Topic 27: The uses of alternatives in economic theory

In this section we will work with matrix equations where it is convenient to index the rows and columns by various finite sets, and not just natural numbers. You can cope. We will also make use of the **Kronecker delta**,

$$\delta_{a,b} = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{if } a \neq b. \end{cases}$$

27.1 Revealed preference and utility maximization

Here is an abstract formal model of choice behavior. There is a nonempty finite set $X = \{x_1, \dots, x_m\}$ of objects. We have a finite set of observations of a subject making choices from various subsets of X. That is, we have a list $\mathcal{B} = (B_1, \dots, B_n)$, where each B_i is nonempty subset of X, and for each B_i we observe that subject chose some $x \in B_i$. We denote this choice by writing $x = c(B_i)$. The function $c\colon \mathcal{B} \to X$ is called a **choice function** We want to know under what conditions we can guarantee that there is a **utility function** $u\colon X \to \boldsymbol{R}$ that rationalizes the choice function c in the sense that for each $i = 1, \dots, n$,

$$x = c(B_i) \implies (\forall y \in B_i)\ [x \neq y \implies u(x) > u(y)]. \tag{1}$$

Note that the way I have formulated this problem is not the most general rationalization framework. I have required that for each observation the subject chooses exactly one object, but I do allow for $B_i = B_j$, that is, we have more than one observation with the same set. Under my notion of rationalization, if the choice were made by maximizing utility over the set, the choice would have to be the same. I essentially do not allow for indifference. You might want to allow for indifference, but it is still an interesting question as to whether we can rationalize the choice without it.

The first step is to define the **strict revealed preference relation** S by

$$x \mathrel{S} y \text{ if there exists some } B_i \text{ with } x, y \in B_i,\ x \neq y,\ \&\ x = c(B_i).$$

That is, $x \mathrel{S} y$ if x is observed to be chosen from some set contains y and $y \neq x$. Knowing the relation S is not the same as knowing c, it contains less information, but nonetheless S determines whether c is rational in the sense of (1).

We say that the observations satisfy the **Strong Axiom of Revealed Preference** if the revealed preference relation S has no cycles.

27.1.1 Theorem *The observations are rational in the sense of* (1) *if and only if they satisfy the Strong Axiom of Revealed Preference.*

Sketch of proof: We start by constructing a matrix A with columns indexed by X and rows indexed by the relation S, viewed as a set of ordered pairs. That is the rows of A are indexed by $\{(x, y) : x \; S \; y\}$. In row (x, y) put 1 in column x and -1 in column y with a zero in every other column. Then (1) is equivalent to the existence of a vector $u \in \boldsymbol{R}^X$ satisfying

$$Au \gg 0.$$

By Gordan's Alternative 25.3.9 the alternative to this is that there exist a $p \in \boldsymbol{R}^S$ such that

$$pA = 0, \quad p > 0.$$

We now show that this alternative is equivalent to violating the Strong Axiom.

Since $p > 0$, there is some row r_1 of A with $p_{r_1} > 0$. Let this row be indexed by a pair (x_1, x_2) that is, $x_1 \; S \; x_2$. So the row r_1 has 1 in column x_1 and -1 in column x_2. Since $p \cdot A^{x_2} = 0$, there must be some other row r_2 with $p_{r_2} > 0$ and the row r_2, column x_2 entry must be an offsetting 1. That means that r_2 must be an ordered pair (x_2, x_3) with $x_2 \; S \; x_3$. Then the row r_2, column x_3 entry is -1, $p_{r_2} > 0$, and $p \cdot A^{x_3} = 0$. Thus there is a row r_3 where the row r_3, column x_3 entry is 1. This row's ordered pair is this of the form $x_3 \; S \; x_4$. Since X is finite, we eventually repeat some x, which by renumbering if need be, forms a cycle with

$$x_1 \; S \; x_2 \; S \cdots S \; x_k \; S \; x_1,$$

which violates the Strong Axiom. ∎

27.2 Subjective probability

The main references here are Scott [29] and Kraft, Pratt, and Seidenberg [21].

The modern approach to uncertainty, as formalized by Kolmogorov, has as its fundamentals:

S, a set of **states of the world**.
$\mathcal{E}$, a collection of **events**.
p, a **probability** on $\mathcal{E}$.

The **states** are assumed to be exhaustive and mutually exclusive. What you choose as the set of states is a modeling decision. *For the purpose of these notes S is assumed to be finite.*

The collection $\mathcal{E}$ of **events** is usually assumed to be an **algebra** of subsets of S. That is, $\mathcal{E}$ satisfies:

i. $S \in \mathcal{E}$, $\varnothing \in \mathcal{E}$.

ii. If $E \in \mathcal{E}$, then $E^c \in \mathcal{E}$.

iii. If $E, F \in \mathcal{E}$, then $E \cap F \in \mathcal{E}$ and $E \cup F \in \mathcal{E}$.

A **probability** p on an algebra $\mathcal{E}$ is a function that satisfies the following properties:

i. For each $E \in \mathcal{E}$,

$$0 \leqslant p(E) \leqslant 1, \quad p(S) = 1, \quad \text{and} \quad p(\varnothing) = 0.$$

ii. If $E \cap F = \varnothing$, then

$$p(E \cup F) = p(E) + p(F).$$

A **probability vector** $p \in \boldsymbol{R}^S$ satisfies

$$p_i \geqslant 0,\ i \in S \qquad \text{and} \qquad \sum_{i \in S} p_i = 1.$$

A probability vector defines a probability p on $\mathcal{E} = 2^S$ via

$$p(E) = \sum_{i \in E} p_i.$$

The subjective relative likelihood of an individual is a binary relation on events (subsets of S). We write

$$E \succcurlyeq F$$

to mean that *event E is at least as likely as event F*. As usual, we write $E \succ F$ to mean $E \succcurlyeq F$ & $F \not\succcurlyeq E$, and $E \sim F$ to mean $E \succcurlyeq F$ & $F \succcurlyeq E$. The **graph** of $\succcurlyeq$ is

$$\operatorname{gr} \succcurlyeq = \{(E, F) : E \succcurlyeq F\}.$$

Let us say that the subjective likelihood relation $\succcurlyeq$ is **represented by a probability measure $\boldsymbol{p}$** if

$$E \succcurlyeq F \iff p(E) \geqslant p(F).$$

Savage [28, p. 32] calls such subjective likelihood relation a **qualitative probability** if it satisfies the following obvious necessary conditions to have a representation by a probability p:

C (Completeness) For all E, F, either $E \succcurlyeq F$ or $F \succcurlyeq E$, or both.

T (Transitivity) For all E, F, G,

$$[E \succcurlyeq F \ \&\ F \succcurlyeq G] \quad \Longrightarrow \quad E \succcurlyeq G.$$

A (Additivity) If $E \cap G = \varnothing$ and $F \cap G = \varnothing$, then

$$E \succcurlyeq F \quad \iff \quad E \cup G \succcurlyeq F \cup G.$$

N (Nontriviality) $S \succ \varnothing$, and for every event E, $E \succcurlyeq \varnothing$.

Bruno de Finetti [12] posed the question of whether these conditions were sufficient to guarantee that $\succcurlyeq$ was representable by a probability. The following example due to Kraft, Pratt, and Seidenberg [21] shows that is not the case. (There is an unfortunate typographical error on page 414 of their paper, but it is corrected later on.)

27.2.1 Example (Qualitative probability not representable) Partially define $\succcurlyeq$ on the finite set $\{a, b, c, d, e\}$ by

$$\begin{aligned} \{a,b,d\} \succ \{c,e\} \succ \{a,b,c\} \succ \{b,e\} \succ \{a,d\} \succ \{a,c\} \succ \{b,c,d\} \succ \{e\} \\ \succ \{c,d\} \succ \{a,b\} \succ \{a\} \succ \{b,d\} \succ \{b,c\} \succ \{d\} \succ \{c\} \succ \{b\} \succ \varnothing \end{aligned} \tag{2}$$

This orders seventeen of the thirty-two subsets. Each of the remaining fifteen subsets is a complement of one of these, so if we assign a probability to each of these sets, the probability of the remainder is determined. The complements must be ordered in the reverse order. That is, we must have

$$\begin{aligned} \{a,b,c,d,e\} \succ \{a,c,d,e\} \succ \{a,b,d,e\} \succ \{a,d,e\} \succ \{a,c,e\} \succ \{b,c,d,e\} \succ \{a,b,e\} \\ \succ \{a,b,c,d\} \succ \{a,e\} \succ \{b,d,e\} \succ \{b,c,e\} \succ \{a,c,d\} \succ \{d,e\} \succ \{a,b,d\} \succ \{c,e\} \end{aligned}$$

This specifies a linear order on all the subsets. Checking additivity is simple, but tedious. K–P–S prove a little lemma to simplify things a bit, but I leave to you to verify that the additivity condition A is satisfied. (Their lemma is that under a linear order, if the bottom half of the order satisfies additivity, and the top half consists of the complements of the bottom half ordered in reverse, then the entire order satisfies additivity.)

