Topic 26: Polyhedra and polytopes

This chapter takes a closer look at the geometry of inequalities that was investigated in Chapter 25.

26.1 Solution sets, polyhedra, and polytopes

26.1.1 Definition *A* ***polyhedron*** *is a nonempty finite intersection of closed half spaces. In a finite dimensional space, a polyhedron is simply a solution set as defined in Section 4.1. A* ***polyhedral cone*** *is a cone that is also a polyhedron. A* ***polytope*** *is the convex hull of a nonempty finite set.*

Our goal is to show that in finite-dimensional spaces, the two kinds of sets are essentially the same. Specifically, we shall show that every polytope is a polyhedron and that every polyhedron is the algebraic sum of a polytope and a finitely generated convex cone, and that every bounded polyhedron is a polytope. (We allow finitely generated convex cones to be subspaces, including the degenerate subspace $\{0\}$.) We are also interested in computational methods for transforming one kind of description into the other.

26.2 Finitely generated cones

Recall that a finitely generated convex cone is the convex cone generated by a finite set. Given vectors $x_1, \dots, x_n$ let

$\langle x_1, \dots, x_n \rangle$ denote the finitely generated convex cone generated by $\{x_1, \dots, x_n\}$.

In particular, $\langle x \rangle$ is the ray generated by x. From Lemma 3.1.7 we know that every finitely generated convex cone is closed. Also in Exercise 3.1.6 you essentially proved the following, which is sometimes taken to be the definition of a finitely generated convex cone.

26.2.1 Proposition *The finitely generated convex cone $\langle x_1, \dots, x_n \rangle$ is the sum $\langle x_1 \rangle + \cdots + \langle x_n \rangle$ of rays.*

26.2.2 Lemma *The dual cone of a finitely generated convex cone in $\boldsymbol{R}^{\mathrm{m}}$ is a polyhedron.*

Proof: This is almost trivial. Let C be the finitely generated convex cone

$$C = \langle x_1 \rangle + \cdots + \langle x_n \rangle.$$

The dual cone C^* is defined to be

$$C^* = \{p \in \boldsymbol{R}^{\mathrm{m}} : p \cdot x \leqslant 0 \text{ for all } x \in C\}$$

but I claim that is also the set

$$A = \{p \in \boldsymbol{R}^{\mathrm{m}} : p \cdot x_i \leqslant 0,\ i = 1, \ldots, n\},$$

which is the intersection of half spaces $\bigcap_{i=1}^{n} \{x_i \leqslant 0\}$.

To see this last claim observe that if $p \in C^*$, then *a fortiori* $p \in A$, that is, $C^* \subset A$. On the other hand assume $p \in A$, and $x \in C$. Then x is of the form $x = \alpha_1 x_1 + \cdots + \alpha_n x_n$, where each $\alpha_i \geqslant 0$. Thus $p \cdot x = \alpha_1 p \cdot x_1 + \cdots + \alpha_n p \cdot x_n \leqslant 0$. Since x is an arbitrary element of C, we see that $p \in C^*$. This proves that $A \subset C^*$, so indeed $C^* = A$ is a polyhedron. ∎

The next step is to prove that the a polyhedral cone is also a finitely generated convex cone. This is more subtle than it sounds. We start with the following lemma.

26.2.3 Lemma *A finite dimensional linear subspace of a vector space is a finitely generated convex cone.*

Proof: To see this let M be an n-dimensional subspace of the vector space X, let $b_1, \ldots, b_n$ be a basis for M, and put

$$b_0 = -(b_1 + \cdots + b_n).$$

I claim that every point $x \in M$ is a *nonnegative* linear combination of $b_0, b_1, \ldots, b_n$. To see this, start by writing $x = \sum_{i=1}^{n} \alpha_i b_i$ as a linear combination of the basis vectors. Renumbering if necessary, assume that α_n is the least α_i, so $\alpha_i - \alpha_n \geqslant 0$ for each i. If $\alpha_n \geqslant 0$, there is nothing to do, but if $\alpha_n < 0$, observe that by the definition of b_0 we have $b_0 + b_1 + \cdots + b_n = 0$ so

$$x = \sum_{i=1}^{n} \alpha_i b_i = \sum_{i=1}^{n} \alpha_i b_i - \alpha_n \sum_{i=0}^{n} b_i = \sum_{i=1}^{n} (\alpha_i - \alpha_n) b_i - \alpha_n b_0,$$

which is a nonnegative linear combination of $b_0, \ldots, b_n$. ∎

The next result is an important but rather technical lemma.

26.2.4 Lemma *The intersection of a linear subspace and the nonnegative orthant in $\boldsymbol{R}^{\mathrm{m}}$ is a finitely generated convex cone.*

Figure 26.2.1 shows the intersection of the 2-dimensional subspace orthogonal to $(-1/2, -1/2, 1)$ with the nonnegative orthant of $\boldsymbol{R}^3$. It is the finitely generated convex cone $\langle(2, 0, 1)\rangle + \langle(0, 2, 1)\rangle$.

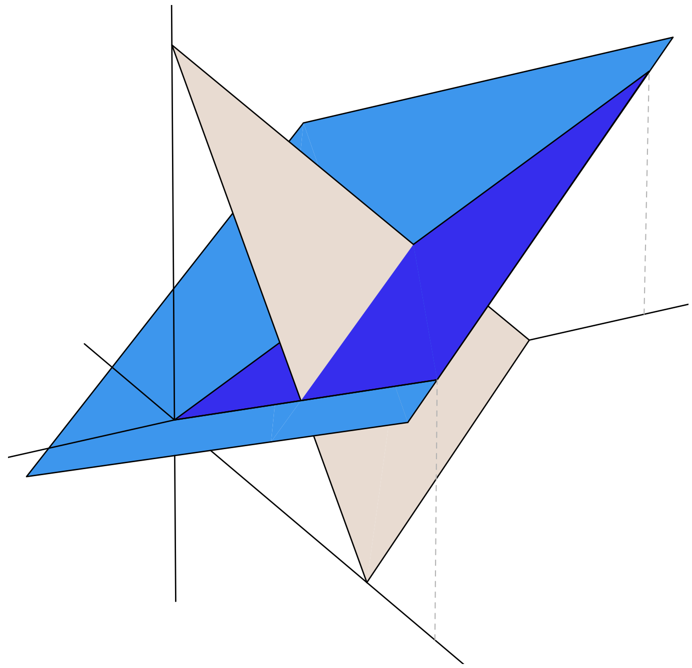

Figure 26.2.1. The intersection of the 2-dimensional subspace orthogonal to $(-1/2, -1/2, 1)$ with the nonnegative orthant of $\boldsymbol{R}^3$ is the finitely generated convex cone $\langle(2,0,1)\rangle + \langle(0,2,1)\rangle$. Also shown is the unit simplex.

