_{i,\ell} \right)}_{x'_{c_i}} \\
&= p_\ell \frac{x_{c_k}}{t_{k,\ell}} + \sum_{\substack{i=1\\ i\neq k}}^{m} p_{c_i} x_{c_i} - \sum_{\substack{i=1\\ i\neq k}}^{m} p_{c_i} \frac{x_{c_k}}{t_{k,\ell}} t_{i,\ell} \\
&= p_\ell \frac{x_{c_k}}{t_{k,\ell}} + \Bigg(\underbrace{\sum_{i=1}^{m} p_{c_i} x_{c_i}}_{p\cdot x} - \cancel{p_{c_k} x_{c_k}} \Bigg) - \left(\sum_{i=1}^{m} p_{c_i} \frac{x_{c_k}}{t_{k,\ell}} t_{i,\ell} - \cancel{p_{c_k} \frac{x_{c_k}}{t_{k,\ell}} t_{k,\ell}} \right)
\end{aligned}$$

The difference is

$$p \cdot x' - p \cdot x = \frac{x_{c_k}}{t_{k,\ell}} \left(p_\ell - \sum_{i=1}^{m} p_{c_i} t_{i,\ell} \right) \tag{5}$$

This suggests the following definition. Given a *tableau*, define

$$\pi_j = \sum_{i=1}^{m} t_{i,j} p_{c_i}, \qquad j = 1, \ldots, n. \tag{6}$$

The j^{th} column A^j is a linear combination $\sum_{i=1}^{m} t_{i,j} A^{c_i}$ of the basis columns $A^{c_1}, \ldots, A^{c_m}$, so the value of the linear combination is given by $\pi_j = \sum_{i=1}^{m} t_{i,j} p_{c_i}$. The value of column j is p_j. By (5), we have:

$p \cdot x' > p \cdot x$ if and only if $\dfrac{x_{c_k}}{t_{k,\ell}} > 0$ and $p_\ell > \pi_\ell$.

29.1.3 Adding a criterion row

Let us keep track of changes in $p \cdot x$ by adding a **criterion row** to the bottom of the *tableau*.

The j^{th} column of the criterion row is $\pi_j - p_j$, for $j = 1, \ldots, n$ and the last column is $p \cdot x = \sum_{i=1}^{m} p_{c_i} x_{c_i}$.

The *tableau* now looks like this.

	A^1	$\cdots$	A^ℓ	$\cdots$	A^n	q
A^{c_1}	$t_{1,1}$	$\cdots$	$t_{1,\ell}$	$\cdots$	$t_{1,n}$	x_{c_1}
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
A^{c_k}	$t_{k,1}$	$\cdots$	$t_{k,\ell}$	$\cdots$	$t_{k,n}$	x_{c_k}
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
A^{c_m}	$t_{m,1}$	$\cdots$	$t_{m,\ell}$	$\cdots$	$t_{m,n}$	x_{c_m}
	$\pi_1 - p_1$	$\cdots$	$\pi_\ell - p_\ell$	$\cdots$	$\pi_n - p_n$	$p \cdot x$

N.B. The criterion row is not part of the *tableau* in the sense that we do not intend to imply that the vector $\pi - p$ is used in the basis. Its elements have a different interpretation, but nonetheless it is easy to update after a replacement operation.

Using (6), after a replacement operation where A^{c_k} is replaced by A^ℓ, the new criterion row must be computed:

$$\begin{aligned}
\pi'_j - p_j &= \sum_{i=1}^m t'_{i,j} p_{c'_i} - p_j \\
&= \sum_{\substack{i=1\\ i\neq k}}^m t'_{i,j} p_{c_i} + t'_{k,j} p_\ell - p_j \qquad \text{(only } c_k \text{ has changed; } c'_k = \ell) \\
&= \sum_{\substack{i=1\\ i\neq k}}^m \left(t_{i,j} - \frac{t_{i,\ell}}{t_{k,\ell}} t_{k,j} \right) p_{c_i} + \frac{t_{k,j}}{t_{k,\ell}} p_\ell - p_j \qquad \text{(use (1) and (3))} \\
&= \sum_{i=1}^m \left(t_{i,j} - \frac{t_{i,\ell}}{t_{k,\ell}} t_{k,j} \right) p_{c_i} - \underbrace{\left(t_{k,j} - \frac{t_{k,\ell}}{t_{k,\ell}} t_{k,j} \right)}_{=0} p_{c_k} + \frac{t_{k,j}}{t_{k,\ell}} p_\ell - p_j \\
&= \underbrace{\sum_{i=1}^m t_{i,j} p_{c_i}}_{\pi_j} - \frac{t_{k,j}}{t_{k,\ell}} \underbrace{\sum_{i=1}^m t_{i,\ell} p_{c_i}}_{\pi_\ell} + \frac{t_{k,j}}{t_{k,\ell}} p_\ell - p_j \\
&= (\pi_j - p_j) - \frac{t_{k,j}}{t_{k,\ell}} (\pi_\ell - p_\ell). \qquad (7)
\end{aligned}$$

Finally, by (5)

$$p \cdot x' = p \cdot x - \frac{x_{c_k}}{t_{k,\ell}} (\pi_\ell - p_\ell). \qquad (8)$$

Comparing equations (7) and (8) to (3), we see that:

> *The updated criterion row is also computed from the* tableau *the same way as any other row!*

Equation (8) also explains why we use $\pi - p$ in the criterion row rather than $p - \pi$.

29.1.4 Choosing the pivot

We want to choose the pivot for the replacement operation to do two things:

1. Increase the value, $p \cdot x'$.
2. Keep $x' \geqq 0$, and

By assumption, before the pivot operation each x_{c_i} (the row i entry under q) satisfies $x_{c_i} \geqslant 0$. It follows from (5) that to increase the value, we should ***choose the column ℓ, so that $\pi_\ell - p_\ell < 0$***, for then $p \cdot x' \geqslant p \cdot x$ and $p \cdot x' > p \cdot x$ provided $x_{c_k} > 0$.

So having chosen a pivot column ℓ, how do we choose the pivot row so that, so that for all $i = 1, \dots, m$,

$$x'_{c'_i} \geqslant 0?$$

For each row i with $t_{i,\ell} > 0$ compute the ratio

$$r_i = \frac{x_{c_i}}{t_{i,\ell}}.$$

By the Replacement Lemma 25.10.1 in order to keep $x' \geqq 0$, we need to ***choose the pivot row k so that***

$$\boldsymbol{t_{k,\ell} > 0 \quad \textit{and} \quad r_k = \min\{r_i : t_{i,\ell} > 0\}.}$$

(Compare this to the proof of Lemma 2.3.3 and Remark 2.3.4.)

29.1.5 The simplex algorithm made explicit

To summarize, the simplex algorithm is this (but there are many variations):

The naïve simplex algorithm

Step 1. Choose the pivot column j so that

$$\pi_j - p_j < 0 \qquad \text{for maximization}$$
$$\pi_j - p_j > 0 \qquad \text{for minimization.}$$

If more than one j has this property, the choice is not crucial, and should be made for convenience.

Step 2. Choose the pivot row k so that

$$t_{k,j} > 0,$$

and

$$r_k = \frac{x_{c_k}}{t_{k,j}} \leqslant \frac{x_{c_i}}{t_{i,j}} = r_i \text{ for all } i \text{ such that } t_{i,j} > 0.$$

If more than one k satisfies the above criteria, Dantzig [4, p. 99n.] cites empirical evidence [15] that choosing the one with the largest $t_{k,j}$ may produce fewer steps.

Step 3. Perform the replacement operation with pivot $t_{k,j}$ on the *tableau*.

Step 4. Repeat Steps 1–3 until a stopping condition is reached. The stopping conditions are: (i) Step 1 cannot be carried out, or (ii) Step 2 cannot be carried out.

- If Step 1 cannot be carried out, then the current x is optimal, and $p \cdot x$ (the criterion row entry in the b column) is the optimal value. (See Proposition 29.3.3 below.)
- If Step 2 cannot be carried out, the problem has no optimum, that is, $p \cdot x$ is unbounded. (See Proposition 29.3.4 below.)

This is the algorithm in a nutshell, but there are several remaining issues:

1. How does one get an initial *tableau*?

   This is answered in section 29.4.

2. Must the algorithm stop?

   The answer is generically yes. But it may cycle and never terminate. This appears not to be common, but there is a simple modification, called the lexicographic simplex algorithm that is guaranteed to stop and not to cycle. This is discussed in section 29.7.

3. What happens if the algorithm stops?

   This is answered in section 29.3. Briefly, it stops at an optimum if there is one, or else it stops and gives a proof that no optimum exists.

29.2 How many steps until the Simplex Algorithm stops?

This is beyond the scope of these notes, but let me say, for those of you who care, that the worst-case behavior of the simplex algorithm is not good. Klee [12] constructs some badly behaved examples. Klee and Minty [13] show by construction that if there are n variables and m constraints, there are problems where the number of steps $S(n, m)$ satisfies

$$S(n+1, m+1) \geqslant 2S(n, m) + 1$$
$$S(n+2, m+k+1) \geqslant kS(n, m) + k - 1.$$

Thus

$$S(n, 2n) \geqslant 2^n - 1.$$

In practice, though, the simplex method is much better behaved. Let $\mu = \min\{n, m\}$. Adler, Megiddo, and Todd [1] show that for a randomly drawn problem, the *expectation* of $S(n, m)$ for the lexicographic self-dual variant of the simplex method is on the order of μ^2. (There result requires certain symmetry conditions on the distribution.) Smale [16] proposes a different probability model in which there is a function c of m such that the expectation of $S(n, m)$ is bounded by $c(m)(\ln n)^{m(m+1)}$. Megiddo [14] refines this by showing that the number of steps is bounded by a function of μ. That is, holding either m or n fixed, the other may grow without changing the upper bound on the expectation. In practice, Dantzig [4, p. 160] reports that it is "rare" to require more than $3m$ steps.

29.3 The stopping conditions

I first turn to the question of whether the simplex algorithm ever stops. A sufficient condition for stopping is the following.

29.3.1 Assumption (Nondegeneracy) *The $m \times n$ matrix A has rank m, and the vector q cannot be written as a linear combination of fewer than m columns of A.*

29.3.2 Proposition *Under the Nondegeneracy Assumption 29.3.1, after each replacement operation in the simplex algorithm, the value $p \cdot x'$ is strictly greater (for a maximization problem) than the previous value $p \cdot x$. Therefore, no basis is* *repeated. Since there are finitely many bases, the algorithm must stop in a finite number of steps.*

Proof: By equations (5–6),

$$p \cdot x' - p \cdot x = \frac{x_{c_k}}{t_{k,j}}(p_j - \pi_j).$$

But we chose k, j so that $\pi_j - p_j < 0$, and $t_{k,j} > 0$. In addition, $x_{c_k} \geqslant 0$ for all k. Nondegeneracy implies that in fact $x_{c_k} > 0$ for all k. Thus $p \cdot x' > p \cdot x$. ∎

The lexicographic simplex algorithm described in section 29.7 will always stop, even in the degenerate case, see Gale [8, Chapter 4, section 7, pp. 123–128] or Dantzig [4, pp. 234–235]. The remainder of the section is devoted to examining the two states in which the algorithm can stop.

29.3.3 Proposition (Gale [8, Theorem 4.2, p. 109]) *Under the Rank Assumption 29.1.1, if the algorithm reaches a tableau with $\pi_j - p_j \geqslant 0$ for all $j = 1, \dots, n$, then x is optimal for a maximization problem; and if the algorithm reaches a tableau with $\pi_j - p_j \leqslant 0$ for all $j = 1, \dots, n$, then x is optimal for a minimization problem.*

Proof: The proof makes use of the dual program

$$\underset{y \in \boldsymbol{R}^{\mathrm{m}}}{\text{minimize}} \ q \cdot y$$

subject to

$$yA \geqq p$$

Given the *tableau*

	A^1	$\cdots$	A^j	$\cdots$	A^n	q
A^{c_1}	$t_{1,1}$	$\cdots$	$t_{1,j}$	$\cdots$	$t_{1,n}$	x_{c_1}
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
A^{c_k}	$t_{k,1}$	$\cdots$	$t_{k,j}$	$\cdots$	$t_{k,n}$	x_{c_k}
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
A^{c_m}	$t_{m,1}$	$\cdots$	$t_{m,j}$	$\cdots$	$t_{m,n}$	x_{c_m}
$\pi - p$	$\pi_1 - p_1$	$\cdots$	$\pi_j - p_j$	$\cdots$	$\pi_n - p_n$	$p \cdot x$

we know that $A^{c_1}, \dots, A^{c_m}$ are linearly independent. Therefore the m equations

$$y \cdot A^{c_i} = p_i, \quad i = 1, \dots, m$$

have a solution y. For $j \notin \{c_1, \dots, c_m\}$, we have from the *tableau* that

$$A^j = \sum_{i=1}^m t_{i,j} A^{c_i}$$

so

$$y \cdot A^j = \sum_{i=1}^m t_{i,j} y \cdot A^{c_i} = \sum_{i=1}^m t_{i,j} p_i = \pi_j \geqslant p_j.$$

That is,

$$A'y \geqq p$$

so y is feasible for the dual. (Remember there are no nonnegativity constraints on y.)