Now to show that this order has no probability representation. From (2) we have

$$\{a\} \succ \{b,d\}, \quad \{c,d\} \succ \{a,b\}, \quad \{b,e\} \succ \{a,d\}$$

so a representation p would have to satisfy

$$p(a) > p(b) + p(d), \quad p(c) + p(d) > p(a) + p(b), \quad p(b) + p(e) > p(a) + p(d).$$

Adding these inequalities, we would have to have

$$p(a) + p(b) + p(c) + p(d) + p(e) > 2p(a) + 2p(b) + 2p(d),$$

or

$$p(c) + p(e) > p(a) + p(b) + p(d),$$

which contradicts $\{a,b,d\} \succ \{c,e\}$. Thus no representation exists. □

K–P–S give a necessary and sufficient condition for a likelihood relation (on a finite set) to be representable by a probability, but their condition is expressed in terms of monomials in the letters representing the elements of the set. The next result, due to Dana Scott [29, Theorem 4.1] gives a friendlier set-theoretic statement. I have replaced Scott's condition (4_B) with a similar condition that is perhaps more transparent. I refer to it as Condition **S**, but there should be a better name. The proof is also mine.

27.2.2 Theorem *Let S be a finite set and let $\mathcal{E}$ be an algebra of subsets of S and let $\succcurlyeq$ be a binary relation on $\mathcal{E}$. For $\succcurlyeq$ to be representable by a probability measure p on $\mathcal{E}$, that is,*

$$E \succcurlyeq F \iff p(E) \geqslant p(F),$$

it is necessary and sufficient that $\succcurlyeq$ satisfy the following three conditions:

N (Nontriviality) $S \succ \varnothing$, and for every event E, $E \succcurlyeq \varnothing$.

C (Completeness) For all $E, F \in \mathcal{E}$, either $E \succcurlyeq F$ or $F \succcurlyeq E$, or both. (Or equivalently, for all $E, F \in \mathcal{E}$, exactly one of $E \succ F$, $F \succ E$, or $E \sim F$ holds.)

*S (Condition **S**) For every finite list $(E_1, F_1), \ldots, (E_n, F_n)$ of pairs of events (where repetitions are allowed),*

$$\left[(E_i \succcurlyeq F_i,\ i = 1, \ldots, n) \ \& \ \sum_{i=1}^{n} \mathbf{1}_{E_i} = \sum_{i=1}^{n} \mathbf{1}_{F_i} \right] \implies E_i \sim F_i,\ i = 1, \ldots, n.$$

Proof: ($\implies$) Assume that $\succcurlyeq$ is representable by p. Then it is obvious that Nontriviality and Completeness must be satisfied.

To see that Condition **S** is also necessary, recall that $\mathbf{1}_E$ is the indicator function of E. That is, $\mathbf{1}_E(s) = 1$ if $s \in E$ and $\mathbf{1}_E(s) = 0$ if $s \notin E$. Thus $\sum_{i=1}^{n} \mathbf{1}_{E_i}(s)$ is the count of the events $E_1, \ldots, E_n$ that contain s. Also observe that for any event E,

$$p(E) = \sum_{s \in E} p(s) = \sum_{s \in S} p(s) \mathbf{1}_E(s).$$

Thus for events $E_1, \ldots, E_n$, we have

$$\sum_{i=1}^{n} p(E_i) = \sum_{i=1}^{n} \left(\sum_{s \in S} p(s) \mathbf{1}_{E_i}(s) \right) = \sum_{s \in S} p(s) \left(\sum_{i=1}^{n} \mathbf{1}_{E_i}(s) \right). \tag{3}$$

In other words, the function $\sum_{i=1}^{n} \mathbf{1}_{E_i}$ is a random variable whose expected value is the sum of probabilities $\sum_{i=1}^{n} p(E_i)$.

Let $(E_1, F_1), \ldots, (E_n, F_n)$ be a list of pairs of events satisfying (i) $E_i \succcurlyeq F_i$, $i = 1, \ldots, n$ and (ii) $\sum_{i=1}^{n} \mathbf{1}_{E_i} = \sum_{i=1}^{n} \mathbf{1}_{F_i}$. By (ii) and (3), we have that

$$\sum_{i=1}^{n} p(E_i) = \sum_{i=1}^{n} p(F_i).$$

From (i), we have $p(E_i) \geqslant p(F_i)$ for each i. Therefore we must actually have $p(E_i) = p(F_i)$, or $E_i \sim F_i$, for each i.

($\Longleftarrow$) We prove the converse by proving its contrapositive. That is, we shall show that if $\succcurlyeq$ is not representable, but satisfies Nontriviality and Completeness, then it must violate Condition **S**.

Consider the following system of inequalities, where the rows of the first matrix are indexed by the graph of $\succ$ and rows are of the second matrix are indexed by the graph of $\succcurlyeq$, and the columns are indexed by the states S.

$$
\begin{array}{c}
\begin{array}{cc} & s \\ E \succ F & \begin{bmatrix} & \vdots & \\ \cdots & \mathbf{1}_E(s) - \mathbf{1}_F(s) & \cdots \\ & \vdots & \end{bmatrix} \end{array} \begin{bmatrix} \vdots \\ p(s) \\ \vdots \end{bmatrix} \gg 0 \\
\begin{array}{cc} & s \\ E \succcurlyeq F & \begin{bmatrix} & \vdots & \\ \cdots & \mathbf{1}_E(s) - \mathbf{1}_F(s) & \cdots \\ & \vdots & \end{bmatrix} \end{array} \begin{bmatrix} \vdots \\ p(s) \\ \vdots \end{bmatrix} \geqq 0
\end{array}
\tag{4}
$$

If the system (4) has a solution p, then the row corresponding to $\{s\} \succcurlyeq \varnothing$ implies $p(s) \geqslant 0$. The row corresponding to $S \succ \varnothing$ implies $\sum_{s \in S} p(s) > 0$. We may normalize p so that it is indeed a probability measure on S. Thus $\succcurlyeq$ is representable if and only if (4) has a solution. We now show that if no solution exists, then Condition **S** is violated.

So suppose (4) does not have a solution. Then by Motzkin's Rational Transposition Theorem 25.3.16 there exist integer-valued nonnegative vectors $k^{\succ}$ (indexed by the graph of $\succ$) and $k^{\succcurlyeq}$ (indexed by the graph of $\succcurlyeq$) such that for each column $s \in S$,

$$\sum_{(E,F):E \succ F} k^{\succ}_{(E,F)} \Big(\mathbf{1}_E(s) - \mathbf{1}_F(s)\Big) + \sum_{(E,F):E \succcurlyeq F} k^{\succcurlyeq}_{(E,F)} \Big(\mathbf{1}_E(s) - \mathbf{1}_F(s)\Big) = 0. \tag{5}$$

Moreover, Motzkin's Theorem guarantees that $k^{\succ}$ is nonzero.

Construct a list of pairs by taking $k^{\succ}_{(E,F)}$ copies of (E,F) for each (E,F) with $E \succ F$ and $k^{\succcurlyeq}_{(E,F)}$ copies of (E,F) for (E,F) with $E \succcurlyeq F$, and enumerate it as $(E_1,F_1), \dots, (E_n,F_n)$.

By construction, for each (E_i,F_i), we have $E_i \succcurlyeq F_i$ and by (5) we have

$$\sum_{i=1}^n \mathbf{1}_{E_i} = \sum_{i=1}^n \mathbf{1}_{F_i}.$$

But since $k^{\succ}$ is nonzero, for at least one pair we have $E_i \succ F_i$, which violates Condition **S**.

This completes the proof. ∎

27.2.3 Remark Note that Condition **S** and Completeness imply Transitivity. We proceed by contraposition. Assume Completeness and that Transitivity fails. That is, there are A, B, C with $A \succcurlyeq B$, $B \succcurlyeq C$, and $C \succ A$. Set

$$\begin{aligned} E_1 &= A, & F_1 &= B, \\ E_2 &= B, & F_2 &= C, \\ E_3 &= C, & F_3 &= A. \end{aligned}$$

Then $E_i \succcurlyeq F_i$ for all i and

$$\sum_{i=1}^{3} \mathbf{1}_{E_i} = \sum_{i=1}^{3} \mathbf{1}_{F_i} = \mathbf{1}_A + \mathbf{1}_B + \mathbf{1}_C.$$

But $E_3 \succ F_3$, which violates Condition **S**.