Proof: Let M be a linear subspace of $\boldsymbol{R}^{\mathrm{m}}$, and let $M_+ = M \cap \boldsymbol{R}^{\mathrm{m}}_+$. If $M = \boldsymbol{R}^{\mathrm{m}}$, then $M_+ = \boldsymbol{R}^{\mathrm{m}}_+$, which is the finitely generated convex cone generated by the unit coordinate vectors. If $M = \{0\}$, then $M_+ = \{0\}$, which is the trivial finitely generated convex cone. So assume M is a proper nontrivial subspace.

- Step 1: M_+ is a cone.

  The orthogonal complement $M^\perp$ of M has a basis $p_1, \dots, p_k$, and

  $$M_+ = \Big\{x \in \boldsymbol{R}^{\mathrm{m}} : x \geqq 0, \ \& \ (\forall i = 1, \dots, k)\ [p_i \cdot x = 0]\Big\}.$$

  From this it is apparent that M_+ is closed under multiplication by positive scalars and is in fact a polyhedral cone.

- Step 2: M_+ is the cone generated by the nonnegative solutions of (1) below.

  If a nonzero vector $\bar{x}$ belongs to M_+, the sum σ of its coordinates, $\sigma = \mathbf{1} \cdot x$, is strictly positive, and $x = (1/\sigma)\bar{x}$ is a nonnegative solution of

$$\underbrace{\begin{bmatrix} \cdots & p_1 & \cdots \\ & \vdots & \\ & \vdots & \\ \cdots & p_k & \cdots \\ \cdots & \mathbf{1} & \cdots \end{bmatrix}}_{A} \begin{bmatrix} \vdots \\ \vdots \\ x \\ \vdots \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \tag{1}$$

where A is the $(k+1) \times m$ matrix whose i^{th} row is p_i, for $i = 1, \dots, k$ and the last row is the row vector $\mathbf{1}$. Since $\bar{x} = \sigma x$, is an arbitrary nonzero element of M_+, the cone M_+ is generated by the set of nonnegative solutions of (1).

Recall that a basic solution to a system $Ax = b$ of equations is a solution that depends on a linearly independent set of columns of A. We next show that every solution of (1) is a linear combination of basic solutions.

- Step 3: Every nonnegative solution of (1) is a linear combination of nonnegative basic solutions of (1).

  The following clever argument is taken from Gale [19, pp. 57–58]. Let x be a nonnegative solution of (1). The proof proceeds by induction on the number of nonzero coordinates of x. Let $\mathbb{P}(n)$ denote the proposition:

  > "If x is a nonnegative solution of (1), and x has at most n nonzero coordinates, then x is a linear combination of basic nonnegative solutions of (1)."

  – If the number of nonzero coordinates of x is 1, say $x_j \neq 0$, then (1) implies then $x_j = 1$ and A^j, the j^{th} column of A, is equal to the nonzero right-hand side of (1). Thus x itself is a basic solution, which proves that $\mathbb{P}(1)$ is true.

  – Now assume the induction hypothesis that $\mathbb{P}(n-1)$ is true. We now show that this implies the truth of $\mathbb{P}(n)$.

    So let $\bar{x}$ be a nonnegative solution of (1) with n nonzero coordinates. To ease notation, assume that we have renumbered things so that the first n components of $\bar{x}$ are nonzero. If the first n columns of A are linearly independent, then x is basic, and we are done.

    So assume that the first n columns $A^1, \dots, A^n$ are dependent, say

    $$\lambda_1 A^1 + \cdots + \lambda_n A^n = 0,$$

    where not all λ_i are zero. By Lemma 2.3.3 there is a solution $x' \geqq 0$ of (1) that depends on a linearly independent subset of the first n columns of A. Since x and x' are both solutions of (1), we have

    $$\mathbf{1} \cdot x = \mathbf{1} \cdot x' = 1.$$

But x' has fewer nonzero components than x since x' depends on a subset of the columns that x depends on. Therefore at least one component j satisfies

$$x'_j > x_j > 0.$$

Setting

$$\mu = \max_{j:x_j>0} x'_j/x_j \quad \text{we see that} \quad \mu > 1.$$

For the sake of concreteness suppose $\mu = x'_1/x_1$.

Since x and x' are both nonnegative, we see that

$$\mu x \geqq x' \quad \text{and} \quad (\mu x)_1 = x'_1.$$

Now set

$$x'' = \frac{1}{\mu - 1}(\mu x - x').$$

I claim that x'' is a nonnegative solution of (1): Clearly $p_i \cdot x'' = 0$ for $i = 1, \dots, k$ since $p_i \cdot x = p_i \cdot x' = 0$. And $\mathbf{1} \cdot x'' = 1$ since $\mathbf{1} \cdot x = \mathbf{1} \cdot x' = 1$. The nonnegativity of x'' follows from $\mu x \geqq x'$. But μ was chosen to make the first coordinates satisfy

$$x''_1 = 0, \quad \text{while} \quad x'_1 > 0,$$

so x'' has at most $n - 1$ nonzero components. Then $\mathbb{P}(n-1)$ implies that x'' is a linear combination of basic solutions. By construction x' is basic, so

$$x = \frac{x' + (\mu - 1)x''}{\mu},$$

is a linear combination of basic solutions.

The Principle of Induction thus shows that $\mathbb{P}[n]$ holds for any n. Thus any nonnegative solution of (1) is a linear combination of basic nonnegative solutions.

- Step 4: Since there are only finitely many independent sets of columns of A, there are finitely many basic solutions of (1), and these generate the cone M_+.

This completes the proof that $\boldsymbol{R}^{\mathrm{m}}_+ \cap M$ is a finitely generated convex cone. ∎

26.2.5 Corollary *The set of nonnegative solutions of a matrix equation $Ax = 0$ is a finitely generated convex cone.*

Proof: The set of solutions to $Ax = 0$ is linear subspace. Use the lemma. ∎

We now come to one of the main results of this section.

26.2.6 Theorem *Every polyhedral cone in $\boldsymbol{R}^{\mathrm{m}}$ is a finitely generated convex cone.*

Proof: Let C be a cone that is the polyhedron $\bigcap_{i=1}^{n} \{p_i \geqslant \alpha_i\}$. Since C is a cone it must be that each $\alpha_i = 0$, so that

$$C = \bigcap_{i=1}^{n} \{p_i \geqslant 0\} = \{x \in \boldsymbol{R}^{\mathrm{m}} : Px \geqq 0\}$$

where P is the $n \times m$ matrix where the rows are the p_i's.

