Topic 28: Linear Programming: Theory

The material for this chapter is based largely on the beautifully written book by David Gale [2].

28.1 Primal and dual linear programs

A **maximum linear program in standard inequality form**[1] is a constrained maximization problem of the form

$$\underset{x}{\text{maximize}}\ p \cdot x \tag{1}$$

subject to

$$Ax \leqq q \tag{2}$$

$$x \geqq 0 \tag{3}$$

where x and p belong to $\mathbf{R}^{\mathrm{n}}$, q belongs to $\mathbf{R}^{\mathrm{m}}$, and A is $m \times n$. Thus there are n variables and m constraints plus n nonnegativity constraints on the variables.

The program is **feasible** if there is some x satisfying the constraints (2) and (3), that is, if the constraint set is nonempty. It is easy to write down inconsistent constraints, so that we are not guaranteed that an arbitrary program is feasible.

Every maximum linear program in standard inequality form has an associated **dual program**, which is the following minimization problem:

$$\underset{y}{\text{minimize}}\ q \cdot y \tag{4}$$

subject to

$$A'y \geqq p \tag{5}$$

$$y \geqq 0. \tag{6}$$

Here A' denotes the transpose of A. The original maximum linear program may be called the **primal** program to distinguish it from the dual. Note that the roles

[1] Gale [2, p. 74] refers to this as the "standard form" of a linear program. However Dantzig [1, p. 60] defines a standard linear program to be one where a linear function is minimized subject to linear equality constraints and nonnegativity constraints.

of p and q are interchanged in the primal and dual program, and that the matrix A is transposed.

The next result describes an important relationship between feasible points for the primal and its dual.

28.1.1 Lemma (Optimality Criterion for LP) *If x is feasible for the primal program and y is feasible for its dual, then*

$$p \cdot x \leqslant q \cdot y.$$

If in addition $p \cdot x = q \cdot y$, then x is optimal and y is optimal for the dual program, and $p \cdot x = q \cdot y = xAy$.

Proof: Suppose x satisfies (2), $Ax \leqq q$, and $y \geqq 0$. Then $yAx \leqslant y \cdot q$. Likewise if y satisfies (5), $A'y = yA \geqq p$, and $x \geqq 0$, then $yAx \geqslant p \cdot x$. Combining these pieces proves the lemma. ∎

If by some means we have found an optimum for the primal and the dual, then:

> The Optimality Criterion gives a simple method to prove that a pair of solutions to the primal and dual solution are indeed optimal.

Most other optimization techniques do not include a proof that the outcome is optimal.

Gale refers to the following corollary as the **Equilibrium Theorem**. It is also known as the **Complementary Slackness Theorem**. It is a simple consequence of the Optimality Criterion (Lemma 28.1.1), and provides a perhaps even simpler method to verify that

28.1.2 Complementary Slackness Theorem *Suppose x and y are feasible for the primal and dual respectively. They are optimal if and only if both*

$$(Ax)_i < q_i \implies y_i = 0 \tag{7}$$

and

$$(A'y)_j > p_j \implies x_j = 0. \tag{8}$$

Proof: Suppose x and y are feasible for the primal and dual respectively. From $Ax \leqq q$ and $y \geqq 0$, we have $yAx \leqslant y \cdot q$ with equality if and only if (7) holds. Similarly, (8) holds if and only if $yAx = x \cdot p$. The conclusion now follows from the Optimality Criterion 28.1.1, which says that x and y are optimal if and only if $p \cdot x = q \cdot y = yAx$. ∎

The Optimality Criterion and the Complementary Slackness Theorem are vacuous unless both programs have feasible points. We shall show below that if $\bar{x}$ is optimal for the primal, then the dual has an optimal solution $\bar{y}$ and that $p \cdot \bar{x} = q \cdot \bar{y}$ (instead of $p \cdot \bar{x} < q \cdot \bar{y}$).

28.2 The primal is the dual of the dual

The dual program can be rewritten as the following maximum LP in standard inequality form:

$$\underset{y}{\text{maximize}} \ -q \cdot y$$

subject to

$$\begin{aligned}(-A)'y &\leqq -p \\ y &\geqq 0,\end{aligned}$$

where A' is the transpose of A. The dual of this program is:

$$\underset{x}{\text{minimize}} \ -p \cdot x$$

subject to

$$\begin{aligned}-Ax &\geqq -q \\ x &\geqq 0,\end{aligned}$$

or

$$\underset{x}{\text{maximize}} \ p \cdot x$$

subject to

$$\begin{aligned}xA &\leqq p \\ x &\geqq 0,\end{aligned}$$

which is the primal program.