27.2.4 Remark Now let's see that Condition **S** and Completeness imply Additivity. So assume $A \cap C = \varnothing$ and $C \cap C = \varnothing$, then we want to show that

$$A \succcurlyeq B \quad \iff \quad A \cup C \succcurlyeq B \cup C.$$

First assume $A \succcurlyeq B$, and suppose $A \cup C \succcurlyeq B \cup C$ fails. Then $B \cup C \succ A \cup C$. Define

$$\begin{aligned} E_1 &= A, & F_1 &= B, \\ E_2 &= B \cup C, & F_2 &= A \cup C. \end{aligned}$$

Since $A \cap C = \varnothing$ we have that $\mathbf{1}_{A \cup C} = \mathbf{1}_A + \mathbf{1}_C$. Similarly, $\mathbf{1}_{B \cup C} = \mathbf{1}_B + \mathbf{1}_C$. So now observe that

$$\sum_{i=1}^{2} \mathbf{1}_{E_i} = \mathbf{1}_A + \mathbf{1}_B + \mathbf{1}_C = \mathbf{1}_B + \mathbf{1}_A + \mathbf{1}_C = \sum_{i=1}^{2} \mathbf{1}_{F_i}.$$

This violates Condition **S**.

For the converse, assume $A \cup C \succcurlyeq B \cup C$, but that $A \succcurlyeq B$ fails, so that $B \succ C$ and define

$$\begin{aligned} E_1 &= A \cup C, & F_1 &= B \cup C, \\ E_2 &= B, & F_2 &= C. \end{aligned}$$

This violates Condition **S**.

This finishes the proof of Additivity.

27.2.5 Remark We now note that the Kraft–Pratt–Seidenberg example violates Condition **S**. The following list of pairs will do. (These are the same pair we used above to show that the relation was not representable.)

$$\begin{aligned} E_1 &= \{a\}, & F_1 &= \{b, d\}, \\ E_2 &= \{c, d\}, & F_2 &= \{a, b\}. \\ E_3 &= \{b, e\}, & F_3 &= \{a, d\}. \\ E_4 &= \{a, b, d\}, & F_4 &= \{c, e\}. \end{aligned}$$

27.2.6 Remark I mentioned above that what I call Condition **S** is not Condition (4_B) of his Theorem 4.1, [29, p.246]. In the notation of this note, condition (4_B) is:

For every finite list $(E_0, F_0), \dots, (E_n, F_n)$ of pairs of events (where repetitions are allowed),

$$\left[(E_i \succcurlyeq F_i,\ i = 1, \dots, n) \ \& \ \sum_{i=0}^{n} \mathbf{1}_{E_i} = \sum_{i=0}^{n} \mathbf{1}_{F_i}\right] \implies F_0 \succcurlyeq E_0.$$

(Pay attention to the fact that his indices run from 1 to n in one place and from 0 to n in another place.)

My Condition **S** does not imply the conclusion $F_0 \succcurlyeq E_0$ in the situation described—it only implies the weaker $E_0 \not\succ F_0$. But in the presence of Completeness, $F_0 \succcurlyeq E_0$ is equivalent to $E_0 \not\succ F_0$.

27.3 Subjective probability and betting

The payoffs for betting are usually described in terms of **odds**. If you wager an amount b on the event E and the odds against E are given by $\lambda(E)$, you receive λb if E occurs and lose b if E fails to occur. We allow λ to take on any value in $[0, \infty]$. The interpretation of $\lambda(E) = \infty$ is that for any positive bet b, if E occurs, then the bettor may name any real number as his payoff. In a frictionless betting market, the odds against E^c are given by

$$\lambda(E^c) = \frac{1}{\lambda(E)},$$

where we use the conventions

$$\frac{1}{\infty} = 0, \quad \frac{1}{0} = \infty.$$

More conveniently, instead of using λ, define

$$q(E) = \frac{1}{1 + \lambda(E)},$$

$$q(E^c) = \frac{1}{1 + \lambda(E^c)} = \frac{1}{1 + \frac{1}{\lambda(E)}} = \frac{\lambda(E)}{1 + \lambda(E)}.$$

Note that

$$q(E) + q(E^c) = 1,$$

and that

$$\lambda(E) = \frac{q(E^c)}{q(E)}.$$

Moreover, if you bet $q(E) = \frac{1}{1+\lambda(E)}$ on E, then your payoff Π in state s is given by

$$\begin{aligned}\Pi(s) &= q(E)\left[\lambda(E)\mathbf{1}_E(s) - \mathbf{1}_{E^c}(s)\right] \\ &= q(E)\left[\frac{q(E^c)}{q(E)}\mathbf{1}_E(s) - \mathbf{1}_{E^c}(s)\right] \\ &= q(E^c)\mathbf{1}_E(s) - q(E)\mathbf{1}_{E^c}(s) \\ &= \Big(1 - q(E)\Big)\mathbf{1}_E(s) - q(E)\Big(1 - \mathbf{1}_E(s)\Big) \\ &= \mathbf{1}_E(s) - q(E).\end{aligned}$$

That is, $q(E)$ is the price of a lottery ticket that pays \$1 in event E. Let's call such a lottery ticket an ***E*-ticket**.[1]

27.3.1 Subjective probability theorem *Either*
(i) The function q is a probability and $\lambda(E) = \frac{q(E^c)}{q(E)}$ for each E.
Or else
*(ii) The odds are **incoherent**, that is, there is a combination of bets that guarantees the bettor will win a positive amount regardless of which state s occurs.*

A set of incoherent odds is also known as a **Dutch book**.

Proof: Condition (ii) is equivalent to

$$S\left\{\overbrace{\begin{bmatrix} & \vdots & \\ & \mathbf{1}_E(s) - q(E) & \\ & \vdots & \end{bmatrix}}^{\mathcal{E}}\right. \begin{bmatrix}\vdots \\ x(E) \\ \vdots\end{bmatrix} \gg 0$$

(where $x(E)$ is the number of E-tickets).

Gordan's Alternative 25.3.9 assets that the alternative is that there is some probability vector $p \in \boldsymbol{R}^S$, such that for each event E,

$$\sum_{s\in S} p(s)\mathbf{1}_E(s) - q(E) = 0,$$

or

$$q(E) = \sum_{s\in E} p(s) = p(E),$$

which is (i). ∎

[1] Young people think an E-ticket is something that lets you on an airplane, but we older Southern Californians know you it's what lets you get on the Matterhorn.

27.4 No-arbitrage and Arrow–Debreu prices

There are only two time periods, "today" ($t = 0$) and "tomorrow" ($t = 1$). There are finitely many possible **states of nature** tomorrow, and exactly one of them will be realized tomorrow. Denote the set of states by S. The state of nature tomorrow is not known today.

There are n **purely financial assets**. A purely financial asset is a contingent claim denominated in dollars (as opposed to commodities).

There is a **spot market** today for assets and each asset has a market price today or **spot price**. The price of asset i today is p_0^i, and it pays $p_1^i(s)$ in state s tomorrow.

The **cash flow vector** of asset i is

$$A^i = \begin{bmatrix} -p_0^i \\ \vdots \\ \vdots \\ p_1^i(s) \\ \vdots \\ \end{bmatrix} \in \boldsymbol{R} \times \boldsymbol{R}^S.$$

The cash flow convention is that positive numbers represent cash received by the owner of the asset and negative quantities represent cash payed out by the owner. Thus the 0^{th} component of A^i is negative if p_0^i is positive, because to purchase a unit of asset i requires a cash payment if the price is positive. If p_0^i is negative, the "asset" i can be interpreted as a loan to the "owner." Thus we allow for borrowing in our framework, but whether or not the borrower defaults must be part of the specification of the payoff of the asset.

It is even possible that one of the assets may be **riskless** in that

$$p_1(s) = c \quad \text{for all } s \in S.$$

That is, the asset pays the same amount in each state of nature. Suppose the riskless asset has spot price p_0 today. Then r defined by

$$(1 + r)p_0 = c \quad \text{or} \quad r = \frac{c}{p_0} - 1,$$

is the **riskless rate of interest**. If the riskless rate of interest is positive, then $p_0 < c$. But as long as p_0 and c are both positive we must have $r > -1$.

A portfolio is defined by the number of units of each asset held. Since there are n assets, a **portfolio** is simply a vector x in $\boldsymbol{R}^{\mathrm{n}}$. The entry x_i indicates the number of units of asset i, which may be either positive or negative. The cash flow vector of the portfolio is just

$$\sum_{i=1}^{n} A^i x_i.$$

If $x_i < 0$, then the i^{th} asset has been sold short or issued by the portfolio holder. We will not rule this out, so a portfolio need not be a nonnegative vector.