Now remember that x is given by $x_j = 0$ for $j \notin \{c_1, \dots, c_m\}$. Thus

$$p \cdot x = \sum_{i=1}^m p_{c_i} x_{c_i} = \sum_{i=1}^m (y \cdot A^{c_i}) x_{c_i} = y \cdot \sum_{i=1}^m A^{c_i} x_{c_i} = y \cdot q,$$

where the last equality comes from the q column of the *tableau*. Now recall that $p \cdot x = q \cdot y$ implies that x is optimal for the primal and y is optimal for the dual. ∎

29.3.4 Proposition *If the algorithm stops with $\pi_j - p_j < 0$, but $t_{i,j} \leqslant 0$ for all $i = 1, \dots, m$, then the primal has no optimum. That is, $p \cdot x$ is unbounded above on the constraint set.*

Proof: If A^j, were already a member of the left-hand basis, say $j = c_i$, then we would have $t_{i,j} = 1$, so we know that A^j is not in the basis.

From the *tableau*

$$A^j = \sum_{i=1}^m t_{i,j} A^{c_i} \quad \text{and} \quad q = \sum_{i=1}^m x_{c_i} A^{c_i}.$$

Thus for any $\lambda > 0$,

$$q = \sum_{i=1}^m x_{c_i} A^{c_i} + \lambda \underbrace{\left(A^j - \sum_{i=1}^m t_{i,j} A^{c_i} \right)}_{=0} = \lambda A^j + \sum_{i=1}^m (x_{c_i} - \lambda t_{i,j}) A^{c_i}. \tag{9}$$

So define $\bar{x}(\lambda)$ by

$$\bar{x}_j = \lambda, \ \bar{x}_i = x_{c_i} - \lambda t_{i,j} \text{ for } i = 1, \dots, m, \text{ and } \bar{x}_i = 0 \text{ otherwise.}$$

By (9), $A\bar{x}(\lambda) = q$ and $\bar{x}(\lambda) \geqq 0$ since each $t_{i,j} \leqslant 0$. But

$$p \cdot \bar{x}(\lambda) = \lambda p_j + \sum_{i=1}^m p_{c_i}(x_{c_i} - \lambda t_{i,j}) = \sum_{i=1}^m p_{c_i} x_{c_i} + \lambda \left(p_j - \sum_{i=1}^m p_{c_i} t_{i,j} \right) = p \cdot x + \lambda (p_j - \pi_j).$$

Since $\lambda > 0$ is arbitrary and $p_j - \pi_j > 0$, we see that $p \cdot \bar{x}(\lambda)$ is unbounded above. Thus no optimum exists. ∎

The following corollary deals minimization problems, where the pivot is chosen to satisfy $\pi_j - p_j > 0$.

29.3.5 Corollary *If the algorithm stops with $\pi_j - p_j > 0$, but $t_{i,j} \leqslant 0$ for all $i = 1, \ldots, m$, then $p \cdot x$ is unbounded below.*

29.4 Phase 1: Finding a starting point

In order to get started with Phase 2, we need to find a nonnegative x with $Ax = q$.

Case 1: $b \geqq 0$.

We can reduce this to an ancillary LP, namely:

$$\underset{z \in \boldsymbol{R}^{\mathrm{m}}}{\text{minimize}} \ \mathbf{1} \cdot z$$

subject to

$$\begin{aligned} Ax + z &= q \\ x &\geqq 0 \\ z &\geqq 0 \end{aligned}$$

This LP has one important property—Phase 1 is trivial. Indeed

$$x = 0, \quad z = q,$$

is a feasible nonnegative solution. Applying Phase 2 to the ancillary problem solves Phase 1.

Case 2: $q \ngeqq 0$.

If $q \ngeqq 0$, setting $z = q$ does not give a nonnegative feasible starting point. But we can fix that as follows. If $q_i < 0$, multiply the i^{th} constraint by -1. Then the constraints become

$$DAx + z = Dq,$$

where D is the diagonal matrix with $d_{ii} = 1$ if $q_i \geqslant 0$ and $d_{ii} = -1$ if $q_i < 0$, so that the right-hand side constants satisfy $Dq \geqq 0$. We now use the simplex algorithm to solve the ancillary problem

$$\underset{z \in \boldsymbol{R}^{\mathrm{m}}}{\text{minimize}} \ \mathbf{1} \cdot z$$

subject to $x \geqq 0$, $z \geqq 0$, and

$$DAx + z = Dq.$$

Phase 1 is also trivial for this LP:

$$x = 0, \quad z = Dq,$$

is a feasible nonnegative solution.

Note that while the solution to the primal remains the same under this transformation, the solution to the dual does not. If y is the solution to the unmodified dual, then Dy is the solution to the modified dual. That is, the solution to the original dual is obtained from the solution to the modified dual by flipping the sign of y_i whenever $q_i < 0$.

29.4.1 Infeasibility

Phase 1 consists of the application of the simplex algorithm as described in Phase 2 to this ancillary problem, starting as described above. If the optimum $(\bar{x}, \bar{z})$ of the ancillary problem has $\bar{z} = 0$, then $\bar{x}$ is feasible for the primal. But if the optimal $\bar{z} \neq 0$ then the primal has no feasible solution.

Note that if all we want to do is find some solution to a system of inequalities, we can stop at the end of Phase 1.

29.4.2 Inequality constraints

Often linear programs are not given with equality constraints, but with inequality constraints, typically like this:

$$\underset{x \in \boldsymbol{R}^{\mathrm{n}}}{\text{maximize}} \; p \cdot x$$

subject to

$$\begin{aligned} Ax &\leqq q \\ x &\geqq 0 \end{aligned}$$

For some kinds of inequality constraints, Phase 1 is trivial. If all m constraints are inequality constraints, introduce slack variables $z_1, \ldots, z_m \geqslant 0$. Let A_i denote the i^{th} row of A. There are four cases, depending on the sense of the inequality and the sign of q_i.

$$\begin{array}{llllllll} \text{Replace} & A_i \cdot x \leqslant q_i & \text{where} & q_i \geqslant 0 & \text{with} & A_i \cdot x + z_i = q_i. \\ \text{Replace} & A_i \cdot x \geqslant -q_i & \text{where} & q_i \geqslant 0 & \text{with} & A_i \cdot x - z_i = -q_i. \end{array}$$

Then an initial feasible solution is given by

$$x = 0, \quad z = q.$$

On the other hand, if we have one of these cases, then Phase 1 is nontrivial, and we have to introduce auxiliary variables u:

$$\begin{array}{llllll} \text{Replace} & A_i \cdot x \leqslant -q_i & \text{where} & q_i \geqslant 0 & \text{with} & -A_i \cdot x - z_i + u_i = q_i. \\ \text{Replace} & A_i \cdot x \geqslant q_i & \text{where} & q_i \geqslant 0 & \text{with} & A_i \cdot x - z_i + u_i = q_i. \end{array}$$

Then an initial feasible solution is given by

$$x = 0, \quad z = 0, \quad u = q$$

but now we must minimize $\mathbf{1} \cdot u$ in order to find a feasible solution of the original problem, where $u = 0$.

29.5 A worked example

The first example illustrates how a problem involving inequalities can combine Phases 1 and 2.

$$\underset{x}{\text{maximize}} \; 2x_1 + 4x_2 + x_3 + x_4$$

subject to $x_1 \geqslant 0, \ldots, x_4 \geqslant 0$, and

$$\begin{aligned} 2x_1 + x_2 &\leqslant 3 \\ x_2 + 4x_3 + x_4 &\leqslant 3 \\ x_1 + 3x_2 + x_4 &\leqslant 4 \end{aligned}$$

To convert this to a problem with equalities, introduce three slack variables z_1, z_2, and z_3, and write the problem as

$$\underset{x}{\text{maximize}} \; 2x_1 + 4x_2 + x_3 + x_4 + 0z_1 + 0z_2 + 0z_3$$

subject to $x_1 \geqslant 0, \ldots, x_4 \geqslant 0$, $z_1, z_2, z_3 \geqslant 0$, and

$$\begin{aligned} 2x_1 + x_2 + z_1 &= 3 \\ x_2 + 4x_3 + x_4 + z_2 &= 3 \\ x_1 + 3x_2 + x_4 + z_3 &= 4 \end{aligned}$$

Since the right-hand side is already nonnegative there is no need to multiply any rows by -1. Moreover, the right-hand side provides a ready made feasible vector: $x_1 = \cdots = x_4 = 0$, $z_1 = 3$, $z_2 = 3$, $z_3 = 4$. The columns corresponding to these three slack variables are simply the three unit coordinate vectors. This makes it especially easy to create a starting *tableau* with these three vectors as the basis in the left-hand margin. But since the three slack variables are in a sense artificial, it is customary to segregate the columns corresponding to them. Finally, note that by introducing three new variables, we must extend the p vector to include three zero components. This makes the computation of the criterion row especially easy. Here then is the initial *tableau*.

Initial *tableau*

p_{c_i}		a^1	a^2	a^3	a^4	e^1	e^2	e^3	q	
0	e^1	2	1	0	0	1	0	0	3	3
0	e^2	0	1	4	1	0	1	0	3	3
0	e^3	1	[3]	0	1	0	0	1	4	$1\frac{1}{3}$
		−2	**−4**	**−1**	**−1**	0	0	0	0	

Notice that the *tableau* is obtained by filling the matrix inequality with an identity matrix to the right. The criterion row $\pi - p$ is just $-p$, as everything is expressed as a linear combination of e^1, e^2, e^3, which have zero prices associated with them. To help you keep track, I have placed the "prices" p_{c_i} associated with each row in the far left margin.

Since we are maximizing, we look for a criterion row entry that is strictly negative. We may as well choose the most negative column, but that is not essential. It corresponds to the column a^2.

> Now to choose the row, look at the ratios of the q column entries (the current x, z) to the positive a^2 entries, and choose the smallest ratio. For convenience I have put these ratios in the right-hand margin.

In this case the smallest is $1\frac{1}{3} < 3$. Thus we want to replace e^3 by a^2, as is indicated by the rectangle around the pivot above.

The new *tableau* is given below, and the next pivot is indicated.

Replace e^3 by a^2:

p_{c_i}		a^1	a^2	a^3	a^4	e^1	e^2	e^3	q	
0	e^1	$1\frac{2}{3}$	0	0	$-\frac{1}{3}$	1	0	$-\frac{1}{3}$	$1\frac{2}{3}$	
0	e^2	$-\frac{1}{3}$	0	**[4]**	$\frac{2}{3}$	0	1	$-\frac{1}{3}$	$1\frac{2}{3}$	$\frac{5}{12}$
4	a^2	$\frac{1}{3}$	1	0	$\frac{1}{3}$	0	0	$\frac{1}{3}$	$1\frac{1}{3}$	
		$\mathbf{-\frac{2}{3}}$	0	$\mathbf{-1}$	$\frac{1}{3}$	0	0	$1\frac{1}{3}$	$5\frac{1}{3}$	

Replace e^2 by a^3:

p_{c_i}		a^1	a^2	a^3	a^4	e^1	e^2	e^3	q	
0	e^1	**[$1\frac{2}{3}$]**	0	0	$-\frac{1}{3}$	1	0	$-\frac{1}{3}$	$1\frac{2}{3}$	1
1	a^3	$-\frac{1}{12}$	0	1	$\frac{1}{6}$	0	$\frac{1}{4}$	$-\frac{1}{12}$	$\frac{5}{12}$	
4	a^2	$\frac{1}{3}$	1	0	$\frac{1}{3}$	0	0	$\frac{1}{3}$	$1\frac{1}{3}$	4
		$\mathbf{-\frac{3}{4}}$	0	0	$\frac{1}{2}$	0	$\frac{1}{4}$	$1\frac{1}{4}$	$5\frac{3}{4}$	

Replace e^1 by a^1:

p_{c_i}		a^1	a^2	a^3	a^4	e^1	e^2	e^3	q
2	a^1	1	0	0	$-\frac{1}{5}$	$\frac{3}{5}$	0	$-\frac{1}{5}$	1
1	a^3	0	0	1	$\frac{3}{20}$	$\frac{1}{20}$	$\frac{1}{4}$	$-\frac{1}{10}$	$\frac{1}{2}$
4	a^2	0	1	0	$\frac{2}{5}$	$-\frac{1}{5}$	0	$\frac{2}{5}$	1
		0	0	0	$\frac{7}{20}$	$\frac{9}{20}$	$\frac{1}{4}$	$1\frac{1}{10}$	$6\frac{1}{2}$

The algorithm stops here because the criterion row has no more negative entries. Note that we have replaced all the unit coordinate vectors by columns of A.

Warning! The solution $(\bar{x}, \bar{z})$ can now be read off from column q, but remember that those numbers are the coefficients on the corresponding left-hand basis element, and that basis is in no particular order, so read them with care! If the basis element in the left-hand column of row i is a^c, then the right-hand column value (under q) is $\bar{x}_c$, the c^{th} coordinate of $\bar{x}$, not x_i, the i^{th} coordinate! If the basis element in the left-hand column of row i is e^c, then the right-hand column value (under q) is $\bar{z}_c$, the c^{th} coordinate of $\bar{z}$, a slack variable.

The solution we have found is

$$\bar{x}_1 = 1,\ \bar{x}_2 = 1,\ \bar{x}_3 = \tfrac{1}{2},\ \bar{x}_4 = 0,$$

and the value $p \cdot \bar{x}$ is $6\frac{1}{2}$.

Let me just verify that this satisfies the constraints:

$$\begin{aligned}
2(1) + 1(1) + 0(\tfrac{1}{2}) + 0(0) &= 2 + 1 + 0 + 0 = 3 \\
0(1) + 1(1) + 4(\tfrac{1}{2}) + 1(0) &= 0 + 1 + 2 + 0 = 3 \\
1(1) + 3(1) + 0(\tfrac{1}{2}) + 1(0) &= 1 + 3 + 0 + 0 = 4
\end{aligned}$$

Now you either have to redo these calculations yourself or put your faith in the computer program that I wrote to produce these *tableaux*. I don't recommend the latter, as I am a notoriously poor programmer. But you don't need to do the former either. Remember that I told you that it is enough to find a solution to the dual that yields the same value. And here is the surprise I have been saving:

The criterion row entries under the unit vectors comprise a solution to the dual program.