Let M be the linear subspace of vectors in $\boldsymbol{R}^{\mathrm{n}}$ of the form Px,

$$M = \{Px \in \boldsymbol{R}^{\mathrm{n}} : x \in \boldsymbol{R}^{\mathrm{m}}\}.$$

The preceding Lemma 26.2.4 shows that $M_+ = \boldsymbol{R}^{\mathrm{n}}_+ \bigcap M$ is a finitely generated convex cone in $\boldsymbol{R}^{\mathrm{n}}$, say

$$M_+ = \langle y_1 \rangle + \cdots + \langle y_r \rangle.$$

In particular, each $y_i \in M$ so we may write

$$y_i = Px_i \quad \text{where } x_i \in \boldsymbol{R}^{\mathrm{m}},\ i = 1, \ldots, r.$$

Now observe that

$$x \in C \iff Px \in M_+ \iff Px \in \langle y_1 \rangle + \cdots + \langle y_r \rangle. \tag{2}$$

Thus $x \in C$ if and only if the vector Px can be written as

$$Px = \sum_{i=1}^{r} \lambda_i y_i = \sum_{i=1}^{r} \lambda_i P x_i, \qquad \lambda_i \geqslant 0,\ i = 1, \ldots, r,$$

or equivalently

$$Px - \sum_{i=1}^{r} \lambda_i P x_i = P\left(x - \sum_{i=1}^{r} \lambda_i x_i\right) = 0. \tag{3}$$

Now the linear subspace $\{z \in \boldsymbol{R}^{\mathrm{m}} : Pz = 0\}$ is a finitely generated convex cone (Corollary 26.2.5), say

$$\{z \in \boldsymbol{R}^{\mathrm{m}} : Pz = 0\} = \langle z_1 \rangle + \cdots + \langle z_s \rangle,$$

so by (3) we may write

$$x - \sum_{i=1}^{r} \lambda_i x_i = \sum_{j=1}^{s} \mu_j z_j, \qquad \mu_j \geqslant 0,\ j = 1, \ldots, s$$

or, rearranging,

$$x = \sum_{i=1}^{r} \lambda_i x_i + \sum_{j=1}^{s} \mu_j z_j \qquad \begin{array}{l} \lambda_i \geqslant 0,\ i = 1, \ldots, r, \\ \mu_j \geqslant 0,\ j = 1, \ldots, s \end{array}.$$

In other words,

$$C = \langle x_1 \rangle + \cdots + \langle x_r \rangle + \langle z_1 \rangle + \cdots + \langle z_s \rangle.$$

∎

26.2.7 Corollary *The dual cone of a finitely generated convex cone in $\boldsymbol{R}^{\mathrm{m}}$ is a finitely generated convex cone.*

Proof: Lemma 26.2.2 asserts that the dual cone of a finitely generated convex cone is polyhedral so Theorem 26.2.6 applies. ∎

Now comes the converse.

26.2.8 Corollary *Every finitely generated convex cone is a polyhedron.*

Proof: By the lemma just proven, if C is a finitely generated convex cone, then C^* a finitely generated convex cone. By Lemma 26.2.2 the dual C^{**} of the finitely generated convex cone C^* is a polyhedron. But $C^{**} = C$. ∎

26.3 Finitely generated cones and alternatives

The next result summarizes properties of finitely generated convex cones. It may be found for instance in Gale [19, Theorem 2.14] or [16].

26.3.1 Proposition (Properties of finitely generated convex cones and their duals) *The following apply to finitely generated convex cones in $\boldsymbol{R}^{\mathrm{m}}$.*

1. *The dual of a finitely generated convex cone is a finitely generated convex cone.*
2. *A finitely generated convex cone is the dual cone of its dual cone.*
3. *The sum of two finitely generated convex cones is a finitely generated convex cone.*
4. *The intersection of two finitely generated convex cones is a finitely generated convex cone.*

The following relations hold for finitely generated convex cones C_1 and C_2 in $\boldsymbol{R}^{\mathrm{m}}$.

5. $(C_1 + C_2)^* = C_1^* \cap C_2^*$.
6. $(C_1 \cap C_2)^* = C_1^* + C_2^*$.

Proof: (1) is just Corollary 26.2.7. The Bipolar Theorem 8.3.3 proves (2). Property (3) follows from the definitions.

Property (5) is true of arbitrary cones in $\boldsymbol{R}^{\mathrm{m}}$: If $p \cdot (x_1 + x_2) \leqslant 0$ for every $x_1 \in C_1$ and $x_2 \in C_2$, then setting $x_2 = 0$ we see that $p \in C_1^*$. Similarly $p \in C_2^*$, and therefore $p \in C_1^* \cap C_2^*$. For the reverse inclusion, if $p \in C_1^* \cap C_2^*$, then $p \cdot x_i \leqslant 0$ for $x_i \in C_i$, $i = 1, 2$. Adding these inequalities gives $p \cdot (x_1 + x_2) \leqslant 0$, that is, $p \in (C_1 + C_2)^*$.

Property (4) follows from the others: If C_1 and C_2 are finitely generated convex cones, then C_1^* and C_2^* are finitely generated convex cones by (1). Therefore $C_1^* + C_2^*$ is a finitely generated convex cone by (3), so $(C_1^* + C_2^*)^*$ is a finitely generated convex cone by 1 again. Observe that

$$(C_1^* + C_2^*)^* = C_1^{**} \cap C_2^{**} = C_1 \cap C_2,$$

where the first follows from (5) and the second from (1). Since the left-hand side is a finitely generated convex cone, so is the right-hand side.

For Property (6), it is clear that $(C_1 \cap C_2)^* \subset C_1^* + C_2^*$ for arbitrary cones in $\boldsymbol{R}^{\mathrm{m}}$. Now observe that if C_1 and C_2 are finitely generated convex cones, then $C_1 \cap C_2 = C_1^{**} \cap C_2^{**}$ is a finitely generated convex cone and by (5) we have

$$C_1 \cap C_2 = C_1^{**} \cap C_2^{**} = (C_1^* + C_2^*)^*.$$

Taking the dual of each side gives

$$(C_1 \cap C_2)^* = (C_1^* + C_2^*)^{**} = C_1^* + C_2^*.$$

∎

While Property (5) above holds for arbitrary cones in $\boldsymbol{R}^{\mathrm{m}}$, Property (6) need not. For example, in $\boldsymbol{R}^2$, let $C_1 = \{(0,0)\} \cup \{(x,y) : x > 0,\ y > 0\}$, and let $C_2 = \{(0,0)\} \cup \{(x,y) : x < 0,\ y > 0\}$. Then $(C_1 \cap C_2)^* = \boldsymbol{R}^2$, but $C_1^* + C_2^* = \{(x,y) : y \leqslant 0\}$.

26.3.2 Exercise Does Property (6) in Lemma above hold for closed convex cones in $\boldsymbol{R}^{\mathrm{m}}$? (Prove it or give a counterexample.) □

Sample answer: The answer is no. Let $A = \{(x,y,z) : z = 1, x > 0, y = 1/x\}$, $B = \{(x,y,z) : z = 1, x < 0, y = -1/x\}$. Let C be the cone generated by A, and let D be the cone generated by B.

Then C and D are closed, but $C + D = \{0\} \cup \{(x,y,z) : z \geqslant 0, y > 0\}$ is not closed.