27.4.1 Definition *An* ***arbitrage portfolio*** *is a portfolio x whose cash flow vector is semi-positive,*

$$\sum_{i=1}^{n} A^i x_i > 0.$$

27.4.2 Assumption (Iron Law of Theoretical Finance) *There are no arbitrage portfolios.*

This law has the following remarkable and useful consequence:

27.4.3 Asset pricing theorem *In this model, either*

(1) *There is an arbitrage portfolio (that is, the Iron Law of Theoretical Finance fails);*

or else

(2) *there are numbers $\pi(s) > 0$, $s \in S$, such that for each asset i,*

$$p_0^i = \sum_{s \in S} \pi(s) p_1^i(s).$$

Proof: In algebraic terms, alternative (1) states that there is some $x \in \boldsymbol{R}^{\mathrm{n}}$ satisfying $Ax > 0$, where A is the $(|S|+1) \times n$ matrix whose i^{th} column is $A^i \in \boldsymbol{R} \times \boldsymbol{R}^S$. If this is not true, then Stiemke's Theorem 25.3.13 states that there is $y \gg 0 \in \boldsymbol{R} \times \boldsymbol{R}^S$ such that for each i,

$$-y_0 p_0^i + \sum_{s \in S} y_s p_1^i(s) = 0.$$

Clearly the numbers

$$\pi(s) = \frac{y_s}{y_0}$$

satisfy alternative (2). It also follows from Stiemke's Theorem that alternatives (1) and (2) are inconsistent. ∎

The numbers $\pi(s)$, $s \in S$ are called **Arrow–Debreu prices**. The price $\pi(s)$ represents the current market price of a payment of \$1 in state s tomorrow. The theorem says that today's price for any asset is computed by summing the market value of its cash flow over all the future states.

27.4.1 Risk neutral probability

Also note that if a risk free asset exists, then the risk free rate of interest r is determined by

$$r = \frac{1}{\sum_{s \in S} \pi(s)} - 1.$$

Even if there is no risk free asset, given Arrow–Debreu prices, we can still formally define a risk free rate of interest.

27.4.4 Definition *The **risk free rate of interest** (given Arrow–Debreu prices π) is defined by the equation*

$$r = \frac{1}{\sum_{s\in S}\pi(s)} - 1 \quad \textit{or} \quad \sum_{s\in S}\pi(s) = \frac{1}{1+r}.$$

Thus the vector $(1+r)\pi$ defines a probability measure μ on S by

$$\mu(A) = (1+r)\sum_{s\in A}\pi(s).$$

The expected value $\boldsymbol{E}_\mu X$ of a random variable X under the measure μ is given by

$$\boldsymbol{E}_\mu X = (1+r)\sum_{s\in S}\pi(s)X(s),$$

so for asset i we have

$$p^i_0 = \frac{1}{1+r}\boldsymbol{E}_\mu p^i_1.$$

> *That is, the price of each asset is just the present discounted value (discounted at the risk-free interest rate) of the expected value of the asset (under the probability measure μ).*

For this reason, the measure μ is called the **risk neutral probability** for the assets. If this probability is used on S, the price of each asset is simply its discounted expected value, and there are no risk premia.

27.5 Statistical inference—the game

Θ is a set of urns, each urn θ describes a probability p_θ on S. A particular urn θ_0 is used to choose signal $s \in S$ according to probability p_{θ_0}. We observe signal $s \in S$. What information does this convey about θ_0? (Statisticians don't call elements of Θ urns, they call them states of the world. In other words, statisticians believe that God does nothing but play dice.)

27.5.1 Conditional probability

The **conditional probability** of event E given event F is

$$p(E|F) = \frac{p(E\cap F)}{p(F)}.$$

Thus

$$p(E|F)p(F) = p(E\cap F) = p(F|E)p(E),$$

Or

$$p(E|F) = \frac{p(E)}{p(F)}\cdot p(F|E),$$

which is known as **Bayes' Law**.

27.5.2 Bayesian updating

Select urn θ_0 according to probability P on Θ, and select s according to p_{θ_0}. Then the probability that $\theta_0 \in T$, given s is

$$P(T|s) = \frac{\sum_{\theta \in T} p_\theta(s) P(\theta)}{\sum_{\theta \in \Theta} p_\theta(s) P(\theta)}.$$

P is known as a **prior**, and $P(\cdot|s)$ is the corresponding **posterior**.

Should Bayes' Law govern our betting behavior? Let's see.

27.5.3 Statistical inference: the game

Freedman and Purves [17] describe statistical inference in terms of the following game.

The Master of Ceremonies chooses an urn, and announces the signal s.

A Bookie posts odds λ against subsets $T \in \mathcal{T}$ of Θ.

Bets are placed.

The MC reveals the urn, and bets are settled.

(In the real world, the MC never tells.)

27.5.4 Strategies

Bookie chooses $q \geqq 0 \in \boldsymbol{R}^{\mathcal{T} \times S}$. For each $s \in S$,

$$q(T, s) + q(T^c, s) = 1.$$

Bettor then chooses $x \in \boldsymbol{R}^{\mathcal{T} \times S}$, and bets

$$x(T, s) q(T, s)$$

on T when s occurs.

Under these strategies, the expected payoff to the bettor when θ is the selected urn is just

$$\sum_{s \in S} \left(\sum_{T \in \mathcal{T}} \Big(\mathbf{1}_T(\theta) - q(T, s) \Big) x(T, s) \right) p_\theta(s).$$

27.5.1 Bayesian updating theorem *Either*

(1) The Bookie chooses some prior P and posts odds according to the posterior $P(\cdot|s)$

Or else

(2) There is a betting strategy that gives the bettor a positive expected payoff regardless of which urn θ is selected.

Proof: (2) is equivalent to

$$\Theta \left\{ \overbrace{\begin{bmatrix} \\ \Big(\mathbf{1}_T(\theta) - q(T,s)\Big) p_\theta(s) \\ \\ \end{bmatrix}}^{\mathcal{T} \times S} \right. \begin{bmatrix} \vdots \\ x(T,s) \\ \vdots \end{bmatrix} \gg 0,$$

The alternative is the existence of a probability vector $P \in \boldsymbol{R}^\Theta$ such that for each (T,s),

$$\sum_{\theta \in \Theta} \Big(\mathbf{1}_T(\theta) - q(T,s)\Big) p_\theta(s) P(\theta) = 0.$$

In other words,

$$\sum_{\theta \in T} p_\theta(s) P(\theta) = \sum_{\theta \in \Theta} q(T,s) p_\theta(s) P(\theta),$$

or

$$q(T,s) = \frac{\sum_{\theta \in T} p_\theta(s) P(\theta)}{\sum_{\theta \in \Theta} p_\theta(s) P(\theta)} = P(T|s),$$

which is (1). ∎

27.6 Dynamic asset pricing

In this model there are three periods: "today" ($t = 0$), "tomorrow" ($t = 1$), and "later" ($t = 2$). The set S of states has the structure $S = U \times V$, where u is revealed tomorrow and v is revealed later. We assume that each asset i pays nothing tomorrow and $p_2^i(u,v)$ later. The **spot price** of asset i today is p_0^i. Its spot price tomorrow in state u will be $p_1^i(u)$.

A **dynamic portfolio** is a vector

$$x = \left((x_0^i)_{i=1,\dots,n}, \Big(x_1^i(u)\Big)_{i=1,\dots,n,\ u \in U} \right) \in \boldsymbol{R}^{\mathrm{n}} \times \boldsymbol{R}^{\mathrm{n} \times |U|}.$$

The dynamic portfolio x is **self-financing** if

$$\sum_{i=1}^n p_0^i x_0^i \leqslant 0,$$

and for each $u \in U$,

$$\sum_{i=1}^n p_1^i(u) \Big(x_1^i(u) - x_0^i\Big) \leqslant 0.$$

The cash flow of a dynamic portfolio x is

$$\begin{array}{c} \\ \hline \vdots \\ u \\ \vdots \\ \hline \vdots \\ (u,v) \\ \vdots \end{array}
\begin{bmatrix} \begin{array}{c} -\sum_{i=1}^n p_0^i x_0^i \\ \hline \vdots \\ -\sum_{i=1}^n p_1^i(u)\Big(x_1^i(u)-x_0^i\Big) \\ \vdots \\ \hline \vdots \\ \sum_{i=1}^n p_2^i(u,v)x_1^i(u) \\ \vdots \end{array} \end{bmatrix}$$

A **dynamic arbitrage portfolio** is a portfolio that has a semi-positive cash flow. Note that this implies that the portfolio is self-financing.