That is,

$$\bar{y}_1 = \frac{9}{20}, \quad \bar{y}_2 = \frac{1}{4}, \quad \bar{y}_3 = 1\frac{1}{10},$$

solves the dual problem, which is

$$\underset{y}{\text{minimize}} \; 3y_1 + 3y_2 + 4y_3$$

subject to

$$\begin{aligned} 2y_1 \quad\quad + \quad y_3 &\geqslant 2 \\ y_1 + \quad y_2 + 3y_3 &\geqslant 4 \\ + 4y_2 \quad\quad &\geqslant 1 \\ y_2 + \quad y_3 &\geqslant 1 \end{aligned}$$

Now it is easy to verify that

$$q \cdot \bar{y} = 3(\tfrac{9}{20}) + 3(\tfrac{1}{4}) + 4(1\tfrac{1}{10}) = 1\tfrac{7}{20} + \tfrac{3}{4} + 4\tfrac{2}{5} = 6\tfrac{1}{2}.$$

has the same value as primal, and I leave it to you to verify the feasibility. But I can tell you right now that the first three inequalities will be satisfied as equalities (since the dual variables $\bar{x}_1, \bar{x}_2, \bar{x}_3$ are strictly positive), and the fourth inequality is likely strict (as $\bar{x}_4 = 0$).

I changed my mind. Here is the verification that $\bar{y}$ is feasible for the dual:

$$\begin{aligned} 2(\tfrac{9}{20}) + 0(\tfrac{1}{4}) + 1(1\tfrac{1}{10}) &= \tfrac{9}{10} + 0 + 1\tfrac{1}{10} = 2 \quad = 2 \\ 1(\tfrac{9}{20}) + 1(\tfrac{1}{4}) + 3(1\tfrac{1}{10}) &= \tfrac{9}{20} + \tfrac{1}{4} + 3\tfrac{3}{10} = 4 \quad = 4 \\ 0(\tfrac{9}{20}) + 4(\tfrac{1}{4}) + 0(1\tfrac{1}{10}) &= \; 0 + 1 + \quad 0 = 1 \quad = 1 \\ 0(\tfrac{9}{20}) + 1(\tfrac{1}{4}) + 1(1\tfrac{1}{10}) &= \; 0 + \tfrac{1}{4} + 1\tfrac{1}{10} = 1\tfrac{7}{20} > 1 \end{aligned}$$

Now either this is an incredibly contrived example, or else there is something magical I haven't yet told you about the simplex algorithm. It's the latter.

29.6 The simplex algorithm solves the dual program too

The simplex algorithm applied to the following sort of problem also computes a solution to the dual program.

$$\underset{x \in \mathbf{R}^n}{\text{maximize}}\ p \cdot x$$

subject to

$$\begin{aligned} Ax &= q \\ x &\geqq 0 \end{aligned}$$

The dual program is

$$\underset{y \in \mathbf{R}^m}{\text{minimize}}\ q \cdot y$$

subject to

$$A'y \geqq p$$

As we saw in the last section, the initial *tableau* can be written

	$A^1 \cdots\cdots A^n$	$e^1 \cdots\cdots e^m$	q
e^1	$a_{1,1} \cdots\cdots a_{1,n}$	$1 \quad 0 \cdots 0$	b_1
$\vdots$	$\vdots \qquad \vdots$	$0 \quad \ddots \quad 0$	$\vdots$
e^m	$a_{m,1} \cdots\cdots a_{m,n}$	$0 \cdots 0 \quad 1$	q_m
$\pi - p$	$-p_1 \cdots\cdots -p_n$	$0 \cdots\cdots 0$	0

Discuss what happens if we can't replace all the e^i's.

Assume for now that the simplex algorithm enables us to replace all the coordinate vectors with columns of A. Without loss of generality, by rearranging the rows and columns of A if necessary, we can assume the algorithm stops in the following configuration, which has the property that $c_i = i$ for $i = 1, \dots, m$.

	$A^1 \cdots\cdots A^m$	$A^{m+1} \cdots\cdots A^n$	$e^1 \cdots\cdots e^m$	q
A^1	$1 \quad 0 \cdots 0$	$t_{1,m+1} \cdots\cdots t_{1,n}$	$s_{1,1} \cdots s_{1,m}$	x_1
$\vdots$	$0 \quad \ddots \quad 0$	$\vdots \qquad \vdots$	$\vdots \qquad \vdots$	$\vdots$
A^m	$0 \cdots 0 \quad 1$	$t_{m,m+1} \cdots\cdots t_{m,n}$	$s_{m,1} \cdots s_{m,m}$	x_m
$\pi - p$	$0 \cdots\cdots 0$	$\pi_{m+1} - p_{m+1} \cdots \pi_n - p_n$	$y_1 \cdots y_m$	$p \cdot x$

There are three key observations to make here.

1. The block $[s_{i,j}]_{\substack{j=1,\dots,m\\ i=1,\dots,m}}$ is the inverse of the block $A_{m,m} = [a_{i,j}]_{\substack{j=1,\dots,m\\ i=1,\dots,m}}$. (Recall the use of the Gauss–Jordan method for inverting a matrix.)

2. By construction of the criterion row, the y_k's satisfy

$$y_k = \sum_{i=1}^{m} s_{i,k} p_i \qquad k = 1, \dots, m.$$

3. For $j > m$, we have $\pi_j \geqslant p_j$. (Otherwise the algorithm would not stop here with an optimal x.)

Thus, as in the proof of Proposition 29.3.3, for $j = 1, \dots, m$ we have

$$y \cdot A^j = \sum_{k=1}^{m} y_k a_{k,j} = \sum_{k=1}^{m} \left(\sum_{i=1}^{m} s_{i,k} p_i \right) a_{k,j} = \sum_{i=1}^{m} p_i \sum_{k=1}^{m} (s_{i,k} a_{k,j}) = \sum_{i=1}^{m} p_i \delta_{i,j} = p_j,$$

where the penultimate equality follows because $[s_{i,j}]$ is the inverse of $A_{m,m}$. For $j > m$,

$$y \cdot A^j = y \cdot \sum_{i=1}^{m} t_{i,j} A^i = \sum_{i=1}^{m} t_{i,j} y \cdot A^i = \sum_{i=1}^{m} t_{i,j} p_i = \pi_j \geqslant p_j,$$

by the third observation. In other words,

$$A'y \geqq p,$$

so y is feasible for the dual program.

In addition,

$$q \cdot y = \left(\sum_{i=1}^{m} x_i A^i \right) \cdot y = \sum_{i=1}^{m} x_i (A^i \cdot y) = \sum_{i=1}^{m} x_i p_i = p \cdot x$$

since $x_j = 0$ for $j > m$. Thus $p \cdot x = q \cdot y$, so y is optimal.

29.6.1 Solving the dual with inequality constraints

The same technique also solves the dual for problems of the form

$$\underset{x \in \mathbf{R}^n}{\text{maximize}} \; p \cdot x$$

subject to

$$\begin{aligned} Ax &\leqq q \\ x &\geqq 0 \end{aligned}$$

The dual program is

$$\underset{y \in \mathbf{R}^m}{\text{minimize}} \; q \cdot y$$

subject to $y \geqq 0$ and

$$A'y \geqq p$$

Instead, we introduce a vector z of slack variables and solve the following problem:

$$\underset{x \in \boldsymbol{R}^{\mathrm{n}},\ z \in \boldsymbol{R}^{\mathrm{m}}}{\text{maximize}}\ p \cdot x + 0 \cdot z$$

subject to

$$\begin{aligned} Ax + Iz &= q \\ x &\geqq 0 \\ z &\geqq 0 \end{aligned}$$

The dual program is

$$\underset{y \in \boldsymbol{R}^{\mathrm{m}}}{\text{minimize}}\ q \cdot y$$

$$\begin{bmatrix} A' \\ I \end{bmatrix} y \geqq \begin{bmatrix} p \\ 0 \end{bmatrix}$$

with no sign constraints on y.

Our algorithm applied to this problem produces vectors $\bar{x}$, $\bar{z}$, and $\bar{y}$ that satisfies $q \cdot \bar{y} = p \cdot \bar{x}(+0 \cdot \bar{z})$, and $\bar{y}[A, I] \geqq [p, 0]$. But this implies $\bar{y}A \geqq p$ and $\bar{y} \geqq 0$, so the computed solution $\bar{y}$ to the dual of the equality case also solves the dual for the inequality case.

29.7 Degeneracy, cycling, and the lexicographic simplex algorithm

Proposition 29.3.2 shows that the simplex algorithm must stop if the linear program is nondegenerate. But verification of nondegeneracy is difficult. This is unfortunate, as the next example shows that the naïve simplex algorithm can cycle and never stop in the degenerate case. However there is a simple modification, the lexicographic simplex algorithm, that will stop even in the degenerate case.

29.7.1 A cycling example

The first example of cycling in the simplex algorithm is due to Hoffman [11]. Beale [2] constructed the following simpler example of cycling. (See also [4, pp. 228–230].) The problem is to

$$\underset{x}{\text{maximize}}\ \tfrac{3}{4}x_1 - 150x_2 + \tfrac{1}{50}x_3 - 6x_4$$

subject to $x \geqq 0$, and

$$\begin{aligned} \tfrac{1}{4}x_1 - 60x_2 - \tfrac{1}{25}x_3 + 9x_4 &\leqslant 0 \\ \tfrac{1}{2}x_1 - 90x_2 - \tfrac{1}{50}x_3 + 3x_4 &\leqslant 0 \\ + \ \ x_3 \qquad &\leqslant 1 \end{aligned}$$

Introducing slack variables z and setting them to the right-hand side constants leads to the *tableau* shown in Table 29.7.1. The algorithm was implemented to choose the pivot column with the most negative value in the criterion row, and when more than one row minimized the ratio, the first row to do so was selected for the pivot. As you can see, the seventh *tableau* is the same as the first, so the algorithm is doomed to repeat itself.

A peculiar (and nongeneric) feature of this problem is that the *tableau* always gives a choice of two pivot rows, and the minimum ratio is always zero. Indeed the proof of Proposition 29.3.2 shows that a zero ratio is necessary for cycling.

29.7.2 The lexicographic simplex algorithm

Dantzig, Orden, and Wolfe [5] provide a pivot choice rule that will not cycle. Their rule for choosing the pivot row is lexicographic. To use it, we need to use an extended *tableau* with an identity matrix spliced in to the left of the q column. (You will probably want this anyway to compute the solution to the dual.) Here is a typical *tableau*:

	A^1	$\cdots$	A^j	$\cdots$	A^n	e^1	$\cdots$	e^m	q
A^{c_1}	$t_{1,1}$	$\cdots$	$t_{1,j}$	$\cdots$	$t_{1,n}$	$s_{1,1}$	$\cdots$	$s_{1,m}$	x_{c_1}
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$
A^{c_k}	$t_{k,1}$	$\cdots$	$t_{k,j}$	$\cdots$	$t_{k,n}$	$s_{k,1}$	$\cdots$	$s_{k,m}$	x_{c_k}
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$
A^{c_m}	$t_{m,1}$	$\cdots$	$t_{m,j}$	$\cdots$	$t_{m,n}$	$s_{m,1}$	$\cdots$	$s_{m,m}$	x_{c_m}
$\pi - p$	$\pi_1 - p_1$	$\cdots$	$\pi_j - p_j$	$\cdots$	$\pi_n - p_n$	y_1	$\cdots$	y_m	$p \cdot x$

The rule for choosing the column k is this

p_{c_i}		a^1	a^2	a^3	a^4	e^1	e^2	e^3	q	
Initial *tableau*										
0	e^1	$\frac{1}{4}$	-60	$-\frac{1}{25}$	9	1	0	0	0	0
0	e^2	$\frac{1}{2}$	-90	$-\frac{1}{50}$	3	0	1	0	0	0
0	e^3	0	0	1	0	0	0	1	1	
		$-\frac{3}{4}$	150	$-\frac{1}{50}$	6	0	0	0	0	
Replace e^1 by a^1:										
$\frac{3}{4}$	a^1	1	-240	$-\frac{4}{25}$	36	4	0	0	0	
0	e^2	0	30	$\frac{3}{50}$	-15	-2	1	0	0	0
0	e^3	0	0	1	0	0	0	1	1	
		0	-30	$-\frac{7}{50}$	33	3	0	0	0	
Replace e^2 by a^2:										
$\frac{3}{4}$	a^1	1	0	$\frac{8}{25}$	-84	-12	8	0	0	0
-150	a^2	0	1	$\frac{1}{500}$	$-\frac{1}{2}$	$-\frac{1}{15}$	$\frac{1}{30}$	0	0	0
0	e^3	0	0	1	0	0	0	1	1	1
		0	0	$-\frac{2}{25}$	18	1	1	0	0	
Replace a^1 by a^3:										
$\frac{1}{50}$	a^3	$3\frac{1}{8}$	0	1	$-262\frac{1}{2}$	$-37\frac{1}{2}$	25	0	0	
-150	a^2	$-\frac{1}{160}$	1	0	$\frac{1}{40}$	$\frac{1}{120}$	$-\frac{1}{60}$	0	0	0
0	e^3	$-3\frac{1}{8}$	0	0	$262\frac{1}{2}$	$37\frac{1}{2}$	-25	1	1	$\frac{2}{525}$
		$\frac{1}{4}$	0	0	-3	-2	3	0	0	
Replace a^2 by a^4:										
$\frac{1}{50}$	a^3	$-62\frac{1}{2}$	10500	1	0	50	-150	0	0	0
-6	a^4	$-\frac{1}{4}$	40	0	1	$\frac{1}{3}$	$-\frac{2}{3}$	0	0	0
0	e^3	$62\frac{1}{2}$	-10500	0	0	-50	150	1	1	
		$-\frac{1}{2}$	120	0	0	-1	1	0	0	
Replace a^3 by e^1:										
0	e^1	$-1\frac{1}{4}$	210	$\frac{1}{50}$	0	1	-3	0	0	
-6	a^4	$\frac{1}{6}$	-30	$-\frac{1}{150}$	1	0	$\frac{1}{3}$	0	0	0
0	e^3	0	0	1	0	0	0	1	1	
		$-1\frac{3}{4}$	330	$\frac{1}{50}$	0	0	-2	0	0	
Replace a^4 by e^2:										
0	e^1	$\frac{1}{4}$	-60	$-\frac{1}{25}$	9	1	0	0	0	
0	e^2	$\frac{1}{2}$	-90	$-\frac{1}{50}$	3	0	1	0	0	
0	e^3	0	0	1	0	0	0	1	1	
		$-\frac{3}{4}$	150	$-\frac{1}{50}$	6	0	0	0	0	

Table 29.7.1. Beale's cycling program.