Let $K = C^*$ and $L = D^*$. Now $(K \cap L)^*$ is closed, but $K^* + L^* = C + D$ is not closed. ∎

26.4 Polytopes

We have seen in the last section that finitely generated convex cones (finite sums of rays) and polyhedral cones (finite intersections of closed half-spaces) are the same objects. In this section we use that equivalence to prove that every polyhedron is the sum of a polytope and a finitely generated convex cone. We do this by taking a set A in $\boldsymbol{R}^{\mathrm{m}}$ and "translating it up" into $\boldsymbol{R}^{\mathrm{m}} \times \boldsymbol{R}$ to get the set $\hat{A} = \{\hat{x} : x \in A\}$, where

$$\hat{x} = (x, 1) \in \boldsymbol{R}^{\mathrm{m}} \times \boldsymbol{R} \quad \text{for} \quad x \in \boldsymbol{R}^{\mathrm{m}}.$$

We then consider the cone C generated by A, and observe that $A = \{x \in \boldsymbol{R}^{\mathrm{m}} : \hat{x} \in C\}$. See Figure 26.4.1. This procedure is called "homogenization," since convex cones are defined by homogeneous linear inequalities. The argument follows Ziegler [35, § 1.1] and rests on two relatively simple lemmas that give us what we need.

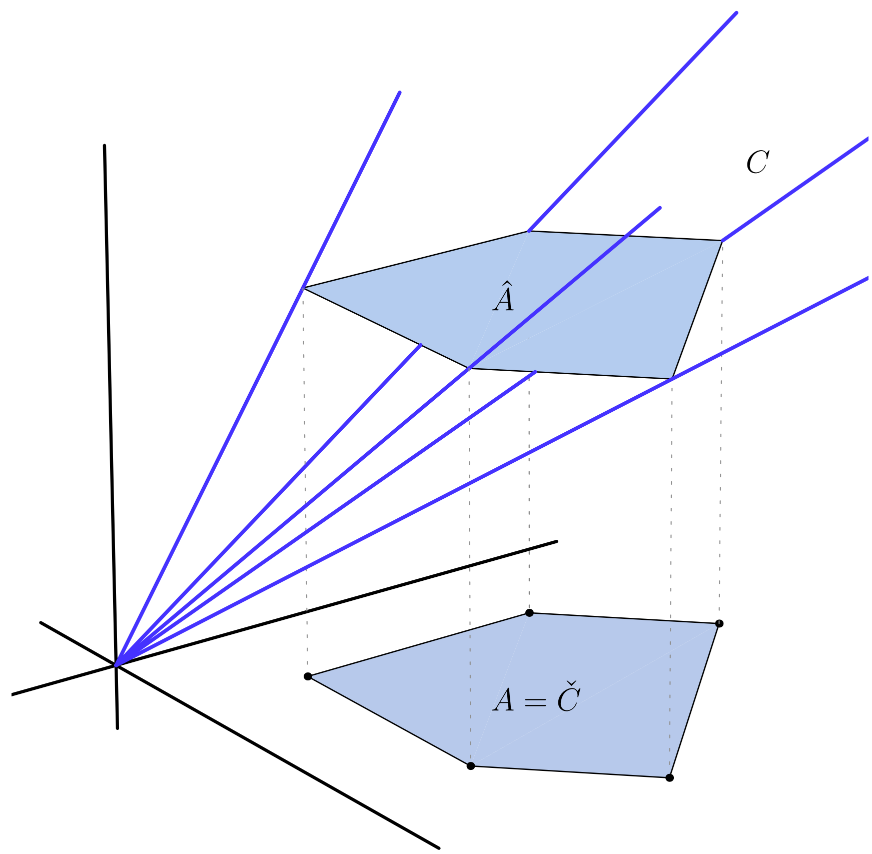

Figure 26.4.1

26.4.1 Lemma *Let P be a polyhedron in $\boldsymbol{R}^{\mathrm{m}} \times \boldsymbol{R}$. Then $\check{P} = \{x \in \boldsymbol{R}^{\mathrm{m}} : \hat{x} \in P\}$ is a polyhedron in $\boldsymbol{R}^{\mathrm{m}}$.*

Proof: Write a typical element in $\boldsymbol{R}^{\mathrm{m}} \times \boldsymbol{R}$ as (x, α) where $x \in \boldsymbol{R}^{\mathrm{m}}$ and $\alpha \in \boldsymbol{R}$. Now P is the intersection of finitely many half-spaces $\{(p_i, \gamma_i) \leqslant \beta_i\}$, $i = 1, \dots, n$. That is,

$$P = \{(x, \alpha)\boldsymbol{R}^{\mathrm{m}} \times \boldsymbol{R} : p_i \cdot x + \gamma_i \alpha \leqslant \beta_i,\ i = 1, \dots, n\}.$$

Then

$$\check{P} = \{x \in \boldsymbol{R}^{\mathrm{m}} : \hat{x} \in P\} = \{x \in \boldsymbol{R}^{\mathrm{m}} : p_i \cdot x + \leqslant \beta_i - \gamma_i,\ i = 1, \dots, n\},$$

which is the intersection of the half-spaces $\{p_i \leqslant \beta_i - \gamma_i\}$. ∎

26.4.2 Corollary *If K is a polytope, then it is a polyhedron.*

Proof: Let $K = \operatorname{co}\{x_1, \dots, x_k\} \subset \boldsymbol{R}^{\mathrm{m}}$ and let C be the finitely generated convex cone generated by $\{\hat{x}_1, \dots, \hat{x}_k\} \subset \boldsymbol{R}^{\mathrm{m}} \times \boldsymbol{R}$. Then by Corollary 26.2.8, C is a polyhedral cone. Now $K = \check{C} = \{x \in \boldsymbol{R}^{\mathrm{m}} : \hat{x} \in C\}$. (Why?) So by Lemma 26.4.1, K is a polyhedron. ∎

26.4.3 Lemma *If C is a finitely generated convex cone in $\boldsymbol{R}^{\mathrm{m}} \times \boldsymbol{R}_+ = \{(x, \alpha) \in \boldsymbol{R}^{\mathrm{m}} \times \boldsymbol{R} : \alpha \geqslant 0\}$, then $\check{C} = \{x \in \boldsymbol{R}^{\mathrm{m}} : \hat{x} \in C\}$ is the sum of a polytope and a finitely generated convex cone.*

Proof: The vectors that generate C can be normalized so that their $m + 1$st coordinate is either zero or one, so we may write C as

$$C = \left\langle \begin{bmatrix} v_1 \\ 1 \end{bmatrix} \right\rangle + \cdots + \left\langle \begin{bmatrix} v_k \\ 1 \end{bmatrix} \right\rangle + \left\langle \begin{bmatrix} y_1 \\ 0 \end{bmatrix} \right\rangle + \cdots + \left\langle \begin{bmatrix} y_n \\ 0 \end{bmatrix} \right\rangle .$$