27.6.1 Dynamic pricing theorem *If (and only if) there are no dynamic arbitrage portfolios, then there are probability measures $\hat{\mu}$ and μ on $S = U \times V$, a "one-period risk-free interest rate" $r_{0,1}$ between periods 0 and 1, a "two-period risk-free interest rate" $r_{0,2}$ between periods 0 and 2, and for each partial state u, there is a "one-period risk-free interest rate" $r_{1,2}(u)$ between period 1 in state u and period 2, such that the following properties are satisfied.*

1. *For each asset i, today's spot price is the expected present discounted value of future prices. Specifically,*

$$p_0^i = \frac{1}{1+r_{0,1}}\, \boldsymbol{E}_{\hat{\mu}}\, p_1^i = \frac{1}{1+r_{0,2}}\, \boldsymbol{E}_{\mu}\, p_2^i.$$

2. *The measures $\hat{\mu}$ and μ have the same conditional probabilities. That is, for every (u,v),*

$$\hat{\mu}(v|u) = \mu(v|u).$$

3. *For each partial state u, for each asset i, tomorrow's spot price $p_1^i(u)$ in state u is the conditional expected present discounted value of the payoffs later. That is,*

$$p_1^i(u) = \frac{1}{1+r_{1,2}(u)}\, \boldsymbol{E}_{\hat{\mu}}(p_2^i \mid u) = \frac{1}{1+r_{1,2}(u)}\, \boldsymbol{E}_{\mu}(p_2^i \mid u).$$

4. *The **term structure** of interest rates and discount factors satisfies*

$$1+r_{0,2} = (1+r_{0,1})\, \boldsymbol{E}_{\mu}(1+r_{1,2}), \qquad \frac{1}{1+r_{0,2}} = \frac{1}{1+r_{0,1}}\, \boldsymbol{E}_{\hat{\mu}}\, \frac{1}{1+r_{1,2}}.$$

Proof: A dynamic portfolio x is a dynamic arbitrage portfolio if it satisfies

$$
\begin{array}{cc}
 & \begin{array}{c|c} j & (i,u) \end{array} \\
\begin{array}{c} 0 \\ \\ u' \\ \\ \\ (u'',v) \\ \\ \end{array} &
\left[\begin{array}{c|ccc}
-p_0^j & 0 & \dots & 0 \\ \hline
\vdots & & & \\
p_1^j(u') & & -p_1^i(u)\delta_{u,u'} & \\
\vdots & & & \\ \hline
\vdots & & & \\
0 & & p_2^i(u,v)\delta_{u,u''} & \\
\vdots & & &
\end{array}\right]
\end{array}
\left[\begin{array}{c} x_0^j \\ \hline \vdots \\ x_1^i(u) \\ \vdots \end{array}\right] > 0.
$$

(Figure 27.6.1 illustrates this matrix inequality for $n = 2$, $U = \{1,2,3\}$, and $V = \{1,2\}$.)

The (Stiemke) alternative is that there is some

$$\pi = \left(\pi_0, \Big(\pi_1(u)\Big)_{u\in U}, \Big(\pi_2(u,v)\Big)_{(u,v)\in U\times V}\right) \gg 0$$

such that for each $j = 1,\dots,n$

$$-p_0^j\pi_0 + \sum_{u\in U} p_1^j(u)\pi_1(u) = 0,$$

and also for each (i,u), $i = 1,\dots,n$, $u \in U$,

$$-p_1^i(u)\pi_1(u) + \sum_{v\in V} p_2^i(u,v)\pi_2(u,v) = 0.$$

This is homogeneous in π, so without loss of generality $\pi_0 = 1$, so we have

$$p_0^i = \sum_{u\in U} p_1^i(u)\pi_1(u). \tag{6}$$

and

$$p_1^i(u) = \sum_{v\in V} p_2^i(u,v)\frac{\pi_2(u,v)}{\pi_1(u)} \tag{7}$$

so that

$$p_0^i = \sum_{(u,v)\in U\times V} p_2^i(u,v)\pi_2(u,v). \tag{8}$$

Thus, we may interpret the $\pi_1(u)$ and $\pi_2(u,v)$ as today's prices for a dollar at the various dates and states of the world. As before we can normalize these prices to define an interest rate and a probability measure.

Equation (8) suggests we define $r_{0,2}$ by

$$(1 + r_{0,2}) \sum_{(u,v)\in U\times V} \pi_2(u,v) = 1. \tag{9}$$

$$
\begin{array}{rc}
 & \begin{array}{cc|cccccc} j{=}1 & j{=}2 & (i,u){=}(1,1) & (i,u){=}(2,1) & (i,u){=}(1,2) & (i,u){=}(2,2) & (i,u){=}(1,3) & (i,u){=}(2,3) \end{array} \\
\begin{array}{r} 0 \\ \hline u'{=}1 \\ u'{=}2 \\ u'{=}3 \\ \hline (u'',v){=}(1,1) \\ (u'',v){=}(1,2) \\ (u'',v){=}(2,1) \\ (u'',v){=}(2,2) \\ (u'',v){=}(3,1) \\ (u'',v){=}(3.2) \end{array}
&
\left[\begin{array}{cc|cccccc}
-p_0^1 & -p_0^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
p_1^1(1) & p_1^2(1) & -p_1^1(1) & -p_1^2(1) & 0 & 0 & 0 & 0 \\
p_1^1(2) & p_1^2(2) & 0 & 0 & -p_1^1(2) & -p_1^2(2) & 0 & 0 \\
p_1^1(3) & p_1^2(3) & 0 & 0 & 0 & 0 & -p_1^1(3) & -p_1^2(3) \\ \hline
0 & 0 & p_2^1(1,1) & p_2^2(1,1) & 0 & 0 & 0 & 0 \\
0 & 0 & p_2^1(1,2) & p_2^2(1,2) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & p_2^1(2,1) & p_2^2(2,1) & 0 & 0 \\
0 & 0 & 0 & 0 & p_2^1(2,2) & p_2^2(2,2) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & p_2^1(3,1) & p_2^2(3,1) \\
0 & 0 & 0 & 0 & 0 & 0 & p_2^1(3,2) & p_2^2(3,2)
\end{array}\right]
\end{array}
\left[\begin{array}{c} x_0^1 \\ x_0^2 \\ \hline x_1^1(1) \\ x_1^2(1) \\ x_1^1(2) \\ x_1^2(2) \\ x_1^1(3) \\ x_1^2(3) \end{array}\right]
>
\left[\begin{array}{c} 0 \\ \hline 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right]
$$

Figure 27.6.1. An arbitrage portfolio for $n = 2$, $U = \{1, 2, 3\}$, and $V = \{1, 2\}$.

It is the riskless rate of interest between periods 0 and 2. The corresponding probability measure μ on $U \times V$ is defined by

$$\mu(u,v) = (1 + r_{0,2})\pi_2(u,v). \tag{10}$$

Then (8) becomes

$$\boxed{p_0^i = \frac{1}{1 + r_{0,2}}\, \boldsymbol{E}_\mu\, p_2^i.} \tag{11}$$

Similarly, equation (6) suggests defining $r_{0,1}$ by

$$(1 + r_{0,1}) \sum_{u \in U} \pi_1(u) = 1. \tag{12}$$

It is the risk free one period rate between periods today and tomorrow. It determines a probability $\hat{\mu}_\bullet$ on U by

$$\hat{\mu}_\bullet(u) = (1 + r_{0,1})\pi_1(u). \tag{13}$$

Then (8) can be rewritten as

$$p_0^i = \frac{1}{1 + r_{0,1}}\, \boldsymbol{E}_{\hat{\mu}_\bullet}\, p_1^i. \tag{14}$$

Equation (7) suggests that for each $u \in U$, we define $r_{1,2}(u)$ by

$$\Big(1 + r_{1,2}(u)\Big) \sum_{v \in V} \frac{\pi_2(u,v)}{\pi_1(u)} = 1. \tag{15}$$

It is the riskless rate of interest at time 1 in state u. (From the point of view of period 0, the rate $r_{1,2}$ is a random variable.) We also have a probability measure $\hat{\mu}(\cdot \mid u)$ on V defined by

$$\hat{\mu}(v \mid u) = \Big(1 + r_{1,2}(u)\Big)\frac{\pi_2(u,v)}{\pi_1(u)}. \tag{16}$$

Therefore

$$p_1^i(u) = \frac{1}{1 + r_{1,2}(u)}\, \boldsymbol{E}_{\hat{\mu}|u}\, p_2^i. \tag{17}$$