Lexicographic rule

Choose the pivot row k so that

$$t_{k,j} > 0$$

and the vector

$$r_k = \left(\frac{x_{c_k}}{t_{k,j}}, \frac{s_{k,1}}{t_{k,j}}, \dots, \frac{s_{k,m}}{t_{k,j}}\right)$$

is lexicographically minimal in $\{r_i : t_{i,j} > 0\}$.

This differs from our previous rule, which only looked at the first component of these vectors. The proof that this rule works is not hard, and may be found in Gale [8, Chapter 4, section 7, pp. 123–128] or Dantzig [4, pp. 234–235]. In practice, it appears that cycling is not a problem. Charnes [3] deals with the problem of cycling by slightly perturbing q.

29.7.3 Lexicographic simplex example

Here is the lexicographic simplex method applied to Beale's example. I have placed the entire r_i vector in the right-hand margin. (This is not computationally efficient—if you have tens of thousands of variables, you don't want to compute these extra ratios unless you need them all to break ties.)

Initial *tableau*

p_{c_i}		a^1	a^2	a^3	a^4	e^1	e^2	e^3	q	
0	e^1	$\frac{1}{4}$	-60	$-\frac{1}{25}$	9	1	0	0	0	0 | 4 0 0
0	e^2	$\boxed{\frac{1}{2}}$	-90	$-\frac{1}{50}$	3	0	1	0	0	0 | 0 2 0
0	e^3	0	0	1	0	0	0	1	1	
		$\mathbf{-\frac{3}{4}}$	150	$\mathbf{-\frac{1}{50}}$	6	0	0	0	0	

Replace e^2 by a^1:

p_{c_i}		a^1	a^2	a^3	a^4	e^1	e^2	e^3	q	
0	e^1	0	-15	$-\frac{3}{100}$	$7\frac{1}{2}$	1	$-\frac{1}{2}$	0	0	
$\frac{3}{4}$	a^1	1	-180	$-\frac{1}{25}$	6	0	2	0	0	
0	e^3	0	0	$\boxed{1}$	0	0	0	1	1	1 | 0 0 1
		0	15	$\mathbf{-\frac{1}{20}}$	$10\frac{1}{2}$	0	$1\frac{1}{2}$	0	0	

Replace e^3 by a^3:

p_{c_i}		a^1	a^2	a^3	a^4	e^1	e^2	e^3	q
0	e^1	0	-15	0	$7\frac{1}{2}$	1	$-\frac{1}{2}$	$\frac{3}{100}$	$\frac{3}{100}$
$\frac{3}{4}$	a^1	1	-180	0	6	0	2	$\frac{1}{25}$	$\frac{1}{25}$
$\frac{1}{50}$	a^3	0	0	1	0	0	0	1	1
		0	15	0	$10\frac{1}{2}$	0	$1\frac{1}{2}$	$\frac{1}{20}$	$\frac{1}{20}$

There is no pivot column, so the current basis is optimal. A solution is

$$\boxed{x = \left(\tfrac{1}{25},\quad 0,\quad 1,\quad 0\right)}$$

Verify that x satisfies the constraints:

$$\begin{aligned}
\tfrac{1}{4}(\tfrac{1}{25}) - 60(0) - \tfrac{1}{25}(1) + 9(0) &= \tfrac{1}{100} + 0 - \tfrac{1}{25} + 0 = -\tfrac{3}{100} < 0\\
\tfrac{1}{2}(\tfrac{1}{25}) - 90(0) - \tfrac{1}{50}(1) + 3(0) &= \tfrac{1}{50} + 0 - \tfrac{1}{50} + 0 = \quad 0 = 0\\
0(\tfrac{1}{25}) + 0(0) + 1(1) + 0(0) &= 0 + 0 + 1 + 0 = \quad 1 = 1\,.
\end{aligned}$$

Thus a solution is

$$\boxed{x = \left(\tfrac{1}{25},\quad 0,\quad 1,\quad 0\right)}$$

Check the value of $p \cdot x$:

$$\tfrac{3}{4}(\tfrac{1}{25}) - 150(0) + \tfrac{1}{50}(1) - 6(0) = \tfrac{3}{100} + 0 + \tfrac{1}{50} + 0 = \tfrac{1}{20}.$$

A solution to the dual is

$$\boxed{y = \left(0, \quad 1\tfrac{1}{2}, \quad \tfrac{1}{20}\right).}$$

Recall that the dual problem is

$$\underset{y}{\text{minimize}} \;\; y_3$$

subject to $y \geqq 0$ and

$$\begin{aligned}
\tfrac{1}{4}y_1 + \tfrac{1}{2}y_2 \qquad &\geqslant \; 3/4 \\
-60y_1 - 90y_2 \qquad &\geqslant -150 \\
-\tfrac{1}{25}y_1 - \tfrac{1}{50}y_2 + y_3 &\geqslant \; 1/50 \\
9y_1 + 3y_2 \qquad &\geqslant \; -6.
\end{aligned}$$

Check that the value of the dual is

$$0(0) + 0(1\tfrac{1}{2}) + 1(\tfrac{1}{20}) = 0 + 0 + \tfrac{1}{20} = \tfrac{1}{20}.$$

Now verify the feasibility of the dual.

$$\begin{array}{rcccccl}
\tfrac{1}{4}(0) + \tfrac{1}{2}(1\tfrac{1}{2}) + 0(\tfrac{1}{20}) & = & 0 + \tfrac{3}{4} + 0 & = & \tfrac{3}{4} & = & \tfrac{3}{4} \\
-60(0) - 90(1\tfrac{1}{2}) + 0(\tfrac{1}{20}) & = & 0 - 135 + 0 & = & -135 & > & -150 \\
-\tfrac{1}{25}(0) - \tfrac{1}{50}(1\tfrac{1}{2}) + 1(\tfrac{1}{20}) & = & 0 - \tfrac{3}{100} + \tfrac{1}{20} & = & \tfrac{1}{50} & = & \tfrac{1}{50} \\
9(0) + 3(1\tfrac{1}{2}) + 0(\tfrac{1}{20}) & = & 0 + 4\tfrac{1}{2} + 0 & = & 4\tfrac{1}{2} & > & -6\,.
\end{array}$$

29.8 The simplex algorithm and vertexes

The simplex algorithm works by taking a feasible solution expressed as a linear combination of the columns of A and one-by-one replacing elements of this basis until an optimal basis is found. It turns out this rather abstract explanation has a nice geometric interpretation. According to Proposition 28.7.1 in Section 28.7, basic feasible solutions to

$$\begin{aligned}
Ax &= q \\
x &\geqq 0
\end{aligned}$$

are vertices of the polyhedron of feasible solutions. We shall show below in ****** that a pivot operation takes us from a basic solution vertex to a *neighboring* solution vertex. But first let's look at a simple example that I can partially draw. Add this result!!!

Consider the following inequality-form linear program with two variables and nine inequality constraints:

$$\underset{x}{\text{maximize}} \quad x_1 + x_2$$

subject to $x \geqq 0$, and

$$\begin{aligned} -4x_1 + \ x_2 &\leqslant 0 \\ x_1 - 4x_2 &\leqslant 0 \\ -2x_1 + \ x_2 &\leqslant 2 \\ x_1 - 2x_2 &\leqslant 2 \\ -x_1 + 2x_2 &\leqslant 10 \\ 2x_1 - \ x_2 &\leqslant 10 \\ -x_1 + 3x_2 &\leqslant 17 \\ 3x_1 - \ x_2 &\leqslant 17 \\ x_1 + \ x_2 &\leqslant 15 \end{aligned} \tag{10}$$

The solution set of the inequalities is the polytope P shown in Figure 29.8.1.

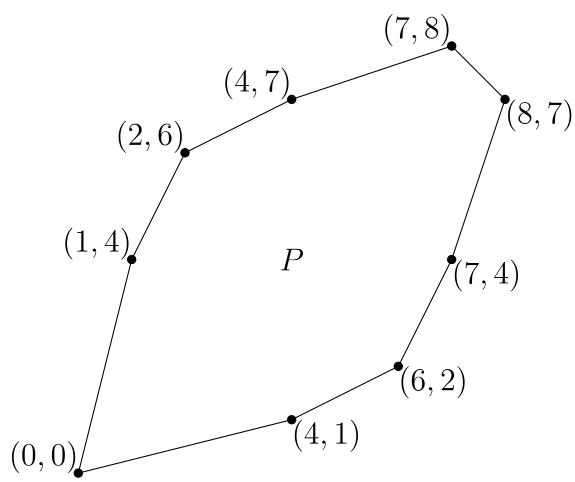

Figure 29.8.1. The polytope P is the set of feasible solutions to the linear program (10).

The dual program is

$$\underset{y}{\text{minimize}} \quad 2y_3 + 2y_4 + 10y_5 + 10y_6 + 17y_7 + 17y_8 + 15y_9$$

subject to $x \geqq 0$, and

$$\begin{aligned} -4y_1 + y_2 - 2y_3 + y_4 - y_5 - y_6 + 3y_7 - y_8 + y_9 &\geqslant 1 \\ y_1 - 4y_2 + y_3 - 2y_4 + 2y_5 + 2y_6 - y_7 + 3y_8 + y_9 &\geqslant 1 \end{aligned} \tag{11}$$

But to use the simplex method, we want to rewrite this as system of equations rather than inequalities, so we add nine slack variables $z_1, \dots, z_9$ and write the constraints as

$$\begin{aligned}
-4x_1 + x_2 + z_1 &= 0 \\
x_1 - 4x_2 + z_2 &= 0 \\
-2x_1 + x_2 + z_3 &= 2 \\
x_1 - 2x_2 + z_4 &= 2 \\
-x_1 + 2x_2 + z_5 &= 10 \\
2x_1 - x_2 + z_6 &= 10 \\
-x_1 + 3x_2 + z_7 &= 17 \\
3x_1 - x_2 + z_8 &= 17 \\
x_1 + x_2 + z_9 &= 15
\end{aligned} \tag{12}$$

or

$$x_1 a^1 + x_2 a^2 + z_1 e^1 + \cdots + z_9 e^9 = q, \tag{13}$$

where

$$a^1 = \begin{bmatrix} -4 \\ 1 \\ -2 \\ 1 \\ -1 \\ 2 \\ -1 \\ 3 \\ 1 \end{bmatrix}, \quad a^2 = \begin{bmatrix} 1 \\ -4 \\ 1 \\ -2 \\ 2 \\ -1 \\ 3 \\ -1 \\ 1 \end{bmatrix}, \quad q = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 2 \\ 10 \\ 10 \\ 17 \\ 17 \\ 15 \end{bmatrix},$$

and e^i is i^{th} unit coordinate vector in $\boldsymbol{R}^9$. We are now in an eleven-dimensional space, but the solutions to (13) and (10) are in one-to-one correspondence. Indeed, the set $\hat{P}$ of solutions to (13) constitute a two-dimensional polytope in $\boldsymbol{R}^{11}$ given as follows. Define the (linear) mapping $z \colon P \to \boldsymbol{R}^9$ by solving (12) for each z_i in terms of $x = (x_1, x_2)$, and then the set $\hat{P}$ solutions is just

$$\hat{P} = \{\big(x, z(x)\big) : x \in P\}.$$

A point (x, z) in $\hat{P}$ is a vertex of $\hat{P}$ if and only if x is a vertex of P and $z = z(x)$.

A starting point for the equality version is to write q as linear combination of the standard basis vectors by setting $x_1 = x_2 = 0$ and $z_i = q_i$, for $i = 1, \ldots, 9$.

The following is the initial *tableau*. The criterion row is simply $-p$, and we search it for ***negative*** entries. As you can see, there are two negative entries in the criterion row. Let's start by choosing to pivot on the first one.