Then you can verify that

$$\check{C} = \{x \in \boldsymbol{R}^{\mathrm{m}} : \hat{x} \in C\} = \operatorname{co}\{v_1, \dots, v_k\} + \operatorname{cone}\{y_1, \dots, y_n\},$$

which is the sum of a polytope and a finitely generated convex cone. ∎

26.4.4 Corollary *If P is a polyhedron, then it is the sum of a polytope and a finitely generated convex cone.*

Proof: Let P be the polyhedron $\{x \in \boldsymbol{R}^{\mathrm{m}} : p_i \cdot x \leqslant \beta_i,\ i = 1, \dots, n\}$. Then

$$C = \{(x, \alpha) \in \boldsymbol{R}^{\mathrm{m}} \times \boldsymbol{R} : \alpha \geqslant 0,\ p_i \cdot x - \alpha \beta_i \leqslant 0,\ i = 1, \dots, n\}$$

is a polyhedral cone in $\boldsymbol{R}^{\mathrm{m}} \times \boldsymbol{R}_+$. Therefore by Theorem 26.2.6, C is a finitely generated convex cone. You can verify that

$$P = \check{C} = \{x \in \boldsymbol{R}^{\mathrm{m}} : \hat{x} \in C\}.$$

Therefore by Lemma 26.4.3, P is the sum of a polytope and a finitely generated convex cone. ∎

26.4.5 Corollary *If P is a bounded polyhedron, then it is a polytope.*

This seems like a good time to recall Proposition 2.6.7, which we reprint here.

26.4.6 Proposition *Every polytope is the convex hull of the set of its extreme points.*

26.5 Extreme rays of finitely generated convex cones

This whole section is really inelegant. Find a better way to exposit this. I need a lot more pictures.

26.5.1 Definition *Let C be a convex cone in a vector space. Recall that a ray $R \subset C$ is called an **extreme ray** if whenever $x \in R$ can be written as a convex combination of points y, z of C, then in fact y and z also belong to R.*

The condition that y and z belong to the same ray implies that they are linearly dependent, and the definition is often written in terms of linear dependence.

Not all closed convex cones have extreme rays. For instance, if C is a nontrivial linear subspace, then C has no extreme rays.

However if C is a pointed cone (that is, $-C \cap C = \{0\}$) in $\boldsymbol{R}^{\mathrm{m}}$ then it has extreme rays, and indeed it is the closed convex hull of its extreme rays. I won't prove that general result here, but the monograph by Phelps [30] has an elegant exposition.

26.5.2 Definition *We say that a set A of vectors is **positively independent** if any strictly positive linear combination of vectors in A is nonzero. In other words A is positively independent if whenever $x_i \in A$ and $\lambda_i \geqslant 0$, $i = 1, \dots, n$,*

$$\lambda_1 x_1 + \cdots + \lambda_n x_n = 0 \implies \lambda_1 = \cdots = \lambda_n = 0.$$

The next result is a restatement of Gordan's Alternative 25.3.9 in terms of positive independence.

26.5.3 Lemma *The set $\{x_1, \dots, x_n\}$ is positively independent if and only if there is some nonzero p satisfying $p \cdot x_i < 0$ for $i = 1, \dots, n$.*

Proof: ($\implies$) Assume positive independence. Then 0 does not belong to the convex hull $K = \operatorname{co}\{x_1, \dots, x_n\}$. Thus by the Strong Separating Hyperplane Theorem, there is some nonzero p satisfying $0 = p \cdot 0 < p \cdot x$ for all $x \in K$. This is the p we want.

($\impliedby$) Assume $p \cdot x_i < 0$ for $i = 1, \dots, n$, let $\lambda_i \geqslant 0$ for all i, and assume that $\lambda_1 x_1 + \cdots + \lambda_n x_n = 0$. Then

$$0 = p \cdot 0 = \lambda_1 \underbrace{p \cdot x_1}_{<0} + \cdots + \lambda_n \underbrace{p \cdot x_n}_{<0},$$

which implies $\lambda_1 = \cdots = \lambda_n = 0$. ∎

26.5.4 Proposition (Properties of pointed finitely generated convex cones) *Let $x_1, \dots, x_n$ be nonzero vectors in $\boldsymbol{R}^{\mathrm{m}}$, and let $C = \langle x_1, \dots, x_n \rangle$ be the finitely generated convex cone they generate. (Hence C is nondegenerate.)* Relate Gordan's Alternative 25.3.9.

1. *The cone C is pointed if and only if $x_1, \dots, x_n$ are positively independent.*

2. *If C is pointed, then it has nondegenerate extreme rays, and each is of the form $\langle x_i \rangle$ for some i. That is, every extreme ray is one of the generators. (But not every x_i need be extreme.) Moreover, the cone C is the convex hull of its extreme rays.*

3. *The dual cone C^* of C is the polyhedron defined by*

$$C^* = \{p \in \boldsymbol{R}^{\mathrm{m}} : p \cdot x_i \leqslant 0 \text{ for all } i = 1, \dots, n.\}$$

If C is pointed,

$$C^* = \{p \in \boldsymbol{R}^{\mathrm{m}} : p \cdot x_i \leqslant 0 \text{ for all } i \text{ such that } \langle x_i \rangle \text{ is an extreme ray of } C\}$$

4. *If C is any cone that spans $\boldsymbol{R}^{\mathrm{m}}$, then C^* is pointed.*

Proof: (1) Assume first that C is pointed. Let and $\lambda_i \geqslant 0$, $i = 1, \dots, n$ and $\lambda_1 x_1 + \cdots + \lambda_n x_n = 0$. If some $\lambda_i > 0$, say $i = 1$, then the nonzero point $\lambda_1 x_1 = -(\lambda_2 x_2 + \cdots + \lambda_n x_n) = 0$ belongs to $-C \cap C$, a contradiction.

Assume $x_1, \dots, x_n$ are positively independent, and let x belong to $-C \cap C$. Then $x = \lambda_1 x_1 + \cdots + \lambda_n x_n = -(\mu_1 x_1 + \cdots + \mu_n x_n)$, so $0 = x - x = (\lambda_1 + \mu_1)x_1 + \cdots + (\lambda_n + \mu_n)x_n$, so by positive independence we conclude that $\lambda_i = \mu_i = 0$, for all i, which implies $x = 0$. Thus C is pointed.

(2) Assume that C is pointed. By part (1), the vectors $x_1, \dots, x_n$ are positively independent. Renumbering if necessary, let $x_1, \dots, x_k$ be a minimal (smallest in cardinality) subset of $x_1, \dots, x_n$ satisfying $C = \langle x_1, \dots, x_k \rangle$. I claim that the extreme rays of C are precisely $\langle x_1 \rangle, \dots, \langle x_k \rangle$.