Now define the measure $\hat{\mu}$ on $U \times V$ by

$$\hat{\mu}(u,v) = \hat{\mu}(v \mid u)\hat{\mu}_\bullet(u). \tag{18}$$

Then $\hat{\mu}_\bullet$ is the marginal of $\hat{\mu}$ on U and $\hat{\mu}(\cdot \mid u)$ is the conditional probability on V given u. So (14) becomes

$$\boxed{p_0^i = \frac{1}{1 + r_{0,1}}\, \boldsymbol{E}_{\hat{\mu}}\, p_1^i.}$$

and (17) becomes

$$\boxed{p^i_1(u) = \frac{1}{1+r_{1,2}(u)}\,\boldsymbol{E}_{\hat\mu}(p^i_2 \mid u).} \tag{19}$$

Also observe that

$$\begin{aligned}
\hat\mu(u,v) &= \hat\mu(v \mid u)\hat\mu_\bullet(u) && (18)\\
&= (1+r_{0,1})\pi_1(u)\big(1+r_{1,2}(u)\big)\frac{\pi_2(u,v)}{\pi_1(u)} && \text{equations (13) and (16)} \quad (20)\\
&= (1+r_{0,1})\big(1+r_{1,2}(u)\big)\pi_2(u,v).
\end{aligned}$$

What is the relationship between $\hat\mu$ and μ? From (20) and (10) we have

$$\mu(u,v) = \frac{1+r_{0,2}}{(1+r_{0,1})\big(1+r_{1,2}(u)\big)}\hat\mu(u,v) \tag{21}$$

Conditioning on u then gives

$$\begin{aligned}
\mu(v \mid u) = \frac{\mu(u,v)}{\sum_{v'}\mu(u,v')} &= \frac{\dfrac{1+r_{0,2}}{(1+r_{0,1})\big(1+r_{1,2}(u)\big)}\hat\mu(u,v)}{\sum_{v'}\dfrac{1+r_{0,2}}{(1+r_{0,1})\big(1+r_{1,2}(u)\big)}\hat\mu(u,v')}\\
&= \frac{\hat\mu(u,v)}{\sum_{v'}\hat\mu(u,v')} = \hat\mu(v \mid u).
\end{aligned}$$

Another way to see this is to note that (10) implies

$$\mu(v \mid u) = \pi_2(u,v)/\sum_{v'}\pi_2(u,v')$$

and equations (15) and (16) imply

$$\hat\mu(v \mid u) = \pi_2(u,v)/\sum_{v'}\pi_2(u,v').$$

Either way

$$\boxed{\mu(v \mid u) = \hat\mu(v \mid u).}$$

Thus (19) can also be written as

$$\boxed{p^i_1(u) = \frac{1}{1+r_{1,2}(u)}\,\boldsymbol{E}_{\mu}(p^i_2 \mid u).}$$

Summing both sides of (21) over $U \times V$ gives

$$\boldsymbol{E}_{\hat\mu}\,\frac{1+r_{0,2}}{(1+r_{0,1})(1+r_{1,2})} = 1.$$

In other words, the term structure satisfies

$$\boxed{\frac{1}{1+r_{0,2}} = \frac{1}{1+r_{0,1}} \boldsymbol{E}_{\hat{\mu}} \frac{1}{1+r_{1,2}}.}$$

On the other hand, rewriting (21) as

$$(1+r_{0,1})\Big(1+r_{1,2}(u)\Big)\mu(u,v) = (1+r_{0,2})\hat{\mu}(u,v)$$

and summing, we see that

$$\boxed{1+r_{0,2} = (1+r_{0,1})\, \boldsymbol{E}_{\mu}(1+r_{1,2})}$$

∎

27.7 Stochastic dominance and expected utility

In this section we consider lotteries over monetary prizes. Given a finite set $m_1 < \cdots < m_n$ of monetary prizes. A **lottery** is a probability distribution over the prizes. Lotteries thus correspond to probability vectors in $\boldsymbol{R}^{\mathrm{n}}$. We say that q **stochastically dominates** p if for each $k = 1, \ldots, n$,

$$\sum_{i=1}^{k} q_i \leqslant \sum_{i=1}^{k} p_i,$$

and $p \neq q$ (so that there is strict inequality for some i). That is, q always assigns lower probability than p to smaller prizes. Intuitively one should prefer a stochastically dominating lottery. The next result is based on Border [6] and Ledyard [22].

27.7.1 Expected utility theorem *Suppose p and q are distinct probability vectors. Either*

(1) There are $u_1 < \cdots < u_n$ such that

$$\sum_{i=1}^{n} u_i p_i > \sum_{i=1}^{n} u_i q_i$$

Or else

(2) q stochastically dominates p.

That is, as long as your choice is not dominated, you act as if you maximize the expected utility of some strictly increasing utility.

Proof: (1) is equivalent to

$$\begin{bmatrix} p_1 - q_1 & p_2 - q_2 & p_3 - q_3 & \cdots & p_{n-1} - q_{n-1} & p_n - q_n \\ -1 & +1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & -1 & +1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix} \gg 0.$$

The alternative is: $y = (y_0, y_1, \ldots, y_{n-1}) > 0$ and

$$\begin{aligned} y_0(p_1 - q_1) & & - y_1 &= 0 \\ y_0(p_2 - q_2) &+ y_1 & - y_2 &= 0 \\ &\vdots & \vdots & \\ y_0(p_{n-1} - q_{n-1}) &+ y_{n-2} & - y_{n-1} &= 0 \\ y_0(p_n - q_n) &+ y_{n-1} & &= 0. \end{aligned}$$

It is easy to see that $y_0 > 0$, for if $y_0 = 0$, everything unravels and $y = 0$, a contradiction.

Write $x_i = \frac{y_i}{y_0} \geqslant 0$, $i = 1, \ldots, n-1$. Then

$$\begin{aligned} p_1 - q_1 & & - x_1 &= 0 \\ p_2 - q_2 &+ x_1 & - x_2 &= 0 \\ &\vdots & \vdots & \\ p_{n-1} - q_{n-1} &+ x_{n-2} & - x_{n-1} &= 0 \\ p_n - q_n &+ x_{n-1} & &= 0. \end{aligned}$$

In other words,

$$\begin{array}{rcrcrl} p_1 & - & q_1 & = & x_1 & \geqslant 0 \\ (p_1+p_2) & - & (q_1+q_2) & = & x_2 & \geqslant 0 \\ & \vdots & & & & \vdots \\ \sum_{i=1}^{n-1} p_i & - & \sum_{i=1}^{n-1} q_i & = & x_{n-1} & \geqslant 0 \\ p_n & - & q_n & = & -x_{n-1} & \leqslant 0 \end{array}$$

which, since p and q are distinct, is just (2). ∎

27.8 Harsanyi's utilitarianism theorem

27.8.1 Exercise Use an appropriate theorem of the alternative (or a separating hyperplane theorem if you can't make the first approach work) to prove the following result due to Harsanyi [18].

Let S be a finite set of states of the world. Let $\mathcal{P}$ be the set of probability measures on S. Let N be a finite society of individuals. Each individual i has a Bernoulli utility function u_i on S, and evaluates elements of $\mathcal{P}$ by means of expected utility:

$$U_i(p) = \sum_{s \in S} u_i(s)p(s).$$

Society also has a Bernoulli utility function u on S, and evaluates elements of $\mathcal{P}$ by means of expected utility:

$$U(p) = \sum_{s \in S} u(s)p(s).$$

Assume the social and individual von Neumann–Morgenstern utilities satisfy the following unanimous indifference condition: for all $p, q \in \mathcal{P}$,

$$(\,\forall i \in N\,)\ [\,U_i(p) = U_i(q)\,] \quad \Longrightarrow \quad U(p) = U(q).$$

Prove that there exist real numbers α_i, $i \in N$, and β such that for all $s \in S$,

$$u(s) = \beta + \sum_{i \in N} \alpha_i u_i(s)$$

so for all $p \in \mathcal{P}$,

$$U(p) = \beta + \sum_{i \in N} \alpha_i U_i(p).$$

□

27.9 Core of a TU game

An n-person game starts with a set $N = \{1, \ldots, n\}$ of **players**. A **coalition** is a nonempty subset of N. Denote the set of coalitions by $\mathcal{N}$. Given a player i, let $S(i) = \{S \in \mathcal{N} : i \in S\}$. A **transferable utility (TU) game** is described by its **characteristic function** $v \colon \mathcal{N} \to \boldsymbol{R}$. The idea is that for every coalition S, the number $v(S)$ represents a total payoff (utility) that can be transferred from the coalition to its members. A vector $x \in \boldsymbol{R}^{\mathrm{n}}$ is an **allocation** (sometimes called an **imputation**) if

$$\sum_{i \in N} x_i = v(N).$$

That is, it is an allocation of the payoff available to the **grand coalition** N. An allocation x belongs to the **core** of v provided

$$\sum_{i \in S} x_i \geqslant v(S), \quad S \in \mathcal{N}.$$

In other words, just as in the case of the core of an economy, no coalition can make all its members better off than they are with allocation x.