Initial *tableau*, with first pivot boxed :

p_{c_i}		$\boldsymbol{a^1}$	a^2	e^1	e^2	e^3	e^4	e^5	e^6	e^7	e^8	e^9	q	
0	e^1	-4	1	1	0	0	0	0	0	0	0	0	0	
0	$\boldsymbol{e^2}$	[1]	-4	0	1	0	0	0	0	0	0	0	0	**0**
0	e^3	-2	1	0	0	1	0	0	0	0	0	0	2	
0	e^4	1	-2	0	0	0	1	0	0	0	0	0	2	2
0	e^5	-1	2	0	0	0	0	1	0	0	0	0	10	
0	e^6	2	-1	0	0	0	0	0	1	0	0	0	10	5
0	e^7	-1	3	0	0	0	0	0	0	1	0	0	17	
0	e^8	3	-1	0	0	0	0	0	0	0	1	0	17	$5\frac{2}{3}$
0	e^9	1	1	0	0	0	0	0	0	0	0	1	15	15
		$\boldsymbol{-1}$	$\boldsymbol{-1}$	0	0	0	0	0	0	0	0	0	0	

Note that a^1 and a^2 are missing from the initial basis (given in the left-hand margin), so $(x_1, x_2) = (0, 0)$.

Replace e^2 by a^1, and choose new pivot:

p_{c_i}		a^1	$\boldsymbol{a^2}$	e^1	e^2	e^3	e^4	e^5	e^6	e^7	e^8	e^9	q	
0	e^1	0	-15	1	4	0	0	0	0	0	0	0	0	
1	$\boldsymbol{a^1}$	1	-4	0	1	0	0	0	0	0	0	0	0	
0	e^3	0	-7	0	2	1	0	0	0	0	0	0	2	
0	$\boldsymbol{e^4}$	0	[2]	0	-1	0	1	0	0	0	0	0	2	**1**
0	e^5	0	-2	0	1	0	0	1	0	0	0	0	10	
0	e^6	0	7	0	-2	0	0	0	1	0	0	0	10	$1\frac{3}{7}$
0	e^7	0	-1	0	1	0	0	0	0	1	0	0	17	
0	e^8	0	11	0	-3	0	0	0	0	0	1	0	17	$1\frac{6}{11}$
0	e^9	0	5	0	-1	0	0	0	0	0	0	1	15	3
		0	$\boldsymbol{-5}$	0	1	0	0	0	0	0	0	0	0	

Now a^1 has been added to the new basis (given in the left-hand margin), but its coefficient (read from the column under q) is zero, so still we have $(x_1, x_2) = (0, 0)$.

Now replace e^4 by a^2 and choose next pivot:

p_{c_i}		a^1	a^2	e^1	$\boldsymbol{e^2}$	e^3	e^4	e^5	e^6	e^7	e^8	e^9	q	
0	e^1	0	0	1	$-3\frac{1}{2}$	0	$7\frac{1}{2}$	0	0	0	0	0	15	
1	$\boldsymbol{a^1}$	1	0	0	-1	0	2	0	0	0	0	0	4	
0	e^3	0	0	0	$-1\frac{1}{2}$	1	$3\frac{1}{2}$	0	0	0	0	0	9	
1	$\boldsymbol{a^2}$	0	1	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	0	0	0	0	1	
0	e^5	0	0	0	0	0	1	1	0	0	0	0	12	
0	$\boldsymbol{e^6}$	0	0	0	$\boxed{1\frac{1}{2}}$	0	$-3\frac{1}{2}$	0	1	0	0	0	3	**2**
0	e^7	0	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	1	0	0	18	36
0	e^8	0	0	0	$2\frac{1}{2}$	0	$-5\frac{1}{2}$	0	0	0	1	0	6	$2\frac{2}{5}$
0	e^9	0	0	0	$1\frac{1}{2}$	0	$-2\frac{1}{2}$	0	0	0	0	1	10	$6\frac{2}{3}$
		0	0	0	$\boldsymbol{-1\frac{1}{2}}$	0	$2\frac{1}{2}$	0	0	0	0	0	5	

Now we have brought both a^1 and a^2 into the basis, with coefficients $x_1 = 4$ and $x_2 = 1$. Note that $(4, 1)$ is a vertex of the polytope in Figure 29.8.1. (We also have $z_1 = 15$, $z_3 = 9$, etc.)

Next replace e^6 by e^2 and get the next pivot:

p_{c_i}		a^1	a^2	e^1	e^2	e^3	$\boldsymbol{e^4}$	e^5	e^6	e^7	e^8	e^9	q	
0	e^1	0	0	1	0	0	$-\frac{2}{3}$	0	$2\frac{1}{3}$	0	0	0	22	
1	$\boldsymbol{a^1}$	1	0	0	0	0	$-\frac{1}{3}$	0	$\frac{2}{3}$	0	0	0	6	
0	e^3	0	0	0	0	1	0	0	1	0	0	0	12	
1	$\boldsymbol{a^2}$	0	1	0	0	0	$-\frac{2}{3}$	0	$\frac{1}{3}$	0	0	0	2	
0	e^5	0	0	0	0	0	1	1	0	0	0	0	12	12
0	$\boldsymbol{e^2}$	0	0	0	1	0	$-2\frac{1}{3}$	0	$\frac{2}{3}$	0	0	0	2	
0	e^7	0	0	0	0	0	$1\frac{2}{3}$	0	$-\frac{1}{3}$	1	0	0	17	$10\frac{1}{5}$
0	$\boldsymbol{e^8}$	0	0	0	0	0	$\boxed{\frac{1}{3}}$	0	$-1\frac{2}{3}$	0	1	0	1	**3**
0	e^9	0	0	0	0	0	1	0	-1	0	0	1	7	7
		0	0	0	0	0	$\boldsymbol{-1}$	0	1	0	0	0	8	

Now we have $(x_1, x_2) = (6, 2)$, which is the next vertex of the polytope.

Replace e^8 by e^4 and get next pivot:

p_{c_i}		a^1	a^2	e^1	e^2	e^3	e^4	e^5	$\boldsymbol{e^6}$	e^7	e^8	e^9	q	
0	e^1	0	0	1	0	0	0	0	-1	0	2	0	24	
1	$\boldsymbol{a^1}$	1	0	0	0	0	0	0	-1	0	1	0	7	
0	e^3	0	0	0	0	1	0	0	1	0	0	0	12	12
1	$\boldsymbol{a^2}$	0	1	0	0	0	0	0	-3	0	2	0	4	
0	e^5	0	0	0	0	0	0	1	5	0	-3	0	9	$1\frac{4}{5}$
0	$\boldsymbol{e^2}$	0	0	0	1	0	0	0	-11	0	7	0	9	
0	e^7	0	0	0	0	0	0	0	8	1	-5	0	12	$1\frac{1}{2}$
0	$\boldsymbol{e^4}$	0	0	0	0	0	1	0	-5	0	3	0	3	
0	$\boldsymbol{e^9}$	0	0	0	0	0	0	0	$\boxed{4}$	0	-3	1	4	**1**
		0	0	0	0	0	0	0	$\boldsymbol{-4}$	0	3	0	11	

Now we have $(x_1, x_2) = (7, 4)$, which is the next vertex of the polytope.

Finally, replace e^9 by e^6 to get:

p_{c_i}		a^1	a^2	e^1	e^2	e^3	e^4	e^5	e^6	e^7	e^8	e^9	q
0	$\boldsymbol{e^1}$	0	0	1	0	0	0	0	0	0	$1\frac{1}{4}$	$\frac{1}{4}$	25
1	$\boldsymbol{a^1}$	1	0	0	0	0	0	0	0	0	$\frac{1}{4}$	$\frac{1}{4}$	8
0	e^3	0	0	0	0	1	0	0	0	0	$\frac{3}{4}$	$-\frac{1}{4}$	11
1	$\boldsymbol{a^2}$	0	1	0	0	0	0	0	0	0	$-\frac{1}{4}$	$\frac{3}{4}$	7
0	e^5	0	0	0	0	0	0	1	0	0	$\frac{3}{4}$	$-1\frac{1}{4}$	4
0	$\boldsymbol{e^2}$	0	0	0	1	0	0	0	0	0	$-1\frac{1}{4}$	$2\frac{3}{4}$	20
0	e^7	0	0	0	0	0	0	0	0	1	1	-2	4
0	$\boldsymbol{e^4}$	0	0	0	0	0	1	0	0	0	$-\frac{3}{4}$	$1\frac{1}{4}$	8
0	$\boldsymbol{e^6}$	0	0	0	0	0	0	0	1	0	$-\frac{3}{4}$	$\frac{1}{4}$	1
		0	0	0	0	0	0	0	0	0	0	1	15

There is no pivot column, so the current basis with $(x_1, x_2) = (8, 7)$ is optimal. A solution is

$$(x_1, x_2) = (8, 7) \text{ and } z = (25, 20, 11, 8, 4, 1, 4, 0, 0).$$
The value is 15.

You can see that after the initial steps, we moved along the vertexes of the solution polytope for the inequality program in x to a solution at a vertex. Actually, we moved along adjacent vertices of the equality polyhedron in the eleven-dimensional (x, z)-space. I wish I could draw a picture to illustrate that, but I can't. I'll give an algebraic proof below in Proposition 29.9.2.

By the way, the solution to the dual can be read off the bottom row under the $e^1, \dots, e^9$ vectors. It is $y = (0, 0, 0, 0, 0, 0, 0, 0, 1)$ and gives the value 15.

Now let's go back and see what would happened if we had chosen the other eligible column for the first pivot.

Initial *tableau*, with first pivot boxed :

p_{c_i}		a^1	$\boldsymbol{a^2}$	e^1	e^2	e^3	e^4	e^5	e^6	e^7	e^8	e^9	q	
0	$\boldsymbol{e^1}$	−4	[1]	1	0	0	0	0	0	0	0	0	0	**0**
0	e^2	1	−4	0	1	0	0	0	0	0	0	0	0	
0	e^3	−2	1	0	0	1	0	0	0	0	0	0	2	2
0	e^4	1	−2	0	0	0	1	0	0	0	0	0	2	
0	e^5	−1	2	0	0	0	0	1	0	0	0	0	10	5
0	e^6	2	−1	0	0	0	0	0	1	0	0	0	10	
0	e^7	−1	3	0	0	0	0	0	0	1	0	0	17	$5\frac{2}{3}$
0	e^8	3	−1	0	0	0	0	0	0	0	1	0	17	
0	e^9	1	1	0	0	0	0	0	0	0	0	1	15	15
		−1	**−1**	0	0	0	0	0	0	0	0	0	0	

As above, the initial x is $(x_1, x_2) = (0, 0)$.

Replace e^1 by a^2 and select next pivot:

p_{c_i}		$\boldsymbol{a^1}$	a^2	e^1	e^2	e^3	e^4	e^5	e^6	e^7	e^8	e^9	q	
1	$\boldsymbol{a^2}$	-4	1	1	0	0	0	0	0	0	0	0	0	
0	e^2	-15	0	4	1	0	0	0	0	0	0	0	0	
0	$\boldsymbol{e^3}$	2	0	-1	0	1	0	0	0	0	0	0	2	**1**
0	e^4	-7	0	2	0	0	1	0	0	0	0	0	2	
0	e^5	7	0	-2	0	0	0	1	0	0	0	0	10	$1\frac{3}{7}$
0	e^6	-2	0	1	0	0	0	0	1	0	0	0	10	
0	e^7	11	0	-3	0	0	0	0	0	1	0	0	17	$1\frac{6}{11}$
0	e^8	-1	0	1	0	0	0	0	0	0	1	0	17	
0	e^9	5	0	-1	0	0	0	0	0	0	0	1	15	3
		$\boldsymbol{-5}$	0	1	0	0	0	0	0	0	0	0	0	

And still $(x_1, x_2) = (0, 0)$.

Replace e^3 by a^1 and select next pivot:

p_{c_i}		a^1	a^2	$\boldsymbol{e^1}$	e^2	e^3	e^4	e^5	e^6	e^7	e^8	e^9	q	
1	$\boldsymbol{a^2}$	0	1	-1	0	2	0	0	0	0	0	0	4	
0	e^2	0	0	$-3\frac{1}{2}$	1	$7\frac{1}{2}$	0	0	0	0	0	0	15	
1	$\boldsymbol{a^1}$	1	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	0	0	0	0	0	1	
0	e^4	0	0	$-1\frac{1}{2}$	0	$3\frac{1}{2}$	1	0	0	0	0	0	9	
0	$\boldsymbol{e^5}$	0	0	$1\frac{1}{2}$	0	$-3\frac{1}{2}$	0	1	0	0	0	0	3	**2**
0	e^6	0	0	0	0	1	0	0	1	0	0	0	12	
0	e^7	0	0	$2\frac{1}{2}$	0	$-5\frac{1}{2}$	0	0	0	1	0	0	6	$2\frac{2}{5}$
0	e^8	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	0	0	1	0	18	36
0	e^9	0	0	$1\frac{1}{2}$	0	$-2\frac{1}{2}$	0	0	0	0	0	1	10	$6\frac{2}{3}$
		0	0	$\boldsymbol{-1\frac{1}{2}}$	0	$2\frac{1}{2}$	0	0	0	0	0	0	5	

Now we have brought both a^1 and a^2 into the basis, with coefficients $x_1 = 1$ and $x_2 = 4$. Note that $(1, 4)$ is a vertex of the polytope in Figure 29.8.1. (We also have $z_1 = 0$, $z_2 = 15$, etc.)