To see this, suppose $x_i = y + z$ where $y, z \in C$. To ease notation, renumber so that $i = 1$. Write

$$y = \sum_{i=1}^{k} \lambda_i x_i \quad \text{and} \quad z = \sum_{i=1}^{k} \mu_i x_i, \quad \text{where} \quad \lambda_i, \mu_i \geqslant 0, \; i = 1, \dots, k. \tag{4}$$

Then

$$(1 - \lambda_1 - \mu_1)x_1 = \sum_{i=2}^{k} (\lambda_i + \mu_i) x_i. \tag{5}$$

There are three cases to consider. (i) If $1 - \lambda_i - \mu_i = 0$, then positive independence implies that $\lambda_i = \mu_i = 0$ for $i = 2, \dots, k$. So (4) implies that y and z are both multiples of x_1, and so linearly dependent. Thus the ray $\langle x_1 \rangle$ is an extreme ray. (ii) If $1 - \lambda_i - \mu_i > 0$, we may divide (5) by it and conclude that x_1 is a nonnegative linear combination of $x_2, \dots, x_k$, contradicting the minimality hypothesis, so this case is ruled out. (iii) If $1 - \lambda_i - \mu_i < 0$, we may divide (5) by it and conclude that $-x_1$ is a nonnegative linear combination of $x_2, \dots, x_k$, so $-C \cap C$ contains x_1, contradicting the hypothesis of pointedness. Thus every ray $\langle x_i \rangle$, $i = 1, \dots, k$ is extreme.

To see that no other ray is extreme, suppose that x is a nonzero point in C that is not any of the rays $\langle x_1 \rangle, \dots, \langle x_p \rangle$. Since these rays generate C it must be that x is a nonnegative linear combination of $x_1, \dots, x_k$ with at least two nonzero coefficients, which shows that x does not lie on an extreme ray.

This also shows that C is the convex hull of its extreme rays.

(3) Let $C' = \{p \in \boldsymbol{R}^{\mathrm{m}} : p \cdot x_i \leqslant 0,\ i = 1, \dots, n\}$. Clearly $C^* \subset C'$. The reverse inclusion is not much harder—if $p \in C'$ and $x \in C$, then $x = \sum_{i=1}^n \lambda_i x_i$, with $\lambda_i \geqslant 0$ so $p \cdot x = \sum_{i=1}^n \lambda_i p \cdot x_i \leqslant 0$, so $p \in C^*$. The result for pointed cones follows from part (2).

(4) Assume that C spans $\boldsymbol{R}^{\mathrm{m}}$ and let $p \in -C^* \cap C^*$. Then $p \cdot x \leqslant 0$ and $-p \cdot x \leqslant 0$ for all x in C. Thus $p \cdot x = 0$ for all x in C, and since C spans $\boldsymbol{R}^{\mathrm{m}}$, we have $p \cdot x = 0$ for all $x \in \boldsymbol{R}^{\mathrm{m}}$. This implies $p = 0$. Thus C^* is pointed. ∎

The next theorem characterizes the extreme rays of a finitely generated convex cone and is due to Weyl [33, 34]. See also Gerstenhaber [20].

26.5.5 Weyl's Facet Lemma *Let C be the finitely generated convex cone $\langle x_1, \dots, x_n \rangle$ in $\boldsymbol{R}^{\mathrm{m}}$. Then nonzero $p \in C^*$ belongs to an extreme ray of C^* if and only if* $\dim \operatorname{span}\{x_i : p \cdot x_i = 0\} = m - 1$.

Proof: (cf. Gale [19, Theorem 2.16]) Let $p \in C^*$ be nonzero, let $I^0 = \{i : p \cdot x_i = 0\}$, let $I^- = \{i : p \cdot x_i < 0\}$, and let $M = \operatorname{span}\{x_i : i \in I^0\}$.

($\implies$) Assume that $p \in C^*$ belongs to an extreme ray of C^* and assume that $\dim M < m - 1$. Then there is exists q independent of p satisfying $q \cdot x_i = 0$ for $i \in I^0$. For $\varepsilon > 0$ small enough we have $(p \pm \varepsilon q) \cdot x_i < 0$ for all $i \in I^-$, so $p \pm \varepsilon q \in C^*$. But $p + \varepsilon q$ and $p - \varepsilon q$ are linearly independent: If $\alpha(p + \varepsilon q) + \beta(p - \varepsilon q) = (\alpha + \beta)p + (\alpha - \beta)\varepsilon q = 0$, the independence of p and q implies $\alpha + \beta = \alpha - \beta = 0$, which in turn implies $\alpha = \beta = 0$. Thus we have written p as the sum of two independent vectors in C^*, so it is not extreme.

($\impliedby$) Assume that $\dim M = m - 1$. Then $L = \{q : q \cdot x_i = 0,\ i \in I^0\}$ is one-dimensional as $\dim M + \dim L = m$. Thus if $p = p_1 + p_2$ for $p_1, p_2 \in C^*$ we have $(p_1 + p_2) \cdot x_i = 0$, $p_j \cdot x_i \leqslant 0$, so $p_j \cdot x_i = 0$ for all $i \in I^0$. Thus $p, p_1, p_2 \in L$, which is one-dimensional, so p_1 and p_2 are dependent, proving that p is extreme. ∎

26.6 How many extreme rays can a dual cone have?

It is easy to see that if C is a pointed cone in $\boldsymbol{R}^2$ that spans $\boldsymbol{R}^2$ (that is, it has more than one ray), then in fact it has two extreme rays. It is also easy to see then that its dual cone also has two extreme rays. The same is true in $\boldsymbol{R}^3$, but seeing it takes a little more work. This might tempt you to believe that it is true in general, but that is not the case. Here is an example. Prove or give a cite.

26.6.1 Example Consider the finite convex cone C in $\boldsymbol{R}^4$ generated by the 5 columns of the 4×5 matrix

$$A = \begin{array}{c} \begin{array}{ccccc} a_1 & a_2 & a_3 & a_4 & a_5 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 9 & 16 & 25 \\ 1 & 8 & 27 & 64 & 125 \end{bmatrix} \end{array}$$

Then the cone C is just $C = \{Ax : x \geqq 0\}$.

It is easy to verify that every subset of $\{a_1, \dots, a_5\}$ of size four is linearly independent. Thus the cone C spans $\boldsymbol{R}^4$. It is also easy to see that C is pointed (that is, it contains no lines, only half-lines), as it is a subset of the nonnegative cone.

I claim that the dual cone C^* is generated by the 6 points $p_1, \dots, p_6$ that make up the 6 columns of the 4×6 matrix

$$P = \begin{array}{c} \begin{array}{cccccc} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 \end{array} \\ \begin{bmatrix} -60 & -30 & -10 & 6 & 12 & 20 \\ 47 & 31 & 17 & -11 & -19 & -29 \\ -12 & -10 & -8 & 6 & 8 & 10 \\ 1 & 1 & 1 & -1 & -1 & -1 \end{bmatrix} \end{array}$$

That is, $C^* = \{Pz : z \geqq 0\}$. Moreover, I claim that the cone C has five extreme rays (generated by $a_1, \dots, a_5$), and C^* has six extreme rays (generated by $p_1, \dots, p_6$).