28.3 Lagrangeans for linear programs

There is another important relationship between the primal and the dual—they have the same Lagrangeans.

Let us start by examining the Lagrangean for the primal program. Let A_i denote the i^{th} row of A and rewrite (2) as

$$q_i - A_i \cdot x \geqslant 0, \quad i = 1, \dots, m,$$

and let y_i denote the Lagrange multiplier for this constraint. The constraint functions are affine and so concave. Incorporate (3) by setting the domain $X = \boldsymbol{R}^{\mathrm{n}}_{+}$. The **Lagrangean** can then be written as

$$L(x; y) = p \cdot x + \sum_{i=1}^{m} y_i(q_i - A_i \cdot x) = p \cdot x + q \cdot y - yAx. \tag{9}$$

Here when I write yAx, y is treated as a row matrix and x is treated as a column matrix.

For the dual problem, let us write (5) as

$$p_j - A^j \cdot y \geqslant 0, \quad j = 1, \dots, m,$$

where A^j is the j^{th} column of A (or the j^{th} row of A'). Since the constraint function $p_j - A^j \cdot y$ is affine in y, it is convex. Thus we have the situation discussed in Section 10.4, so we want to write the Lagrangean with a minus sign in front of the Lagrange multipliers. Letting x_j denote the Lagrange multiplier on the j^{th} constraint, the Lagrangean for the dual (as a convex minimization problem) is

$$L'(y; x) = q \cdot y - \sum_{j=1}^{n} x_j (A^j \cdot y - p_j) = q \cdot y + p \cdot x - yAx, \tag{10}$$

which is the same expression as the Lagrangean for the primal. So $(x; y)$ is a saddlepoint of the primal Lagrangean L if and only if $(y; x)$ is a reverse saddlepoint of dual Lagrangean L'.

28.4 The saddlepoint theorem for linear programs

By the easy half of the Saddlepoint Theorem 10.3.2, if $(\bar{x}, \bar{y})$ is a saddlepoint of $L(x, y) = p \cdot x + q \cdot y - xAy$ over $\boldsymbol{R}^{\mathrm{n}}_{+} \times \boldsymbol{R}^{\mathrm{m}}_{+}$, then $\bar{x}$ is optimal for the primal program and $\bar{y}$ is optimal (minimal) for the dual program. In particular, if there is a saddlepoint, then both programs are feasible. If we knew that both programs satisfied Slater's Condition, then the Saddlepoint Theorem 10.3.6 would assert that any pair of optimal solutions would be a saddlepoint of the Lagrangean. *Remarkably, for the linear programming case, we do not need Slater's Condition—all we need is that both programs are feasible.*

28.4.1 Fundamental Duality Theorem of LP *If both a maximum linear program in standard inequality form and its dual are feasible, then both have optimal solutions, and the values of the two programs are the same.*

If one of the programs is infeasible, neither has an optimum.

Proof: (Gale [2]) The proof of the first part proceeds like this. The existence of an optimum for both the primal and the dual can be formulated as the existence of a solution to a set of linear inequalities. By Farkas's Lemma, this is equivalent to the nonexistence of a solution to the alternative. The fact that both the primal and dual programs are feasible guarantees that the alternative has no solution.

So for the first part, assume both the primal and the dual are feasible.

- In light of the Optimality Criterion 28.1.1, if x and y are feasible for the primal and dual respectively, then $p \cdot x \leqslant q \cdot y$, so if $p \cdot x \geqslant q \cdot y$, we must have $p \cdot x = q \cdot y$.

Thus both the primal and the dual have optimal solutions if and only if there is a solution $(x, y) \geqq 0$ to the inequalities

$$\begin{aligned} Ax &\leqq q \\ -A'y &\leqq -p \\ y \cdot q - x \cdot p &\leqslant 0, \end{aligned}$$

or, in matrix form

$$\begin{bmatrix} A & 0 \\ 0 & -A' \\ -p & q \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \leqq \begin{bmatrix} q \\ -p \\ 0 \end{bmatrix}. \tag{11}$$

- Either these inequalities have a solution, or else by Farkas's Lemma 25.3.1, there is a nonnegative vector $\begin{bmatrix} u \ v \ \alpha \end{bmatrix} \geqq 0$, where $u \in \boldsymbol{R}^{\mathrm{m}}_{+}$, $v \in \boldsymbol{R}^{\mathrm{n}}_{+}$, and $\alpha \in \boldsymbol{R}_{+}$, satisfying

$$\begin{bmatrix} u & v & \alpha \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & -A' \\ -p & q \end{bmatrix} \geqq \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \tag{12}$$

and

$$\begin{bmatrix} u & v & \alpha \end{bmatrix} \begin{bmatrix} q \\ -p \\ 0 \end{bmatrix} < 0. \tag{13}$$

Note that (12) and (13) are *homogeneous*. That is, if $\begin{bmatrix} u \ v \ \alpha \end{bmatrix}$ is a solution, then so is $\begin{bmatrix} \lambda u \ \lambda v \ \lambda\alpha \end{bmatrix}$ for any $\lambda > 0$. Thus, if a solution exists, we can find a solution where either Case 1: $\alpha = 0$, or Case 2: $\alpha = 1$.