Replace e^5 by e^1, get next pivot:

p_{c_i}		a^1	a^2	e^1	e^2	$\boldsymbol{e^3}$	e^4	e^5	e^6	e^7	e^8	e^9	q	
1	$\boldsymbol{a^2}$	0	1	0	0	$-\frac{1}{3}$	0	$\frac{2}{3}$	0	0	0	0	6	
0	e^2	0	0	0	1	$-\frac{2}{3}$	0	$2\frac{1}{3}$	0	0	0	0	22	
1	$\boldsymbol{a^1}$	1	0	0	0	$-\frac{2}{3}$	0	$\frac{1}{3}$	0	0	0	0	2	
0	e^4	0	0	0	0	0	1	1	0	0	0	0	12	
0	$\boldsymbol{e^1}$	0	0	1	0	$-2\frac{1}{3}$	0	$\frac{2}{3}$	0	0	0	0	2	
0	e^6	0	0	0	0	1	0	0	1	0	0	0	12	12
0	$\boldsymbol{e^7}$	0	0	0	0	$\boxed{\frac{1}{3}}$	0	$-1\frac{2}{3}$	0	1	0	0	1	**3**
0	e^8	0	0	0	0	$1\frac{2}{3}$	0	$-\frac{1}{3}$	0	0	1	0	17	$10\frac{1}{5}$
0	e^8	0	0	0	0	1	0	-1	0	0	0	1	7	7
		0	0	0	0	$\boldsymbol{-1}$	0	1	0	0	0	0	8	

Now $(x_1, x_2) = (2, 6)$, which is the next vertex of the solution polytope.

Replace e^7 by e^3, get next pivot:

p_{c_i}		a^1	a^2	e^1	e^2	e^3	e^4	$\boldsymbol{e^5}$	e^6	e^7	e^8	e^9	q	
1	$\boldsymbol{a^2}$	0	1	0	0	0	0	-1	0	1	0	0	7	
0	e^2	0	0	0	1	0	0	-1	0	2	0	0	24	
1	$\boldsymbol{a^1}$	1	0	0	0	0	0	-3	0	2	0	0	4	
0	e^4	0	0	0	0	0	1	1	0	0	0	0	12	12
0	$\boldsymbol{e^1}$	0	0	1	0	0	0	-11	0	7	0	0	9	
0	e^6	0	0	0	0	0	0	5	1	-3	0	0	9	$1\frac{4}{5}$
0	$\boldsymbol{e^3}$	0	0	0	0	1	0	-5	0	3	0	0	3	
0	e^8	0	0	0	0	0	0	8	0	-5	1	0	12	$1\frac{1}{2}$
0	$\boldsymbol{e^9}$	0	0	0	0	0	0	$\boxed{4}$	0	-3	0	1	4	**1**
		0	0	0	0	0	0	$\boldsymbol{-4}$	0	3	0	0	11	

This gives $(x_1, x_2) = (4, 7)$, which continues on to the next vertex of the solution polytope.

Replace e^9 by e^5 to get:

p_{c_i}		a^1	a^2	e^1	e^2	e^3	e^4	e^5	e^6	e^7	e^8	e^9	q
1	$\boldsymbol{a^2}$	0	1	0	0	0	0	0	0	$\frac{1}{4}$	0	$\frac{1}{4}$	8
0	e^2	0	0	0	1	0	0	0	0	$1\frac{1}{4}$	0	$\frac{1}{4}$	25
1	$\boldsymbol{a^1}$	1	0	0	0	0	0	0	0	$-\frac{1}{4}$	0	$\frac{3}{4}$	7
0	e^4	0	0	0	0	0	1	0	0	$\frac{3}{4}$	0	$-\frac{1}{4}$	11
0	$\boldsymbol{e^1}$	0	0	1	0	0	0	0	0	$-1\frac{1}{4}$	0	$2\frac{3}{4}$	20
0	e^6	0	0	0	0	0	0	0	1	$\frac{3}{4}$	0	$-1\frac{1}{4}$	4
0	$\boldsymbol{e^3}$	0	0	0	0	1	0	0	0	$-\frac{3}{4}$	0	$1\frac{1}{4}$	8
0	e^8	0	0	0	0	0	0	0	0	1	1	-2	4
0	$\boldsymbol{e^5}$	0	0	0	0	0	0	1	0	$-\frac{3}{4}$	0	$\frac{1}{4}$	1
		0	0	0	0	0	0	0	0	0	0	1	15

There is no pivot column, so the current basis is optimal. We finish at the vertex $(x_1, x_2) = (7, 8)$, so a solution is

$$x = (7, 8) \text{ and } z = (20, 25, 8, 11, 1, 4, 0, 4, 0).$$
The value is 15.

29.9 The Simplex Algorithm jumps to an adjacent vertex

we now show that the pivot operation in the Simplex Algorithm moves from vertex in the solution polyhedron to an adjacent vertex. But first we have to define what we mean by an adjacent vertex.

29.9.1 Definition *If x and y are vertices of a polyhedron P, then x and y are* ***adjacent*** *of P if the line segment $[x, y]$ is an* ***edge****, meaning an extreme subset of P.*

29.9.2 Proposition *Let x be a vertex of the contraint polyhedron*

$$C = \{z \geqq 0 : Az = q\}.$$

The replacement operation used by the Simplex Algorithm in a tableau for

$$Ax = q$$

with a basis in the left-hand column produces a vertex x' that is adjacent to $\bar{x}$.

The following proof is taken from Dantzig [4, Theorem 4, pp. 155–156].

Proof: We know from Proposition 28.7.1 that x is a basic nonnegative solution to $Ax = q$. As in that proof, by rearranging the columns of A we may assume without loss of generality that for some k, $x = (x_1, \dots, x_k, 0, \dots, 0)$, and that $A^1, \dots, A^k$ are independent, $x_j > 0$ for $j = 1, \dots, k$, $x_j = 0$ for $j > k$ and

$$q = \sum_{j=1}^{k} x_j A^j.$$

Again, by renumbering if we must, we may assume the replacement operation replaces A^1 by A^{k+1}, which yields the vector $x' = (0, x'_2, \dots, x'_{k+1}, 0, \dots, 0)$, where $x'_j > 0$ for $j > k+1$, and

$$q = \sum_{j=2}^{k+1} x'_j A^j.$$

By construction of the Simplex Algorithm, the point x' is a basic nonnegative solution, that is, a vertex of C.

Aside: Here is a finicky point. We know that $x'_{k+1} > 0$. Why? Because if $z'_{k+1} = 0$, then $q = \sum_{j=2}^{k+1} x'_j A^j = \sum_{j=1}^{k} x'_j A^j$. But $A_1, \dots, A^j$ are independent, so the coordinates of q are unique, so $x' = x$. But $x_1 > 0$ and $x'_1 = 0$, a contradiction. Therefore $x'_{k+1} > 0$. We will use this below when we define the scalar μ.

Now since $A^1, \dots, A^k$ are independent, by the Replacement Lemma 25.7.1 there is a unique solution $a = (\bar\alpha_1, \dots, \bar\alpha_k) \neq 0$ to

$$\sum_{j=1}^{k} \alpha_j A^j = A^{k+1}. \tag{14}$$

Next let $y = (1-\lambda)x + \lambda x'$ belong to the segment $[x, x']$. Then $y_1 = (1-\lambda)x_1$, $y_{k+1} = \lambda x'_{k+1'}$, and $y_j = 0$ for $j > k+1$. This suggests we define

$$M = \{z \in C : z_j = 0,\ j > k+1\}.$$

Tt is clear that $x, x' \in M$, so we have just shown that the segment $[x, x']$ is included in M.

Now let $z \geqq 0$ be an arbitrary vector in M. Since z belongs to C we have

$$\sum_{j=1}^{k+1} z_j A^j = q = \sum_{j=1}^{k} x_j A^j,$$

so

$$\sum_{j=1}^{k} (x_j - z_j) A^j = z_{k+1} A^{k+1}. \tag{15}$$

There are two cases: (1) $z_{k+1} = 0$, and (2) $z_{k+1} > 0$.

In Case (1), $z_{k+1} = 0$, since $A^1, \dots, A^k$ are linearly independent, (15) implies that $z = x$. In Case (2), divide (15) by z_{k+1} to get

$$\sum_{j=1}^{k} \frac{x_j - z_j}{z_{k+1}} A^j = A^{k+1}.$$

Then by (14), for any such z we must have $\frac{x_j - z_j}{z_{k+1}} = \bar{\alpha}_j$, or

$$z_j = x_j - \bar{\alpha}_j z_{k+1}, \quad j = 1, \dots, k. \tag{16}$$

That is, z is determined by z_{k+1}. Note that (16) holds even if $z_{k+1} = 0$.

As a special case, letting x' replace z, we have

$$x'_j = x_j - \bar{\alpha}_j x'_{k+1}, \quad j = 1, \dots, k. \tag{17}$$

Set

$$\mu = \frac{z_{k+1}}{x'_{k+1}}, \quad \text{so} \quad z_{k+1} = \mu x'_{k+1},$$

and multiply (17) by μ to get

$$\mu x'_j = \mu x_j - \mu \bar{\alpha}_j x'_{k+1} = \mu x_j - \bar{\alpha}_j z_{k+1}$$

and subtract this from (16) to get

$$z_j - \mu x'_j = x_j - \mu x_j - \bar{\alpha}_j z_{k+1} + \mu \bar{\alpha}_j x'_{k+1},$$

which reduces to

$$z_j = (1-\mu) x_j + \mu x'_j, \quad j = 1, \dots, k.$$

Also note that

$$z_{k+1} = \mu x'_{k+1} = \mu x'_{k+1} + (1-\mu) \underbrace{x_{k+1}}_{=0},$$

and of course

$$z_j = (1-\lambda) x_j + \lambda x'_j = 0, \quad j > k+1.$$

This proves that any $z \in M \subset C$ satisfies

$$z = (1-\mu) x + \mu x'$$

and so lies on the line through x and x'. But x and x' are extreme points of C, so in fact z lies in the segment $[x, x'] \subset M$.[2] This proves that $[x, x']$ is an extreme subset of C, so x' is adjacent to x. ∎

[2] If say $\mu > 1$, then x' is a convex combination of x and z, contradicting the result that x' is a vertex.

29.10 The Simplex Algorithm is a steepest ascent method

Cf. Dantzig [4, Section 7.2, pp. 56–160].
****

29.11 Why is it called the Simplex Algorithm?

See Dantzig [4, Section 7.3].
****

29.12 More worked examples

Just as a picture is worth a thousand words, a good example is worth several pages of dense notation.

29.12.1 Minimization with equality constraints

Consider the following problem.

$$\underset{x}{\text{minimize}} \quad x_1 + 6x_2 - 7x_3 + x_4 + 5x_5$$

subject to $x \geqq 0$, and

$$\begin{aligned} 5x_1 - 4x_2 + 13x_3 - 2x_4 + x_5 &= 20 \\ x_1 - x_2 + 5x_3 - x_4 + x_5 &= 8 \end{aligned}$$

Since the constraints take the form of equalities, no slack variables are necessary, but there is no obvious starting point. So in Phase I, we introduce nonnegative artificial variables u_1 and u_2, and proceed to solve the ancillary problem

$$\underset{u}{\text{minimize}} \; u_1 + u_2$$

subject to $x \geqq 0$, $u \geqq 0$, and

$$\begin{aligned} 5x_1 - 4x_2 + 13x_3 - 2x_4 + x_5 + u_1 \phantom{{}+u_2} &= 20 \\ x_1 - x_2 + 5x_3 - x_4 + x_5 \phantom{{}+u_1} + u_2 &= 8 \end{aligned}$$

Since we require that $u \geqq 0$, the minimum of $u_1 + u_2 \geqslant 0$, with equality only if $u_1 = u_2 = 0$. Thus if the solution to this LP has value zero, we will have succeeded in finding a feasible solution to the original problem. The virtue of this ancillary problem is that there is an obvious starting point: set $x = 0$, and setting $u = (20, 8)$ (that is, set u to the right-hand side). The criterion row is based on the artificial price vector indicated in the left margin of the *tableau*, and is searched for ***positive*** entries.

Here is the initial *tableau*.

p_{c_i}		a^1	a^2	a^3	a^4	a^5	e^1	e^2	q	
Initial *tableau*										
1	u^1	5	-4	[13]	-2	1	1	0	20	$1\frac{7}{13}$
1	u^2	1	-1	5	-1	1	0	1	8	$1\frac{3}{5}$
		6	-5	**18**	-3	**2**	1	1	28	
Replace u^1 by a^3:										
0	a^3	$\frac{5}{13}$	$-\frac{4}{13}$	1	$-\frac{2}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	0	$1\frac{7}{13}$	20
1	u^2	$-\frac{12}{13}$	$\frac{7}{13}$	0	$-\frac{3}{13}$	[$\frac{8}{13}$]	$-\frac{5}{13}$	1	$\frac{4}{13}$	$\frac{1}{2}$
		$-\frac{12}{13}$	$\mathbf{\frac{7}{13}}$	0	$-\frac{3}{13}$	$\mathbf{\frac{8}{13}}$	$-\frac{5}{13}$	1	$\frac{4}{13}$	
Replace u^2 by a^5:										
0	a^3	$\frac{1}{2}$	$-\frac{3}{8}$	1	$-\frac{1}{8}$	0	$\frac{1}{8}$	$-\frac{1}{8}$	$1\frac{1}{2}$	
0	a^5	$-1\frac{1}{2}$	$\frac{7}{8}$	0	$-\frac{3}{8}$	1	$-\frac{5}{8}$	$1\frac{5}{8}$	$\frac{1}{2}$	
		0	0	0	0	0	0	0	0	

According to this, the value (found in the lower right-hand corner) is zero, so we have indeed found a feasible solution to the original problem, namely

$$x = \left(0, \quad 0, \quad 1\tfrac{1}{2}, \quad 0, \quad \tfrac{1}{2}\right).$$

I leave it to you to check that x does indeed satisfy the constraints.