Proof: We shall use Weyl's Lemma 26.5.5 to find the extreme rays of C^*. In our example $m = 4$ and $n = 5$. We shall use the "brute force" approach and look at *all* subsets of $\{a_1, \dots, a_5\}$ of rank 3. Since any four vectors belonging to $\mathcal{A}$ are linearly independent, a subset of $\mathcal{A}$ has rank 3 if and only if it has three elements. Fortunately there are only $\binom{5}{3} = 10$ of these subsets, so it is feasible to enumerate them by hand. Each subset B of size three determines a one-dimensional subspace in $\boldsymbol{R}^4$ (a line) consisting of vectors orthogonal to each element of B (the **orthogonal complement** of B). It is straightforward to solve for this subspace, and I have done so. Points p_i taken from each of these ten lines are used for the columns of the 4×10 matrix

$$\hat{P} = \begin{array}{c} \begin{array}{cccccccccc} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 & p_9 & p_{10} \end{array} \\ \begin{bmatrix} -60 & -30 & -10 & 6 & 12 & 20 & -40 & -24 & -15 & -8 \\ 47 & 31 & 17 & -11 & -19 & -29 & 38 & 26 & 23 & 14 \\ -12 & -10 & -8 & 6 & 8 & 10 & -11 & -9 & -9 & -7 \\ 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

(Note that you have seen $p_1, \ldots, p_6$ before.) Now construct the 5×10 matrix whose elements are the inner products $p_j \cdot a_i$:

$$
A'\hat{P} = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{array}
\begin{array}{c}
\begin{array}{cccccccccc} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 & p_9 & p_{10} \end{array} \\
\begin{bmatrix}
-24 & -8 & 0 & 0 & 0 & 0 & -12 & -6 & 0 & 0 \\
-6 & 0 & 0 & 0 & -2 & -6 & 0 & 0 & 3 & 0 \\
0 & 0 & -4 & 0 & 0 & -4 & 2 & 0 & 0 & -2 \\
0 & -2 & -6 & -6 & 0 & 0 & 0 & 0 & -3 & 0 \\
0 & 0 & 0 & -24 & -8 & 0 & 0 & 6 & 0 & 12
\end{bmatrix}
\end{array}
$$

For the first six columns, all the entries are nonpositive, so $p_1, \ldots, p_6$ each belong to C^*. However for columns 7 through 10, there are entries of both signs. This means that for $i = 7, \ldots, 10$, no nonzero multiple of p_j belongs to C^*.

Further inspection shows that

$$
\begin{aligned}
\{a_i : p_1 \cdot a_i = 0\} &= \{a_3, a_4, a_5\} \\
\{a_i : p_2 \cdot a_i = 0\} &= \{a_2, a_3, a_5\} \\
\{a_i : p_3 \cdot a_i = 0\} &= \{a_1, a_2, a_5\} \\
\{a_i : p_4 \cdot a_i = 0\} &= \{a_1, a_2, a_3\} \\
\{a_i : p_5 \cdot a_i = 0\} &= \{a_1, a_3, a_4\} \\
\{a_i : p_6 \cdot a_i = 0\} &= \{a_1, a_4, a_5\} \\
\{a_i : p_7 \cdot a_i = 0\} &= \{a_2, a_4, a_5\} \\
\{a_i : p_8 \cdot a_i = 0\} &= \{a_2, a_3, a_4\} \\
\{a_i : p_9 \cdot a_i = 0\} &= \{a_1, a_3, a_5\} \\
\{a_i : p_{10} \cdot a_i = 0\} &= \{a_1, a_2, a_4\}
\end{aligned}
$$

This accounts for all subsets of $\{a_1, \ldots, a_5\}$ of rank 3. So Weyl's Facet Lemma shows that C^* is generated by $p_1, \ldots, p_6$, which lie on distinct extreme rays of C^*.

As an aside, you should verify that

$$
\begin{aligned}
\{p_j : p_j \cdot a_1 = 0\} &= \{p_3, p_4, p_5, p_6\} && \text{has rank 3} \\
\{p_j : p_j \cdot a_2 = 0\} &= \{p_2, p_3, p_4\} && \text{has rank 3} \\
\{p_j : p_j \cdot a_3 = 0\} &= \{p_1, p_2, p_4, p_5\} && \text{has rank 3} \\
\{p_j : p_j \cdot a_4 = 0\} &= \{p_1, p_5, p_6\} && \text{has rank 3} \\
\{p_j : p_j \cdot a_5 = 0\} &= \{p_1, p_2, p_3, p_6\} && \text{has rank 3,}
\end{aligned}
$$

confirming that $a_1, \ldots, a_5$ are on distinct extreme rays of $C^{**} = C$. ∎

The points $a_1, \dots, a_5$ are multiples of five distinct nonzero points on the moment curve in $\boldsymbol{R}^4$. The **moment curve** in $\boldsymbol{R}^{\mathrm{m}}$ is the set of points of the form $(t, t^2, \dots, t^m)$, for $t > 0$. A polytope defined by points on the moment curve is called a **cyclic polytope**. See G. M. Ziegler [35, Example 0.6, pp. 10–13] for more on cyclic polytopes. McMullen [26] proves that the cyclic polytopes have the most faces for a given number of vertexes.

I used T. Christof and A. Loebel's computer program PORTA [4, 5] to compute the dual cone and the facets of C. The program uses the **Fourier–Motzkin Elimination Algorithm** described below with extensions due to N. V. Chernikova [2, 3] to efficiently find the six extreme rays of C^*. That left me with only four subsets of rank 3 to find the orthogonal complement by hand. After finding two by hand, I used Mathematica 5.0 to compute $p_7, \dots, p_{10}$ and all the inner products $p_j \cdot a_i$, and its `MatrixRank` function to double check the ranks. Feel free to check any of these computations by hand. □

> The moral of this example is that you should not trust your intuition about polyhedra in dimensions greater than three.

26.7 Fourier–Motzkin elimination

The next result is a generalization of Lemma 26.4.1.

26.7.1 Proposition (Projections of polyhedra) *Let C be a polyhedron in $\boldsymbol{R}^{\mathrm{m+1}}$. Then its projection on $\{z \in \boldsymbol{R}^{\mathrm{m+1}} : z_{\mathrm{m+1}} = 0\}$ is a polyhedron.*

Proof: We can write C as the set of $z \in \boldsymbol{R}^{\mathrm{m+1}}$ whose components satisfy a system of inequalities

$$\begin{aligned} \alpha_{1,1}z_1 + \cdots + \alpha_{1,m}z_m + \alpha_{1,m+1}z_{m+1} &\leqslant \beta_1 \\ &\vdots \\ \alpha_{i,1}z_1 + \cdots + \alpha_{i,m}z_m + \alpha_{i,m+1}z_{m+1} &\leqslant \beta_i \\ &\vdots \\ \alpha_{n,1}z_1 + \cdots + \alpha_{n,m}z_m + \alpha_{n,m+1}z_{m+1} &\leqslant \beta_n. \end{aligned} \tag{6}$$