- We shall show that this latter set of inequalities does not have a solution: Suppose by way of contradiction that (12) and (13) have a nonnegative solution. Rewriting (12), we have

$$aA \geqq \alpha p \tag{14}$$

and

$$Av \leqq \alpha q, \tag{15}$$

while (13) becomes

$$u \cdot q < v \cdot p. \tag{16}$$

- We have assumed that each program is feasible, so let $\bar{x} \geqq 0$ be some feasible solution for the primal, that is, $A\bar{x} \leqq q$. Then premultiplying each side by u gives

$$uA\bar{x} \leqslant u \cdot q \tag{17}$$

since $u \geqq 0$. Similarly let $\bar{y} \geqq 0$ be feasible for the dual, that is, $A'\bar{y} \geqq p$. Then premultiplying each side by v gives $vA'\bar{y} \geqslant v \cdot p$ since $v \geqq 0$, so rearranging gives

$$\bar{y}Av \geqslant v \cdot p. \tag{18}$$

- Case 1: $\alpha = 0$. Then (14) becomes $uA \geqq 0$, which implies

$$uA\bar{x} \geqslant 0,$$

since $\bar{x} \geqq 0$. Also premultiplying (15) by $\bar{y}$ implies

$$\bar{y}Av \leqslant 0,$$

since $\bar{y} \geqq 0$. Combining this with (17) and (18) yields

$$u \cdot q \geqslant uA\bar{x} \geqslant 0 \geqslant \bar{y}Av \geqslant v \cdot p,$$

which contradicts (16).

- Case 2: $\alpha = 1$. In this case, (14) becomes $uA \geqq p$ and (15) becomes $Av \leqq q$, which imply that v is feasible for the primal program and u is feasible for the dual. Therefore, by Lemma 28.1.1, $q \cdot u \geqslant p \cdot v$, which again contradicts (16).

- Since both Case 1 and Case 2 lead to contradictions, we have shown that if both programs are feasible, then both have optimal solutions and both programs have the same value.

- For the second part of the theorem, we have to show that if one program is infeasible, then neither has an optimum. It could be that both programs are infeasible, so certainly neither has an optimal solution. The case of interest is that one program is infeasible, while the other is feasible. In this case we shall show that the objective in the feasible program is unbounded.

- For concreteness, suppose that the primal program is infeasible, but the dual is feasible. Since the primal is assumed infeasible, the system $Ax \leqq q$ has no nonnegative solution, so again by Farkas' Lemma 25.3.1, there is a

$$\tilde{y} \geqq 0 \quad \text{satisfying} \quad \begin{array}{c} \tilde{y}A = A'\tilde{y} \geqq 0 \\ \text{and} \\ q \cdot \tilde{y} < 0. \end{array}$$

Now let $z \geqq 0$ be any feasible solution to the dual, that is, $A'z \geqq p$ and let $\lambda > 0$. Then $(z + \lambda\tilde{y}) \geqq 0$ and

$$A'(z + \lambda\tilde{y}) = \underbrace{A'z}_{\geqq p} + \lambda \underbrace{A'\tilde{y}}_{\geqq 0} \geqq p,$$

so $z + \lambda\tilde{y}$ is feasible for the dual. But

$$q \cdot (z + \lambda\tilde{y}) = q \cdot z + \lambda \underbrace{q \cdot \tilde{y}}_{<0} \to -\infty \quad \text{as} \quad \lambda \to \infty.$$

That is, $\tilde{y}$ is a direction that is feasible and in which the objective function is unbounded below. Therefore no optimal (minimizing) solution exists for the dual.

A similar argument works if the dual is infeasible, but the primal is feasible. ∎

We are now in a position to state these results in the form of a saddlepoint theorem.