A vector $\pi \in \boldsymbol{R}_+^{\mathcal{N}}$ is called a vector of **balancing weights** if

$$\sum_{S(i)} \pi_S = 1, \quad i \in N.$$

The family $\{S \in \mathcal{N} : \pi_S > 0\}$ is called a **balanced family** of coalitions.

A TU game v is a **balanced game** if for every vector π of balancing weights, we have

$$\sum_{S \in \mathcal{N}} \pi_S v(S) \leqslant v(N).$$

27.9.1 Theorem *A TU games has a nonempty core if and only if it is balanced.*

Proof: The core is nonempty if and only if the following system of inequalities has a solution $x \in \boldsymbol{R}^{\mathrm{n}}$:

$$\begin{aligned} \sum_{i \in S} x_i &\geqslant v(S), \quad S \in \mathcal{N} \\ -\sum_{i \in N} x_i &\geqslant -v(S), \end{aligned}$$

or in matrix form

$$\begin{array}{c} \\ S\in\mathcal{N} \\ \\ N \end{array}\begin{array}{c} i\in N \\ \left[\begin{array}{ccc} & \vdots & \\ \cdots & \mathbf{1}_S(i) & \cdots \\ & \vdots & \\ \hline -1 & \cdots & -1 \end{array}\right] \end{array}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \geqq \left[\begin{array}{c} \vdots \\ v(S) \\ \vdots \\ \hline -v(N) \end{array}\right]$$

where $\mathbf{1}_S(i) = 1$ if $i \in S$ and $\mathbf{1}_S(i) = 0$ if $i \notin S$. Note that this system includes two inequalities involving $v(N)$. By Corollary 25.3.5, the alternative is the existence of a vector $p = [\dots, \pi(S), \dots; \alpha] \in \boldsymbol{R}^{\mathcal{N}} \times \boldsymbol{R}$ satisfying $p > 0$ and

$$[\dots, \pi(S), \dots; \alpha]\left[\begin{array}{ccc} & \vdots & \\ \cdots & \mathbf{1}_S(i) & \cdots \\ & \vdots & \\ \hline -1 & \vdots & -1 \end{array}\right] = 0, \quad [\dots, p_S, \dots; \alpha] \cdot \left[\begin{array}{c} \vdots \\ v(S) \\ \vdots \\ \hline -v(N) \end{array}\right] > 0.$$

This can be written out as

$$\begin{aligned} \sum_{S\in\mathcal{N}} \pi(S)\mathbf{1}_S(i) &= \alpha, \quad i = 1, \dots, n \\ \sum_{S\in\mathcal{N}} \pi(S)v(S) &> \alpha v(N). \end{aligned}$$

Since $p > 0$ and $\mathbf{1}_S(i) > 0$ for each $i \in S$ any solution of the alternative must have $\alpha > 0$, and so without loss of generality we may normalize p so that $\alpha = 1$. The alternative above then becomes

$$\begin{aligned} \sum_{S(i)} \pi(S) &= 1, \quad i = 1, \dots, n \\ \sum_{S\in\mathcal{N}} \pi(S)v(S) &> v(N). \end{aligned}$$

This says that $\{\pi(S) : S \in \mathcal{N}\}$ is a family of balancing weights and that the game v is not balanced.

Since these alternatives are mutually exclusive, the core is empty if and only if the game is not balanced. ∎

27.10 Reduced form auctions

This section is based on Border [7], and is currently missing some key parts.

An auction is an institution (set of rules) for selling an object to one of a group of potential buyers or *bidders*. The following problem arises naturally in the study of auctions, see Maskin and Riley [23], Matthews [24], and Border [5].

Let T be a finite nonempty set of *types*, let λ be a probability vector on T, and let $P\colon T \to [0,1]$. For convenience assume $\lambda \gg 0$. A **symmetric auction** is a vector $\boldsymbol{p}$ of functions $p_i\colon T^N \to [0,1]$ satisfying

$$\sum_{i=1}^{N} p_i(\boldsymbol{t}) \leqslant 1 \tag{F}$$

for each $\boldsymbol{t} \in T^N$, and for each permutation π on $\{1,\dots,N\}$, each profile $\boldsymbol{t} \in \boldsymbol{T}$, and each $i = 1,\dots,N$,

$$p_i(\boldsymbol{t}_1,\dots,\boldsymbol{t}_N) = p_{\pi^{-1}(i)}(\boldsymbol{t}_{\pi(1)},\dots,\boldsymbol{t}_{\pi(N)}). \tag{S}$$

Here $p_i(\boldsymbol{t})$ is the probability that bidder i wins the auction in profile $\boldsymbol{t}$. The feasibility condition (**F**) is just that the probability of selling the object cannot exceed unity. Conditions (**S**) is a symmetry condition that says a bidder's number does not matter, only his type.

From bidder i's point of view, what is important to him about the auction $\boldsymbol{p}$ is the conditional probability that he wins given his type. Assuming types are independently and identically distributed according to the probability measure λ, this is given by

$$P(\tau) = \sum_{\boldsymbol{t}^{-i} \in T^{N-1}} p_i(\tau, \boldsymbol{t}^{-i}) \lambda^{N-1}(\boldsymbol{t}^{-i}). \tag{R}$$

Here $\boldsymbol{t}^{-i}$ for a typical element of T^{N-1}

If P is the reduced form of some auction $\boldsymbol{p}$, we may also say that $\boldsymbol{P}$ is *implementable*.

The question is, when is a function $P\colon T \to [0,1]$ implementable?

27.10.1 Theorem (Maskin–Riley–Matthews–Border) *For an independently and identically distributed environment, a function $P\colon T \to [0,1]$ is the reduced form of a symmetric auction if and only if for every subset A of T, it satisfies the Maskin–Riley–Matthews (MRM) condition*

$$N \sum_{\tau \in A} P(\tau)\lambda(\tau) \leqslant 1 - \lambda(A^c)^N. \tag{MRM}$$

27.10.1 An example

Instead of proving the general theorem, I deal with a special case. The general theorem is merely an exercise in keeping your subscripts straight. See Border [7] for all the gory details, or see Border [5] for the symmetric case with an arbitrary (possibly infinite) set T of types.

27.10.2 Example Consider the case of $N = 3$ bidders, and 2 types, $T = \{1, 2\}$, with probabilities $\lambda(1) > 0$, $\lambda(2) > 0$. Given a potential reduced form $P = (P_1, P_2)$, $0 \leqslant P_i \leqslant 1$, $i = 1, 2$, we wish to find a symmetric auction function $p\colon T^3 \to [0, 1]$ satisfying the following (in)equalities:

$$
\begin{aligned}
p(1;1,1)\lambda(1)^2 + p(1;1,2)\lambda(1)\lambda(2) + p(1;2,1)\lambda(1)\lambda(2) + p(1;2,2)\lambda(2)^2 &= P_1 \\
p(2;1,1)\lambda(1)^2 + p(2;1,2)\lambda(1)\lambda(2) + p(2;2,1)\lambda(1)\lambda(2) + p(2;2,2)\lambda(2)^2 &= P_2 \\
p_1(1,1,1) + p_2(1,1,1) + p_3(1,1,1) = p(1;1,1) + p(1;1,1) + p(1;1,1) &\leqslant 1 \\
p_1(1,1,2) + p_2(1,1,2) + p_3(1,1,2) = p(1;1,2) + p(1;1,2) + p(2;1,1) &\leqslant 1 \\
p_1(1,2,1) + p_2(1,2,1) + p_3(1,2,1) = p(1;2,1) + p(2;1,1) + p(1;2,1) &\leqslant 1 \\
p_1(1,2,2) + p_2(1,2,2) + p_3(1,2,2) = p(1;2,2) + p(2;1,2) + p(2;2,1) &\leqslant 1 \\
p_1(2,1,1) + p_2(2,1,1) + p_3(2,1,1) = p(2;1,1) + p(1;2,1) + p(1;1,2) &\leqslant 1 \\
p_1(2,1,2) + p_2(2,1,2) + p_3(2,1,2) = p(2;1,2) + p(1;2,2) + p(2;1,2) &\leqslant 1 \\
p_1(2,2,1) + p_2(2,2,1) + p_3(2,2,1) = p(2;2,1) + p(2;2,1) + p(1;2,2) &\leqslant 1 \\
p_1(2,2,2) + p_2(2,2,2) + p_3(2,2,2) = p(2;2,2) + p(2;2,2) + p(2;2,2) &\leqslant 1
\end{aligned}
$$