In Phase II, we now proceed with the original minimization problem. To do so, we must recalculate the $\pi - p$ criterion row, and search for ***positive*** entries. Here is the new initial *tableau*.

p_{c_i}		a^1	a^2	a^3	a^4	a^5	e^1	e^2	q	
Initial *tableau*										
-7	a^3	$\frac{1}{2}$	$-\frac{3}{8}$	1	$-\frac{1}{8}$	0	$\frac{1}{8}$	$-\frac{1}{8}$	$1\frac{1}{2}$	
5	a^5	$-1\frac{1}{2}$	[$\frac{7}{8}$]	0	$-\frac{3}{8}$	1	$-\frac{5}{8}$	$1\frac{5}{8}$	$\frac{1}{2}$	$\frac{4}{7}$
		-12	**1**	0	-2	0	-4	9	-8	
Replace a^5 by a^2:										
-7	a^3	$-\frac{1}{7}$	0	1	$-\frac{2}{7}$	$\frac{3}{7}$	$-\frac{1}{7}$	$\frac{4}{7}$	$1\frac{5}{7}$	
6	a^2	$-1\frac{5}{7}$	1	0	$-\frac{3}{7}$	$1\frac{1}{7}$	$-\frac{5}{7}$	$1\frac{6}{7}$	$\frac{4}{7}$	
		$-10\frac{2}{7}$	0	0	$-1\frac{4}{7}$	$-1\frac{1}{7}$	$-3\frac{2}{7}$	$7\frac{1}{7}$	$-8\frac{4}{7}$	

Notice that in Phase II, I never pivot on a column corresponding to the artificial variables (look at that nice fat 9 in the criterion row of the first *tableau*), because they may not be used in a bona fide solution. Why, then you might ask, do I keep them in the *tableau*? The answer is that they compute the solution to the dual.

We can read a solution from the final *tableau* above:

$$x = \left(0, \quad \tfrac{4}{7}, \quad 1\tfrac{5}{7}, \quad 0, \quad 0\right)$$

Let me verify that the constraints are satisfied:

$$\begin{aligned} 5(0) - 4(\tfrac{4}{7}) + 13(1\tfrac{5}{7}) - 2(0) + 1(0) &= 0 - 2\tfrac{2}{7} + 22\tfrac{2}{7} + 0 + 0 = 20 = 20 \\ 1(0) - 1(\tfrac{4}{7}) + 5(1\tfrac{5}{7}) - 1(0) + 1(0) &= 0 - \tfrac{4}{7} + 8\tfrac{4}{7} + 0 + 0 = 8 = 8 \end{aligned}$$

The value is $-8\frac{4}{7}$.

We can also read off a solution to the dual:

$$y = \left(-3\tfrac{2}{7}, \quad 7\tfrac{1}{7}\right).$$

Recall that the dual problem is

$$\underset{y}{\text{maximize}} \ 20y_1 + 8y_2$$

subject to

$$\begin{aligned} 5y_1 + y_2 &\leqslant 1 \\ -4y_1 - y_2 &\leqslant 6 \\ 13y_1 + 5y_2 &\leqslant -7 \\ -2y_1 - y_2 &\leqslant 1 \\ y_1 + y_2 &\leqslant 5 \end{aligned}$$

Verify the feasibility of the dual.

$$\begin{aligned} 5(-3\tfrac{2}{7}) + 1(7\tfrac{1}{7}) &= -16\tfrac{3}{7} + 7\tfrac{1}{7} = -9\tfrac{2}{7} < 1 \\ -4(-3\tfrac{2}{7}) - 1(7\tfrac{1}{7}) &= 13\tfrac{1}{7} - 7\tfrac{1}{7} = 6 = 6 \\ 13(-3\tfrac{2}{7}) + 5(7\tfrac{1}{7}) &= -42\tfrac{5}{7} + 35\tfrac{5}{7} = -7 = -7 \\ -2(-3\tfrac{2}{7}) - 1(7\tfrac{1}{7}) &= 6\tfrac{4}{7} - 7\tfrac{1}{7} = -\tfrac{4}{7} < 1 \\ 1(-3\tfrac{2}{7}) + 1(7\tfrac{1}{7}) &= -3\tfrac{2}{7} + 7\tfrac{1}{7} = 3\tfrac{6}{7} < 5 \end{aligned}$$

29.12.2 An example with a negative right-hand side constant

Consider the problem

$$\underset{x}{\text{maximize}}\ 2x_1 - 3x_2 + x_3 + x_4$$

subject to $x \geqq 0$, and

$$\begin{aligned} x_1 + 2x_2 + x_3 + x_4 &= 3 \\ x_1 - 2x_2 + 2x_3 + x_4 &= -2 \\ 3x_1 - x_2 \qquad - x_4 &= -1 \end{aligned}$$

Rewrite the constraints as

$$\begin{aligned} x_1 + 2x_2 + x_3 + x_4 &= 3 \\ -x_1 + 2x_2 - 2x_3 - x_4 &= 2 \\ -3x_1 + x_2 \qquad + x_4 &= 1 \end{aligned}$$

This has no effect on the primal, but the dual is different. This form has the virtue that the following ancillary problem has an obvious starting feasible point.

$$\underset{y}{\text{minimize}}\quad u_1 + u_2 + u_3$$

subject to $x \geqq 0$, $u \geqq 0$, and

$$\begin{aligned} x_1 + 2x_2 + x_3 + x_4 + u_1 \qquad\qquad &= 3 \\ -x_1 + 2x_2 - 2x_3 - x_4 \qquad + u_2 \qquad &= 2 \\ -3x_1 + x_2 \qquad + x_4 \qquad\qquad + u_3 &= 1 \end{aligned}$$

A feasible starting point is given by setting $x = 0$, and setting $u = (3, 2, 1)$.

Here is the initial *tableau*.

Initial *tableau*:

	a^1	a^2	a^3	a^4	e^1	e^2	e^3	q	
e^1	1	2	1	1	1	0	0	3	$1\frac{1}{2}$
e^2	-1	$\boxed{2}$	-2	-1	0	1	0	2	1
e^3	-3	1	0	1	0	0	1	1	1
	-3	$\mathbf{5}$	-1	1	0	0	0	6	

Replace e^2 by a^2 to get:

	a^1	a^2	a^3	a^4	e^1	e^2	e^3	q	
e^1	2	0	3	2	1	-1	0	1	$\frac{1}{3}$
a^2	$-\frac{1}{2}$	1	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	1	
e^3	$-2\frac{1}{2}$	0	$\boxed{1}$	$1\frac{1}{2}$	0	$-\frac{1}{2}$	1	0	0
	$-\frac{1}{2}$	0	$\mathbf{4}$	$\mathbf{3\frac{1}{2}}$	0	$-2\frac{1}{2}$	0	1	

Replace e^3 by a^3 to get:

	a^1	a^2	a^3	a^4	e^1	e^2	e^3	q	
e^1	$\boxed{9\frac{1}{2}}$	0	0	$-2\frac{1}{2}$	1	$\frac{1}{2}$	-3	1	$\frac{2}{19}$
a^2	-3	1	0	1	0	0	1	1	
a^3	$-2\frac{1}{2}$	0	1	$1\frac{1}{2}$	0	$-\frac{1}{2}$	1	0	
	$\mathbf{9\frac{1}{2}}$	0	0	$-2\frac{1}{2}$	0	$-\frac{1}{2}$	-4	1	

Replace e^1 by a^1 to get:

	a^1	a^2	a^3	a^4	e^1	e^2	e^3	q
a^1	1	0	0	$-\frac{5}{19}$	$\frac{2}{19}$	$\frac{1}{19}$	$-\frac{6}{19}$	$\frac{2}{19}$
a^2	0	1	0	$\frac{4}{19}$	$\frac{6}{19}$	$\frac{3}{19}$	$\frac{1}{19}$	$1\frac{6}{19}$
a^3	0	0	1	$\frac{16}{19}$	$\frac{5}{19}$	$-\frac{7}{19}$	$\frac{4}{19}$	$\frac{5}{19}$
	0	0	0	0	-1	-1	-1	0

Since the value is 0, we have found a feasible starting point for the original problem.

Now to maximize. But first we must recalculate the $\pi - p$ criterion row.

Here is the new initial *tableau*.

Initial *tableau*:

	a^1	a^2	a^3	a^4	e^1	e^2	e^3	q	
a^1	1	0	0	$-\frac{5}{19}$	$\frac{2}{19}$	$\frac{1}{19}$	$-\frac{6}{19}$	$\frac{2}{19}$	
a^2	0	1	0	$\frac{4}{19}$	$\frac{6}{19}$	$\frac{3}{19}$	$\frac{1}{19}$	$1\frac{6}{19}$	$6\frac{1}{4}$
a^3	0	0	1	$\boxed{\frac{16}{19}}$	$\frac{5}{19}$	$-\frac{7}{19}$	$\frac{4}{19}$	$\frac{5}{19}$	$\frac{5}{16}$
	0	0	0	$\mathbf{-1\frac{6}{19}}$	$-\frac{9}{19}$	$-\frac{14}{19}$	$-\frac{11}{19}$	$-3\frac{9}{19}$	

Replace a^3 by a^4 to get:

a^1	1	0	$\frac{5}{16}$	0	$\frac{3}{16}$	$-\frac{1}{16}$	$-\frac{1}{4}$	$\frac{3}{16}$
a^2	0	1	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{4}$	0	$1\frac{1}{4}$
a^4	0	0	$1\frac{3}{16}$	1	$\frac{5}{16}$	$-\frac{7}{16}$	$\frac{1}{4}$	$\frac{5}{16}$
	0	0	$1\frac{9}{16}$	0	$-\frac{1}{16}$	$-1\frac{5}{16}$	$-\frac{1}{4}$	$-3\frac{1}{16}$

Thus a solution is

$$x = \left(\tfrac{3}{16},\quad 1\tfrac{1}{4},\quad 0,\quad \tfrac{5}{16}\right)$$

Verify the constraints are satisfied:

$$\begin{aligned}
1(\tfrac{3}{16}) + 2(1\tfrac{1}{4}) + 1(0) + 1(\tfrac{5}{16}) &= \tfrac{3}{16} + 2\tfrac{1}{2} + 0 + \tfrac{5}{16} = 3 = 3 \\
1(\tfrac{3}{16}) - 2(1\tfrac{1}{4}) + 2(0) + 1(\tfrac{5}{16}) &= \tfrac{3}{16} - 2\tfrac{1}{2} + 0 + \tfrac{5}{16} = -2 = -2 \\
3(\tfrac{3}{16}) - 1(1\tfrac{1}{4}) + 0(0) - 1(\tfrac{5}{16}) &= \tfrac{9}{16} - 1\tfrac{1}{4} + 0 - \tfrac{5}{16} = -1 = -1
\end{aligned}$$

The value is $-3\frac{1}{16}$.

According to the criterion row we see that a solution to the dual is $y = \left(-\frac{1}{16},\quad -1\frac{5}{16},\quad -\frac{1}{4}\right)$. But this is a solution to the modified dual, not the original dual. To convert it we must flip the signs on the components corresponding to negative right-hand sides in the original problem. These are the second and third components. Thus a solution to the original dual is

$$y = \left(-\tfrac{1}{16},\quad 1\tfrac{5}{16},\quad \tfrac{1}{4}\right).$$

Recall that the original dual problem is

$$\underset{y}{\text{minimize}}\ 3y_1 - 2y_2 - y_3$$

subject to

$$\begin{aligned}
y_1 + y_2 + 3y_3 &\geqslant 2 \\
2y_1 - 2y_2 - y_3 &\geqslant -3 \\
y_1 + 2y_2 &\geqslant 1 \\
y_1 + y_2 - y_3 &\geqslant 1
\end{aligned}$$

Check that the value of the dual solution is

$$3(-\tfrac{1}{16}) - 2(1\tfrac{5}{16}) - 1(\tfrac{1}{4}) = -\tfrac{3}{16} - 2\tfrac{5}{8} - \tfrac{1}{4} = -3\tfrac{1}{16}.$$

Now verify the feasibility of the dual solution for the original dual.

$$\begin{aligned}
1(-\tfrac{1}{16}) + 1(1\tfrac{5}{16}) + 3(\tfrac{1}{4}) &= -\tfrac{1}{16} + 1\tfrac{5}{16} + \tfrac{3}{4} = 2 = 2\\
2(-\tfrac{1}{16}) - 2(1\tfrac{5}{16}) - 1(\tfrac{1}{4}) &= -\tfrac{1}{8} - 2\tfrac{5}{8} - \tfrac{1}{4} = -3 = -3\\
1(-\tfrac{1}{16}) + 2(1\tfrac{5}{16}) + 0(\tfrac{1}{4}) &= -\tfrac{1}{16} + 2\tfrac{5}{8} + 0 = 2\tfrac{9}{16} > 1\\
1(-\tfrac{1}{16}) + 1(1\tfrac{5}{16}) - 1(\tfrac{1}{4}) &= -\tfrac{1}{16} + 1\tfrac{5}{16} - \tfrac{1}{4} = 1 = 1
\end{aligned}$$

29.12.3 A tricky point with negative right-hand side constants

If the constraints are inequality constraints and the right-hand side has negative values, simply adding slack variables does not immediately lead to a feasible point, so Phase 1 cannot be combined with Phase 2.