28.4.2 Saddlepoint Theorem for Linear Programming *The following are equivalent.*

1. *The Lagrangean*
$$L(x, y) = p \cdot x + q \cdot y - xAy$$
has a saddlepoint over $\boldsymbol{R}^{\mathrm{n}}_{+} \times \boldsymbol{R}^{\mathrm{m}}_{+}$.
2. *The primal has an optimal solution.*
3. *The dual has an optimal solution.*
4. *Both the primal and dual are feasible.*

Also, the following are equivalent.

a. $(\bar{x}, \bar{y})$ *is a saddlepoint of the Lagrangean.*
b. $\bar{x}$ *is optimal for the primal and* $\bar{y}$ *is optimal for the dual.*
c. $\bar{x}$ *is feasible for the primal program and* $\bar{y}$ *is feasible for the dual, and* $p \cdot \bar{x} = q \cdot \bar{y}$.

Proof:

- By the easy half of the Saddlepoint Theorem 10.3.2, if $(\bar{x}, \bar{y})$ is a saddlepoint of the Lagrangean, then $\bar{x}$ is optimal for the primal and $\bar{y}$ is optimal for the dual. Thus (1) $\implies$ (2) & (3) and $(a) \implies (b)$.
- By the Optimality Criterion 28.1.1 if $\bar{x}$ is optimal for the primal and $\bar{y}$ is optimal for the dual, then $(\bar{x}, \bar{y})$ is a saddlepoint of the Lagrangean. It follows that $(b) \implies (c) \implies (a)$ and (2) & (3) $\implies$ (1).
- Clearly (2) & (3) $\implies$ (4).
- The Fundamental Duality Theorem 28.4.1 implies (2) $\iff$ (3) and that (4) $\implies$ (2) & (3).

This finishes the proof of the Saddlepoint Theorem for Linear Programing. ∎

You may suspect that it is possible to combine linear constraints with more general concave constraints that satisfy Slater's Condition. This is indeed the case as Uzawa [4] has shown. (See also Moore [3].)

28.5 Other formulations

Not every linear program comes to us already in standard inequality form, nor is the inequality form always the easiest to work with. There are other forms, some of which have names, and all of which can be translated into one another. In fact, we just translated a standard minimum inequality form into a standard maximum inequality form above. Each of these forms also has a dual, and the program and its dual satisfy the Fundamental Duality Theorem of LP 28.4.1. That is, if both a linear program (in any form) and its dual are feasible, then both have optimal solutions, and the values of the two programs are the same. If one of the programs is infeasible, the other has no optimal solution. Table 28.6.1 summarizes these forms and their dual programs. The characterization of the general form implies the other results.

28.5.1 The general form

Let us start with a linear program in **general maximum form**, which allows for both linear inequalities and equations, and optional sign constraints on the components of x. We partition the set $V = \{1, \dots, n\}$ of indices for the variables, and the set $C = \{1, \dots, m\}$ of indices for the constraints.

$$\underset{x}{\text{maximize}}\ \ p \cdot x = \sum_{j=1}^{n} p_j x_j$$

subject to

$$\begin{aligned} A_i \cdot x &\leqslant q_i & i &\in C_L \\ A_i \cdot x &= q_i & i &\in C_E \\ A_i \cdot x &\geqslant q_i & i &\in C_G \\ x_j &\leqslant 0 & j &\in V_N \\ x_j &\ \ \text{free} & j &\in V_F \\ x_j &\geqslant 0 & j &\in V_P, \end{aligned}$$

so $V_N \cup V_F \cup V_P = V$, and $C_L \cup C_E \cup C_P = C$.

We can translate this into standard inequality maximum form as follows. Start by rewriting all the constraints as $\leqslant$ inequalities,

$$\begin{aligned} A_i \cdot x &\leqslant q_i & i &\in C_L \\ A_i \cdot x &\leqslant q_i & i &\in C_E \\ -(A_i) \cdot x &\leqslant -q_i & i &\in C_E \\ -(A_i) \cdot x &\leqslant -q_i & i &\in C_G. \end{aligned}$$

Next replace x by $u - v$, where both $u \geqq 0$ and $v \geqq 0$. This places no sign restrictions on the components of x. To capture the requirement that $x_j \leqslant 0$ for $j \in V_N$, we require $u \cdot e^j \leqslant 0$, where e^j is the j^{th} unit coordinate vector in $\boldsymbol{R}^{\mathrm{n}}$. (Do you see why this works?) Similarly $x_j \geqslant 0$ corresponds to $v \cdot e^j \leqslant 0$. Thus our rewritten problem is

$$\underset{u,v}{\text{maximize}}\ p \cdot (u - v)$$

subject to

$$\begin{aligned}
A_i \cdot (u - v) &\leqslant q_i & i &\in C_L \\
A_i \cdot (u - v) &\leqslant q_i & i &\in C_E \\
-(A_i) \cdot (u - v) &\leqslant -q_i & i &\in C_E \\
-(A_i) \cdot (u - v) &\leqslant -q_i & i &\in C_G \\
u \cdot e^j &\leqslant \ \ 0 & j &\in V_N \\
v \cdot e^i &\leqslant \ \ 0 & j &\in V_P \\
u &\geqq 0 \\
v &\geqq 0.
\end{aligned}$$