Because of symmetry, $p(1;1,2) = p(1;2,1)$ and $p(2;1,2) = p(2;2,1)$, so we can reduce the system to:

$$
\begin{aligned}
p(1;1,1)\lambda(1)^2 + 2p(1;1,2)\lambda(1)\lambda(2) + p(1;2,2)\lambda(2)^2 &= P_1 \\
p(2;1,1)\lambda(1)^2 + 2p(2;1,2)\lambda(1)\lambda(2) + p(2;2,2)\lambda(2)^2 &= P_2 \\
3p(1;1,1) &\leqslant 1 \\
2p(1;1,2) + p(2;1,1) &\leqslant 1 \\
p(1;2,2) + 2p(2;1,2) &\leqslant 1 \\
3p(2;2,2) &\leqslant 1
\end{aligned}
$$

In matrix form this becomes

$$
\begin{array}{cc}
\text{indices} & \begin{array}{ccc|ccc} (1\cdot 11) & (1\cdot 12) & (1\cdot 22) & (2\cdot 11) & (2\cdot 12) & (2\cdot 22) \end{array} \\
\begin{array}{c} (1) \\ (2) \\ \hline (111) \\ (112) \\ (122) \\ (222) \end{array} &
\left[\begin{array}{ccc|ccc}
\lambda(1)^2 & 2\lambda(1)\lambda(2) & \lambda(2)^2 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda(1)^2 & 2\lambda(1)\lambda(2) & \lambda(2)^2 \\ \hline
3 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 3
\end{array}\right]
\left[\begin{array}{c} p_{1\cdot 11} \\ p_{1\cdot 12} \\ p_{1\cdot 22} \\ \hline p_{2\cdot 11} \\ p_{2\cdot 12} \\ p_{2\cdot 22} \end{array}\right]
\begin{array}{c} = \\ = \\ \hline \leqslant \\ \leqslant \\ \leqslant \\ \leqslant \end{array}
\left[\begin{array}{c} P_1 \\ P_2 \\ \hline 1 \\ 1 \\ 1 \\ 1 \end{array}\right]
\end{array}
\tag{22}
$$

Need to explain this!!

Since we eliminated the redundant conditions resulting from symmetry, we may reindex so that what matters is distribution d of types. The new indices are

$$
\begin{array}{c}
\text{indices} \\ \sigma{=}1 \\ \sigma{=}2 \\ m{=}(3,0) \\ m{=}(2,1) \\ m{=}(1,2) \\ m{=}(0,3)
\end{array}
\begin{array}{c}
\begin{array}{cccccc}
{\scriptstyle \tau;d= \atop 1;(2,0)} & {\scriptstyle \tau;d= \atop 1;(1,1)} & {\scriptstyle \tau;d= \atop 1;(0,2)} & {\scriptstyle \tau;d= \atop 2;(2,0)} & {\scriptstyle \tau;d= \atop 2;(1,1)} & {\scriptstyle \tau;d= \atop 2;(0,2)}
\end{array} \\
\left[\begin{array}{ccc|ccc}
\lambda(1)^2 & 2\lambda(1)\lambda(2) & \lambda(2)^2 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda(1)^2 & 2\lambda(1)\lambda(2) & \lambda(2)^2 \\ \hline
3 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 3
\end{array}\right]
\end{array}
\left[\begin{array}{c}
r(1;(2,0)) \\ r(1;(1,1)) \\ r(1;(0,2)) \\ \hline r(2;(2,0)) \\ r(2;(1,1)) \\ r(2;(0,2))
\end{array}\right]
\begin{array}{c}
= \\ = \\ \hline \leqslant \\ \leqslant \\ \leqslant \\ \leqslant
\end{array}
\left[\begin{array}{c}
P_1 \\ P_2 \\ \hline 1 \\ 1 \\ 1 \\ 1
\end{array}\right]
$$

The dual system is:

$$
\begin{aligned}
Z_1\lambda(1)^2 - 3u_{3,0} &\leqslant 0 && (23) \\
2Z_1\lambda(1)\lambda(2) - 2u_{2,1} &\leqslant 0 \\
Z_1\lambda(2)^2 - u_{1,2} &\leqslant 0 \\
Z_2\lambda(1)^2 - u_{2,1} &\leqslant 0 \\
2Z_2\lambda(1)\lambda(2) - 2u_{1,2} &\leqslant 0 \\
Z_2\lambda(2)^2 - 3u_{0,3} &\leqslant 0 && (24) \\
Z_1P_1 + Z_2P_2 - u_{3,0} - u_{2,1} - u_{1,2} - u_{0,3} &> 0 && (25)
\end{aligned}
$$

It is apparent that if the dual system has a solution, then it has a solution with $Z_1, Z_2 > 0$. Renumbering types if necessary, assume

$$Z_1/\lambda(1) \geqslant Z_2/\lambda(2). \tag{26}$$

Fixing Z, we can choose u to make inequalities (23–24) bind. Simply set

$$
\begin{aligned}
u_{3,0} &= Z_1\lambda(1)^2/3 \\
u_{2,1} &= \max\{Z_1\lambda(1)\lambda(2),\ Z_2\lambda(1)^2\} = Z_1\lambda(1)\lambda(2) \\
u_{1,2} &= \max\{Z_1\lambda(2)^2,\ Z_2\lambda(1)\lambda(2)\} = Z_1\lambda(2)^2 \\
u_{0,3} &= Z_2\lambda(2)^2/3,
\end{aligned}
$$

where the maxima are given by (26). Then (25) becomes

$$Z_1P_1 + Z_2P_2 > Z_1\lambda(1)^2/3 + Z_1\lambda(1)\lambda(2) + Z_1\lambda(2)^2 + Z_2\lambda(2)^2/3. \tag{27}$$

Now this can be rewritten as

$$
\begin{aligned}
\frac{Z_1}{\lambda_1}P_1\lambda(1) + \frac{Z_2}{\lambda_2}P_2\lambda(2) &> \frac{Z_1}{3\lambda_1}\Big(\lambda(1)^3 + \lambda(1)^2\lambda(2) + \lambda(1)\lambda(2)^2\Big) + \frac{Z_2}{3\lambda(2)}\lambda(2)^3 \\
&= \frac{Z_1}{3\lambda_1}\left(c((3,0)) + c((2,1)) + c((1,2))\right) + \frac{Z_2}{3\lambda(2)}c((0,3)),
\end{aligned}
$$

where the equalities are taken to define c

Multiply by $3\lambda(1)/Z_1$ to get

$$3\left(P_1\lambda(1) + \frac{Z_2\lambda(1)}{Z_1\lambda(2)}P_2\lambda(2)\right) > \big(c((3,0)) + c((2,1)) + c((1,2))\big) + \frac{Z_2\lambda(1)}{Z_1\lambda(2)}c((0,3)) \quad (28)$$

Case 1. If

$$3P_1\lambda(1) > c((3,0)) + c((2,1)) + c((1,2)),$$

the (**MRM′**) condition is violated for $A = \{1\}$.

Case 2. Otherwise, rearrange (28) as

$$3\frac{Z_2\lambda(1)}{Z_1\lambda(2)}\left(P_2\lambda(2) - c((0,3))\right) > c((3,0)) + c((2,1)) + c((1,2)) - 3P_1\lambda(1).$$

By (26), we have $Z_2\lambda(1)/Z_1\lambda(2) \leqslant 1$, so we can strengthen the inequality by writing

$$3\left(P_2\lambda(2) - c((0,3))\right) > c((3,0)) + c((2,1)) + c((1,2)) - 3P_1\lambda(1)$$

which can be rewritten as

$$3(P_1\lambda(1) + P_2\lambda(2)) > c((3,0)) + c((2,1)) + c((1,2)) + c((0,3)) = 1.$$

This violates the (**MRM′**) condition for $A = \{1,2\}$.

□

27.11 Simple rationality

27.12 Stochastic rationality

See my notes at http://www.hss.caltech.edu/~kcb/Notes/StochasticChoice.pdf, which are based on McFadden and Richter [26]

27.13 Concave rationality

See Afriat [1, 2], Diewert [10], Kannai [20], Richter and Wong [27], Richter and Matzkin [25], and Varian [31, 32, 33].

27.14 Dynamic Bayesian updating

Cf. Heath and Sudderth [19]

27.15 Representative voting

See Fishburn [14, 15].

27.16 Probabilities with given marginals

See Blackwell , and ****.

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