Change the constraints in the previous problem to inequalities.

$$\underset{x}{\text{maximize}}\ 2x_1 - 3x_2 + x_3 + x_4$$

subject to $x \geqq 0$, and

$$\begin{aligned}
x_1 + 2x_2 + x_3 + x_4 &\leqslant 3\\
x_1 - 2x_2 + 2x_3 + x_4 &\leqslant -2\\
3x_1 - x_2 - x_4 &\leqslant -1
\end{aligned}$$

Add nonnegative slack variables to convert the constraints to equalities.

$$\begin{aligned}
x_1 + 2x_2 + x_3 + x_4 + z_1 &= 3\\
x_1 - 2x_2 + 2x_3 + x_4 + z_2 &= -2\\
3x_1 - x_2 - x_4 + z_3 &= -1
\end{aligned}$$

Now multiply the second and third equations by -1 to get

$$\begin{aligned}
x_1 + 2x_2 + x_3 + x_4 + z_1 &= 3\\
-x_1 + 2x_2 - 2x_3 - x_4 - z_2 &= 2\\
-3x_1 + x_2 + x_4 - z_3 &= 1
\end{aligned}$$

In this case setting $x = 0$ and $z = q$ does not give a feasible solution to the primal. To find a nonnegative feasible point, solve the ancillary problem

$$\underset{u}{\text{minimize}}\quad u_1 + u_2 + u_3$$

subject to $x \geqq 0$, $z \geqq 0$, $u \geqq 0$, and

$$\begin{aligned}
x_1 + 2x_2 + x_3 + x_4 + z_1 \quad + u_1 \quad &= 3 \\
-x_1 + 2x_2 - 2x_3 - x_4 - z_2 \quad + u_2 \quad &= 2 \\
-3x_1 + x_2 + x_4 - z_3 \quad + u_3 &= 1
\end{aligned}$$

This problem has a trivial starting point, given by $x = 0$, $z = 0$, and $u = (3, 2, 1)$.

Here is the initial *tableau*.

	a^1	a^2	a^3	a^4	e^1	e^2	e^3	u^1	u^2	u^3	q	

Initial *tableau*:

	a^1	a^2	a^3	a^4	e^1	e^2	e^3	u^1	u^2	u^3	q	
u^1	1	2	1	1	1	0	0	1	0	0	3	$1\frac{1}{2}$
u^2	−1	[2]	−2	−1	0	−1	0	0	1	0	2	1
u^3	−3	1	0	1	0	0	−1	0	0	1	1	1
	−3	**5**	−1	**1**	**1**	−1	−1	0	0	0	6	

Replace u^2 by a^2 to get:

	a^1	a^2	a^3	a^4	e^1	e^2	e^3	u^1	u^2	u^3	q	
u^1	2	0	3	2	1	1	0	1	−1	0	1	$\frac{1}{3}$
a^2	$-\frac{1}{2}$	1	−1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	0	1	
u^3	$-2\frac{1}{2}$	0	[1]	$1\frac{1}{2}$	0	$\frac{1}{2}$	−1	0	$-\frac{1}{2}$	1	0	0
	$-\frac{1}{2}$	0	**4**	**$3\frac{1}{2}$**	**1**	**$1\frac{1}{2}$**	−1	0	$-2\frac{1}{2}$	0	1	

Replace u^3 by a^3 to get:

	a^1	a^2	a^3	a^4	e^1	e^2	e^3	u^1	u^2	u^3	q	
u^1	[$9\frac{1}{2}$]	0	0	$-2\frac{1}{2}$	1	$-\frac{1}{2}$	3	1	$\frac{1}{2}$	−3	1	$\frac{2}{19}$
a^2	−3	1	0	1	0	0	−1	0	0	1	1	
a^3	$-2\frac{1}{2}$	0	1	$1\frac{1}{2}$	0	$\frac{1}{2}$	−1	0	$-\frac{1}{2}$	1	0	
	$9\frac{1}{2}$	0	0	$-2\frac{1}{2}$	**1**	$-\frac{1}{2}$	3	0	$-\frac{1}{2}$	−4	1	

Replace u^1 by a^1 to get:

	a^1	a^2	a^3	a^4	e^1	e^2	e^3	u^1	u^2	u^3	q
a^1	1	0	0	$-\frac{5}{19}$	$\frac{2}{19}$	$-\frac{1}{19}$	$\frac{6}{19}$	$\frac{2}{19}$	$\frac{1}{19}$	$-\frac{6}{19}$	$\frac{2}{19}$
a^2	0	1	0	$\frac{4}{19}$	$\frac{6}{19}$	$-\frac{3}{19}$	$-\frac{1}{19}$	$\frac{6}{19}$	$\frac{3}{19}$	$\frac{1}{19}$	$1\frac{6}{19}$
a^3	0	0	1	$\frac{16}{19}$	$\frac{5}{19}$	$\frac{7}{19}$	$-\frac{4}{19}$	$\frac{5}{19}$	$-\frac{7}{19}$	$\frac{4}{19}$	$\frac{5}{19}$
	0	0	0	0	0	0	0	−1	−1	−1	0

The value is 0, so we have found a feasible starting point for Phase 2. Now to recalculate the $\pi - p$ criterion row and maximize. Here is the new initial *tableau*.

Initial *tableau*:

	a^1	a^2	a^3	a^4	e^1	e^2	e^3	u^1	u^2	u^3	q	
a^1	1	0	0	$-\frac{5}{19}$	$\frac{2}{19}$	$-\frac{1}{19}$	$\frac{6}{19}$	$\frac{2}{19}$	$\frac{1}{19}$	$-\frac{6}{19}$	$\frac{2}{19}$	
a^2	0	1	0	$\frac{4}{19}$	$\frac{6}{19}$	$-\frac{3}{19}$	$-\frac{1}{19}$	$\frac{6}{19}$	$\frac{3}{19}$	$\frac{1}{19}$	$1\frac{6}{19}$	$6\frac{1}{4}$
a^3	0	0	1	$\boxed{\frac{16}{19}}$	$\frac{5}{19}$	$\frac{7}{19}$	$-\frac{4}{19}$	$\frac{5}{19}$	$-\frac{7}{19}$	$\frac{4}{19}$	$\frac{5}{19}$	$\frac{5}{16}$
	0	0	0	$\mathbf{-1\frac{6}{19}}$	$\mathbf{-\frac{9}{19}}$	$\frac{14}{19}$	$\frac{11}{19}$	$-\frac{9}{19}$	$-\frac{14}{19}$	$-\frac{11}{19}$	$-3\frac{9}{19}$	

Replace a^3 by a^4 to get:

	a^1	a^2	a^3	a^4	e^1	e^2	e^3	u^1	u^2	u^3	q	
a^1	1	0	$\frac{5}{16}$	0	$\boxed{\frac{3}{16}}$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{16}$	$-\frac{1}{16}$	$-\frac{1}{4}$	$\frac{3}{16}$	1
a^2	0	1	$-\frac{1}{4}$	0	$\frac{1}{4}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{4}$	0	$1\frac{1}{4}$	5
a^4	0	0	$1\frac{3}{16}$	1	$\frac{5}{16}$	$\frac{7}{16}$	$-\frac{1}{4}$	$\frac{5}{16}$	$-\frac{7}{16}$	$\frac{1}{4}$	$\frac{5}{16}$	1
	0	0	$1\frac{9}{16}$	0	$\mathbf{-\frac{1}{16}}$	$1\frac{5}{16}$	$\frac{1}{4}$	$-\frac{1}{16}$	$-1\frac{5}{16}$	$-\frac{1}{4}$	$-3\frac{1}{16}$	

Replace a^1 by e^1 to get:

	a^1	a^2	a^3	a^4	e^1	e^2	e^3	u^1	u^2	u^3	q
e^1	$5\frac{1}{3}$	0	$1\frac{2}{3}$	0	1	$\frac{1}{3}$	$1\frac{1}{3}$	1	$-\frac{1}{3}$	$-1\frac{1}{3}$	1
a^2	$-1\frac{1}{3}$	1	$-\frac{2}{3}$	0	0	$-\frac{1}{3}$	$-\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{1}{3}$	1
a^4	$-1\frac{2}{3}$	0	$\frac{2}{3}$	1	0	$\frac{1}{3}$	$-\frac{2}{3}$	0	$-\frac{1}{3}$	$\frac{2}{3}$	0
	$\frac{1}{3}$	0	$1\frac{2}{3}$	0	0	$1\frac{1}{3}$	$\frac{1}{3}$	0	$-1\frac{1}{3}$	$-\frac{1}{3}$	-3

Thus a solution is

$$x = (0, \quad 1, \quad 0, \quad 0)$$

Verify the constraints are satisfied:

$$\begin{aligned} 1(0) + 2(1) + 1(0) + 1(0) &= 0 + 2 + 0 + 0 = \quad 2 < \quad 3 \\ 1(0) - 2(1) + 2(0) + 1(0) &= 0 - 2 + 0 + 0 = -2 = -2 \\ 3(0) - 1(1) + 0(0) - 1(0) &= 0 - 1 + 0 + 0 = -1 = -1 \end{aligned}$$

The value is -3.

Note that by relaxing the constraints from equations in the previous section to inequalities, the value has increased.

A solution to the dual is

$$y = \left(0, \quad 1\tfrac{1}{3}, \quad \tfrac{1}{3}\right).$$

This can be read off the criterion row in two places, under the slack variables, or by appropriate sign flips under the auxiliary variables. Recall that the dual problem is

$$\underset{y}{\text{minimize}}\ 3y_1 - 2y_2 - \ y_3$$

subject to

$$\begin{aligned} y_1 + \ y_2 + 3y_3 &\geqslant \ 2 \\ 2y_1 - 2y_2 - \ y_3 &\geqslant -3 \\ y_1 + 2y_2 \qquad &\geqslant \ 1 \\ y_1 + \ y_2 - \ y_3 &\geqslant \ 1 \end{aligned}$$

Check that the value of the dual is

$$3(0) - 2(1\tfrac{1}{3}) - 1(\tfrac{1}{3}) = 0 - 2\tfrac{2}{3} - \tfrac{1}{3} = -3.$$

Now verify the feasibility of the dual.

$$\begin{aligned} 1(0) + 1(1\tfrac{1}{3}) + 3(\tfrac{1}{3}) &= 0 + 1\tfrac{1}{3} + 1 = \ 2\tfrac{1}{3} > \ 2 \\ 2(0) - 2(1\tfrac{1}{3}) - 1(\tfrac{1}{3}) &= 0 - 2\tfrac{2}{3} - \tfrac{1}{3} = -3 = -3 \\ 1(0) + 2(1\tfrac{1}{3}) + 0(\tfrac{1}{3}) &= 0 + 2\tfrac{2}{3} + 0 = \ 2\tfrac{2}{3} > \ 1 \\ 1(0) + 1(1\tfrac{1}{3}) - 1(\tfrac{1}{3}) &= 0 + 1\tfrac{1}{3} - \tfrac{1}{3} = \ 1 = \ 1 \end{aligned}$$

References

[1] I. Adler, N. Megiddo, and M. J. Todd. 1984. New results on the average behavior of simplex algorithms. *Bulletin of the American Mathematical Society* 11(2):378–382. http://projecteuclid.org/euclid.bams/1183552182

[2] E. M. L. Beale. 1955. Cycling in the dual simplex algorithm. *Naval Research Logistics Quarterly* 2(4):269–276. DOI: 10.1002/nav.3800020406

[3] A. Charnes. 1952. Optimality and degeneracy in linear programming. *Econometrica* 20(2):160–170. http://www.jstor.org/stable/1907845

[4] G. B. Dantzig. 1963. *Linear programming and extensions*. Princeton: Princeton University Press.

[5] G. B. Dantzig, A. Orden, and P. Wolfe. 1955. The generalized simplex method for minimizing a linear form under linear inequality constraints. *Pacific Journal of Mathematics* 5(2):183–195. http://projecteuclid.org/Dienst/UI/1.0/Summarize/euclid.pjm/1103044531

[6] R. Dorfman, P. A. Samuelson, and R. M. Solow. 1987. *Linear programming and economic analysis*. New York: Dover. Reprint of the 1958 edition published by McGraw-Hill in New York.

[7] J. Franklin. 2002. *Methods of mathematical economics: Linear and nonlinear programming, fixed point theorems*. Number 37 in Classics in Applied Mathematics. Philadelphia: SIAM. Corrected reprint of the 1980 edition published by Springer–Verlag.

[8] D. Gale. 1960. *Theory of linear economic models*. New York: McGraw-Hill.

[9] ——— . 1969. How to solve linear inequalities. *American Mathematical Monthly* 76(6):589–599. http://www.jstor.org/stable/2316658

[10] ——— . 1989. *Theory of linear economic models*. Chicago: University of Chicago Press. Reprint of the 1960 edition published by McGraw-Hill.

[11] A. J. Hoffman. 1953. Cycling in the simplex algorithm. Report 2974, National Bureau of Standards. December 16, 1953. 7 pp.

[12] V. L. Klee, Jr. 1965. A class of linear programming problems requiring a large number of iterations. *Numerische Mathematik* 7(4):313–321. DOI: 10.1007/BF01436525

[13] V. L. Klee, Jr. and G. G. Minty. 1972. How good is the simplex algorithm? In O. Shisha, ed., *Inequalities III*. New York: Academic Press.

[14] N. Megiddo. 1986. Improved asymptotic analysis of the average number of steps performed by the self-dual simplex algorithm. *Mathematical Programming* 35(2):140–172. DOI: 10.1007/BF01580645

[15] W. Orchard-Hays. 1956. Evolution of computer codes for linear programming. Paper P-810, RAND Corporation, Santa Monica, California.

[16] S. Smale. 1983. On the average number of steps of the simplex method of linear programming. *Mathematical Programming* 27(3):241–262. DOI: 10.1007/BF02591902

[17] G. J. Stigler. 1945. The cost of subsistence. *Journal of Farm Economics* 27(2):303–314. http://www.jstor.org/stable/pdf/10.2307/1231810.pdf