Or in matrix form

$$\underset{u,v}{\text{maximize}}\ (p, -p) \cdot (u, v)$$

subject to

$$\begin{bmatrix} A_L & -A_L \\ A_E & -A_E \\ -A_E & A_E \\ -A_G & A_G \\ I_N & 0_N \\ 0_P & I_P \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \leqq \begin{bmatrix} q_L \\ q_E \\ -q_E \\ -q_G \\ 0_N \\ 0_P \end{bmatrix},$$

where A_L, A_E, and A_G are matrices with n columns and whose rows are the rows A_i of A for $i \in C_L$, $i \in C_E$, and $i \in C_G$ respectively; the rows of I^N and I^P are unit coordinate vectors e^j i $\boldsymbol{R}^{\mathrm{n}}$ for $j \in V_N$ and $j \in V_P$ respectively; and q_L, q_E, and q_P have components q_i for $i \in C_L$, $i \in C_E$, and $i \in C_G$ respectively. The zeros are of the dimension they need to be.

The dual of this maximum problem in standard inequality form is thus the following:

$$\underset{z,\,w}{\text{minimize}}\ (q_L,\, q_E,\, -q_E,\, -q_G,\, 0_N,\, 0_P) \cdot (z_L,\, z_E,\, w_E,\, z_G,\, z_N,\, z_P)$$

subject to $z \geqq 0$, $w \geqq 0$, and

$$\begin{bmatrix} A'_L & A'_E & -A'_E & -A'_G & I'_N & 0'_P \\ -A'_L & -A'_E & A'_E & A'_G & 0'_N & I'_P \end{bmatrix} \begin{bmatrix} z_L \\ z_E \\ w_E \\ z_G \\ z_N \\ z_P \end{bmatrix} \geqq \begin{bmatrix} p \\ -p \end{bmatrix},$$

where z_L, z_E, w_E, z_G, z_N, z_P are of the appropriate dimensions.

Define $y \in \boldsymbol{R}^{\mathrm{m}}$ by

$$y_i = \begin{cases} z_i & i \in C_L \\ z_i - w_i & i \in C_E \\ -z_i & i \in C_G \end{cases}$$

so that

$$\begin{aligned} &y_i \geqslant 0 && i \in C_L \\ &y_i \text{ unsigned} && i \in C_E \\ &y_i \leqslant 0 && i \in C_G \end{aligned}$$

and rewrite the dual as

$$\underset{y}{\text{minimize}} \;\; q \cdot y$$

subject to $z \geqq 0$, $w \geqq 0$, and

$$\begin{bmatrix} A \\ -A \end{bmatrix} y + \begin{bmatrix} I^N & 0 \\ 0 & I^P \end{bmatrix} \begin{bmatrix} z_N \\ z_P \end{bmatrix} \geqq \begin{bmatrix} p \\ -p \end{bmatrix},$$

where A is the $m \times n$ matrix of columns $A^1, \dots, A^n$. What this says is

$$\begin{aligned} Ay + \sum_{j \in N} z_j e^j &\geqq p \\ Ay - \sum_{j \in P} z_j e^j &\leqq p \end{aligned}$$

Since $z \geqq 0$, the dual can be written:

$$\underset{y}{\text{minimize}}\ q \cdot y$$

subject to

$$\begin{aligned} A^j y &\geqslant p_j && j \in V_N \\ A^j y &= p_j && j \in V_F \\ A^j y &\leqslant p_j && j \in V_P \\ y_i &\geqslant 0 && i \in C_L \\ y_i &\ \ \text{free} && i \in C_E \\ y_i &\leqslant 0 && i \in C_G \end{aligned}$$

Recall that the variables in the dual are the Lagrange multipliers for the primal. Thus we see that, the Lagrange multipliers associated with the equality constraints ($j \in C_E$) are not a priori restricted in sign, while the multipliers for the $\leqslant$-inequality constraints ($i \in C_L$) are nonnegative, and the multipliers for the $\geqslant$-inequality constraints ($i \in C_G$) are nonpositive. Since the primal variables are the Lagrange multipliers for the dual program, the nonnegativity constraints ($j \in V_P$) on the primal correspond to $\leqslant$-inequality constraints in the dual, the nonpositivity constraints ($j \in V_N$) on the primal correspond to $\geqslant$-inequality constraints in the dual, and the unrestricted primal variable are associated with equality constraints in the dual.

28.5.2 Canonical (equality) form

There is one more useful form for linear programs, the **canonical** or **equality form**.[2] In it, all the constraints are equations, and all the variables are nonnegative. An LP is in **canonical maximum form** if it is written as:

$$\underset{x}{\text{maximize}}\ p \cdot x$$

subject to

$$\begin{aligned} Ax &= q \\ x &\geqq 0 \end{aligned}$$

To transform an inequality form into the equality form, introduce slack variables $z \in \boldsymbol{R}^{\mathrm{m}}$ and observe that

$$Ax \leqq q \quad \iff \quad Ax + z = q,\ z \geqq 0.$$

[2] The equality form is what Gale [2, p. 75] calls the canonical form, while Dantzig [1, p. 60] calls this the standard form.

Elaborate this.

It follows from the characterization of the dual to the general maximum problem that the dual program can be written as the decidedly non-equality minimum problem

$$\boxed{\begin{array}{l} \qquad\qquad \underset{y}{\text{minimize}}\ q \cdot y \\ \text{subject to} \\ \qquad\qquad A'y \geqq p \end{array}}$$

Define the canonical minimum problem.

Note the lack of sign restrictions on y.

28.5.3 The “altitude” form

Here is an interesting curiosity. We can convert any LP problem to the problem of finding the “highest point” in a constraint set. For instance, consider the standard maximum LP:

$$\underset{x}{\text{maximize}}\ p \cdot x$$

subject to

$$\begin{aligned} Ax &\leqq q \\ x &\geqq 0 \end{aligned}$$

where x and p belong to $\boldsymbol{R}^{\mathrm{n}}$, q belongs to $\boldsymbol{R}^{\mathrm{m}}$, and A is $m \times n$. Introduce a new variable ζ (altitude) and consider the transformed problem

$$\underset{x,\zeta}{\text{maximize}}\ \zeta$$

subject to

$$\begin{aligned} Ax &\leqq q \\ \zeta - p \cdot x &\leqslant 0 \\ x &\geqq 0. \end{aligned}$$

In the transformed problem, the added the constraint is that ζ cannot exceed $p \cdot x$, so in order to maximize ζ, we have to maximize $p \cdot x$, which is the original problem.

28.5.1 Exercise Show that the dual to the altitude form is the same as the dual to the standard maximum problem, namely:

$$\underset{y}{\text{minimize}}\ q \cdot y$$

subject to

$$\begin{aligned} A'y &\geqq p \\ y &\geqq 0. \end{aligned}$$

□

Sample answer: The find the dual of the altitude form, rewrite the primal as

$$\underset{x,\zeta}{\text{maximize}} \begin{bmatrix} 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x & \zeta \end{bmatrix}$$

subject to

$$\begin{bmatrix} A & 0 \\ -p & 1 \end{bmatrix} \begin{bmatrix} x & \zeta \end{bmatrix} \leqq \begin{bmatrix} q \\ 0 \end{bmatrix}$$
$$x \geqq 0.$$

The dual has an extra variable, call it η and can be written as

$$\underset{y,\eta}{\text{minimize}} \begin{bmatrix} q & 0 \end{bmatrix} \cdot \begin{bmatrix} y & \eta \end{bmatrix} = q \cdot y$$

subject to

$$\begin{bmatrix} A' & -p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ \eta \end{bmatrix} \geqq \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$y \geqq 0.$$

These constraints reduce to

$$\begin{aligned} A'y &\geqq \eta p \\ \eta &\geqslant 1 \\ y &\geqq 0. \end{aligned}$$

The only rôle of η is to add restrictions on y, so y is least restricted when $\eta = 1$, and the dual problem reduces to

$$\underset{y}{\text{minimize}} \; q \cdot y \tag{19}$$

subject to

$$A'y \geqq p \tag{20}$$
$$y \geqq 0. \tag{21}$$

■

28.6 Basic optimal solutions

Tie this in with Dantzig, pages 154–156

In the canonical maximization problem, we are looking to write q as a linear combination of the columns of A. In discussing algorithms for solving linear programs it is useful to be able to restrict attention to solutions where q is a linear combination of a linearly independent set of columns. That is, we want to make sure that if an optimum exists, then there is an optimum $\bar{x}$ where $\{A^j : \bar{x}_j > 0\}$ is a linearly independent set, where A^j denotes the j^{th} column of A. We shall call such a $\bar{x}$ a **basic solution** to the linear program. To prove the existence of a basic solution we must first derive and prove the proper version of the Complementary Slackness Theorem for canonical programs.

28.6.1 Theorem (Complementary Slackness for Canonical Form) *Let x and y be feasible for the canonical maximum problem and its dual. That is, $x \geqq 0$, $Ax = q$, and $A'y \geqq p$. Then x and y are optimal if and only for $i = 1, \dots, m$,*

$$A^j \cdot y > p_j \implies x_j = 0.$$

Proof: Let $x \geqq 0$, $Ax = q$, and $A'y \geqq p$.

($\implies$) Assume that x and y are optimal and that $A^j \cdot y > p_j$. Since x and y are optimal, we know by the Fundamental Duality Theorem that $p \cdot x = q \cdot y$. Since $x \geqq 0$ and $A'y \geqq p$, we have that

$$x \cdot A'y \geqslant x \cdot p = q \cdot y.$$

On the other hand $Ax = q$, so

$$q \cdot y = Ax \cdot y = yAx = \sum_{j=1}^{n} (A^j \cdot y)x_j.$$

Combining these we get

$$yAx = p \cdot x \quad \text{or} \quad \sum_{j=1}^{m} (A^j \cdot y - p_j)x_j = 0.$$

Since each $x_j \geqslant 0$ and each $A^j \cdot y - p_j \geqslant 0$, the only way the sum can be zero is that each product term $(A^j \cdot y - p_j)x_j$ is zero. So if $A^j \cdot y - p_j > 0$, then $x_j = 0$.

($\impliedby$) Assume that $x_j = 0$ whenever $A^j \cdot y > p_j$. Then since $A'y \geqslant p$ and $x \geqq 0$, we must have $xA'y = yAx = p \cdot x$. But $Ax = q$ so $q \cdot y = Ax \cdot y = p \cdot y$, which implies that$x$ and y are both optimal. ∎

28.6.2 Theorem *If a canonical (equality constraint) linear programming problem has an optimal solution, then it has a basic optimal solution.*

Proof: Let x and y be an optimal solution to the primal and the dual, so that $Ax = q$ and $A'y \geqq p$, and let $B = \{j : x_j > 0\}$. Then by the contrapositive of Complementary Slackness 28.6.1, $A^j \cdot y = p_j$ for all $j \in B$.

By Lemma 2.3.3 on nonnegative basic solutions, since $Ax = q$, there exists $\tilde{x}$ satisfying $A\tilde{x} = q$ such that $\tilde{B} = \{j : \tilde{x}_j > 0\} \subset B$ and $\{A^j : \tilde{x}_j > 0\}$ is linearly independent. Now for all $j \in \tilde{B} \subset B$ we also have $A^j \cdot y = p_j$. That is, $A^j \cdot y > p_j$ implies $\tilde{x}_j = 0$, so by Complementary Slackness 28.6.1 again, $\tilde{x}$ is also optimal. ∎

28.7 Basic feasible solutions are vertices

The next result has implications for computing optimal solutions via the Simplex Method as discussed in Section 29.7 below, but it is interesting in its own right. Recall (Definition 2.6.1) that **vertex** of a polyhedron or polytope is an extreme point, that is, it is not a proper convex combination of two other points in the set. The next Proposition may be found, for instance, in Dantzig [1, Theorem 3, pp.154–155].

28.7.1 Proposition *Consider the polyhedron*

$$C = \{x : Ax = q,\ x \geqq 0\}.$$

Then $\bar{x}$ is a basic nonnegative solution of $Ax = q$ if and only if it is a vertex of C.

Proof: ($\Longrightarrow$) Assume that $\bar{x}$ is a basic nonnegative solution of $Ax = q$. Then $\{A^j : x_j > 0\}$ is linearly independent. By rearranging the columns of A we may assume without loss of generality that for some k, $\bar{x} = (\bar{x}_1, \dots, \bar{x}_k, 0, \dots, 0)$, and that $A^1, \dots, A^k$ are independent, and

$$q = \sum_{j=1}^{k} x_j A^j.$$

Now let $\bar{x} = (1-\lambda)y + \lambda z$ with $0 < \lambda < 1$, and $y, z \in C$. Since $y, z \geqq 0$, and

Check the case $k = 0$.

$$0 = \bar{x}_j = (1-\lambda)\bar{y}_j + \lambda \bar{z}_j, \quad j > k,$$

we have

$$\bar{x}_j = y_j = z_j = 0, \quad j > k.$$

Since $y, z \in C$ we must have $Ay = Az = q$, or

$$q = \sum_{j=1}^{k} y_j A^j = \sum_{j=1}^{k} z_j A^j.$$

So by independence

$$\bar{x}_j = y_j = z_j, \quad j = 1, \dots, k.$$

Primal program		**Dual program**
General maximum form		General minimum form
maximize$_x$ $p \cdot x$ subject to $A_i \cdot x \leqslant q_i$ $i \in C_L$; $A_i \cdot x = q_i$ $i \in C_E$; $A_i \cdot x \geqslant q_i$ $i \in C_G$; $x_j \leqslant 0$ $j \in V_N$; x_j free $j \in V_F$; $x_j \geqslant 0$ $j \in V_P$,		minimize$_y$ $q \cdot y$ subject to $A^j y \geqslant p_j$ $j \in V_N$; $A^j y = p_j$ $j \in V_F$; $A^j y \leqslant p_j$ $j \in V_P$; $y_i \geqslant 0$ $i \in C_L$; y_i free $i \in C_E$; $y_i \leqslant 0$ $i \in C_G$
Standard maximum form		Standard minimum form
maximize$_x$ $p \cdot x$ subject to $Ax \geqq q$; $x \geqq 0$		minimize$_y$ $q \cdot y$ subject to $A'y \geqq p$; $y \geqq 0$
Canonical maximum form		
maximize$_x$ $p \cdot x$ subject to $Ax = q$; $x \geqq 0$		minimize$_y$ $q \cdot y$ subject to $A'y \geqq p$; y free
Canonical minimum form		
minimize$_x$ $p \cdot x$ subject to $Ax = q$; $x \geqq 0$		maximize$_y$ $q \cdot y$ subject to $A'y \leqq p$; y free

Table 28.6.1. Selected forms of linear programs and their duals.

Thus

$$y = z = \bar{x},$$

so $\bar{x}$ is an extreme point of C, that is, a vertex.

($\Longleftarrow$) Assume that $\bar{x}$ is an extreme point of C. We wish to show that it is a basic nonnegative solution $Ax = q$. So assume by way of contradiction that $\{A^j : \bar{x}_j > 0\}$ is dependent. Again without loss of generality we may assume $\bar{x} = (\bar{x}_1, \dots, \bar{x}_k, 0, \dots, 0)$ and that there are numbers $\alpha_1, \dots, \alpha_k$, not all zero such that

$$\sum_{j=1}^{k} \alpha_j A^j = 0.$$

Thus for every γ,

$$q = \sum_{j=1}^{k} (\bar{x}_j - \gamma\alpha_j) A^j.$$

Let $a = (\alpha_1, \dots, \alpha_k, 0, \dots, 0)$. Then for for every γ we have

$$A(\bar{x} + \gamma a) = q,$$

and $\gamma > 0$ small enough ($0 \leqslant \gamma \leqslant \min\{\bar{x}_j/\alpha_j : j \leqslant k \text{ and } \alpha_j \neq 0\}$) we have $x \pm \gamma a \geqq 0$. Then

$$\bar{x} = \tfrac{1}{2}(\bar{x} + \gamma a) + \tfrac{1}{2}(\bar{x} - \gamma a)$$

is a proper convex combination of distinct elements of C. This contradicts the hypothesis that x is an extreme point, and the conclusion is proved. ∎

28.8 Linear equations as LPs

It is possible to recast the problem of solving linear equations and inequalities as LP problems. Consider the problem of finding a nonnegative solution to a system of equations. That is, find x such that

$$Ax = q$$

$$x \geqq 0.$$

Consider the linear program in equality minimum form:

$$\underset{x,z}{\text{minimize}} \ \mathbf{1} \cdot z$$

subject to

$$\begin{aligned} Ax + z &= q \\ x, z &\geqq 0 \end{aligned}$$

Here $\mathbf{1}$ is the vector whose components are all 1. Without loss of generality we may assume $q \geqq 0$, for if $q_i < 0$ we may multiply AI_i and q_i by -1 without affecting the solution set. Then note that this program is feasible, since $x = 0$, $z = q$ is a nonnegative feasible solution. Since we require $z \geqq 0$, we have $\mathbf{1} \cdot z \geqslant 0$ and $\mathbf{1} \cdot z = 0$ if and only if $z = 0$, in which case $Ax = q$. Thus, if this linear program has value 0 if and only $Ax = q$, $x \geqq 0$ has a solution, and any optimal $(x, 0)$ provides a nonnegative solution to the equation.

At this point you might be inclined to say "so what?" In the next lecture, I describe the simplex algorithm, which is a special version of Gauss–Jordan elimination, that is a reasonably efficient and easily programmable method for solving linear programs. In other words, it also finds nonnegative solutions to linear equations when they exist.

References

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