Define the sets

$$P = \{i : \alpha_{i,m+1} > 0\}, \quad N = \{i : \alpha_{i,m+1} < 0\}, \quad Z = \{i : \alpha_{i,m+1} = 0\}.$$

We may rewrite the system (6) as

$$\begin{aligned} z_{m+1} &\leqslant \frac{\beta_i - \alpha_{i,1}z_1 - \cdots - \alpha_{i,m}z_m}{\alpha_{i,m+1}}, && i \in P \\ z_{m+1} &\geqslant \frac{\beta_i - \alpha_{i,1}z_1 - \cdots - \alpha_{i,m}z_m}{\alpha_{i,m+1}}, && i \in N \\ 0 &\leqslant \beta_i - \alpha_{i,1}z_1 - \cdots - \alpha_{i,m}z_m, && i \in Z. \end{aligned}$$

This system is equivalent to the following system

$$\begin{aligned} \frac{\beta_j - \alpha_{j,1}z_1 - \cdots - \alpha_{j,m}z_m}{\alpha_{j,m+1}} \leqslant z_{m+1} \leqslant \frac{\beta_i - \alpha_{i,1}z_1 - \cdots - \alpha_{i,m}z_m}{\alpha_{i,m+1}}, \quad i \in P,\ j \in N. \\ 0 \leqslant \beta_i - \alpha_{i,1}z_1 - \cdots - \alpha_{i,m}z_m, \quad i \in Z. \end{aligned} \tag{7}$$

Typically (7) will have many more inequalities than (6) (on the order of $n^2/4$ versus n), but we can **eliminate** z_{m+1} from (7), and consider the following system in m variables

$$\begin{aligned} \frac{\beta_j - \alpha_{j,1}z_1 - \cdots - \alpha_{j,m}z_m}{\alpha_{j,m+1}} \leqslant \frac{\beta_i - \alpha_{i,1}z_1 - \cdots - \alpha_{i,m}z_m}{\alpha_{i,m+1}}, \qquad i \in P,\ j \in N. \\ 0 \leqslant \beta_i - \alpha_{i,1}z_1 - \cdots - \alpha_{i,m}z_m, \qquad i \in Z. \end{aligned} \tag{8}$$

Now (8) has a solution in $\boldsymbol{R}^{\mathrm{m}}$ if and only if (6) has a solution in $\boldsymbol{R}^{\mathrm{m}+1}$. Indeed, $(z_1, \ldots, z_m) \in \boldsymbol{R}^{\mathrm{m}}$ is a solution of (8) if and only $(z_1, \ldots, z_m, 0) \in \boldsymbol{R}^{\mathrm{m}+1}$ belongs to the projection of C. Thus the projection is a polyhedron. ∎

The technique of eliminating z_{m+1} from the system of inequalities and expanding the number of inequalities is called **Motzkin elimination** or **Fourier–Motzkin elimination**.[1] If we iterate this procedure, we can reduce a system of inequalities in m variables to a much larger system in 1 variable. It is easy to verify whether this latter system has a solution, and if it does, we know the original solution has a solution. Thus Fourier–Motzkin elimination provides a test for the solvability of a system of inequalities.

26.7.2 Example (Using Fourier–Motzkin elimination) Consider the following system of three inequalities in the two variables x and y.

$$\begin{array}{rcl} x - y \leqslant 0 & & x \leqslant y & & x \leqslant y \\ -x \leqslant -3 & \iff & -x + 3 \leqslant 0 & \iff & -x + 3 \leqslant 0 \\ 2x + 3y \leqslant 6 & & 2x - 6 \leqslant -3y & & -\tfrac{2}{3}x + 2 \geqslant y \end{array}$$

[1] According to Dantzig and Eaves [6], "For years the method was referred to as the *Motzkin* Elimination Method. However, because of the odd grave-digging custom of looking for artifacts in long forgotten papers, it is now known as the *Fourier–Motzkin* Elimination Method and perhaps will eventually be known as the *Fourier–Dines–Motzkin* Elimination Method." They declined, however, to put their money where their collective mouth is and titled their paper "Fourier–Motzkin Elimination and its Dual." Here is some background: In 1826, Fourier [7, 10, 11, 12] used this method of elimination to reduce a special system of inequalities in three variables to a system in two variables. Dines [9] in 1919 and Motzkin [27] in 1934 used this method as a test of the existence of a solution in more general cases. The paper by Dines is expressed in terms of minors of the coefficient matrix and is not very easy to follow. In 1956 Kuhn [25] used Motzkin's method to prove the Farkas Alternative in a very clear and thorough exposition of the technique. I highly recommend Kuhn's paper to anyone interested in pursuing this further.

Combine the first and last inequality in the last system to eliminate y and get the resulting system:

$$\begin{array}{rl} x & \leqslant -\frac{2}{3}x + 2 \\ -x + 3 & \leqslant 0 \end{array} \iff \begin{array}{rl} -2 & \leqslant -\frac{5}{3}x \\ 3 & \leqslant x \end{array} \iff \begin{array}{rl} \frac{6}{5} & \geqslant x \\ 3 & \leqslant x \end{array}$$

These last two reduce to

$$\frac{6}{5} \geqslant 3, \quad \text{oops!}$$

which is false. Therefore the original system is inconsistent. □

26.8 The Double Description Method

We have seen that polyhedra and polytopes are essentially the same things, but given a description of an object as polyhedron, can we recover its vertices? Or given a polytope can we find its bounding hyperplanes? Let's start with the case of cones, because we can use homogenization to reduce the general problem to one for cones.

We know that if C is a finitely generated convex cone in $\boldsymbol{R}^{\mathrm{m}}$, so is its dual C^*. So if C is a finitely generated convex cone, there are finite sets $Y = \{y_1, \ldots, y_n\}$ and $P = \{p_1, \ldots, p_k\}$ and such that

$$C = \langle y_1 \rangle + \cdots + \langle y_n \rangle \quad \text{and} \quad C^* = \langle p_1 \rangle + \cdots + \langle p_k \rangle,$$

in which case we also have

$$C = \Big\{ x : (\forall p \in P)\ [\, p \cdot x \leqslant 0 \,] \Big\} = P^*,$$

which describes C as a polyhedron.

Recall that a closed convex cone C satisfies $C = C^{**}$.

26.8.1 Definition *A pair (P, Y) of finite sets of vectors in $\boldsymbol{R}^{\mathrm{m}}$ is a* ***double description pair*** *for the cone C if P generates C^* and Y generates C. That is,*

$$C = P^* = \operatorname{cone} Y.$$

Note that unless we require $y_1, \ldots, y_n$ and $p_1, \ldots, p_k$ to be distinct extreme rays that this representation is not unique.

26.8.2 Proposition *The pair (P, Y) is a double description for a cone C if and only if the pair (Y, P) is a double description for the cone C^*.*

Proof: This is obvious given that $C^{**} = C$. ■

This definition is *not* the standard definition of a double description pair. It is traditional to think of $P = \{p_1, \dots, p_k\}$ as a $k \times m$ matrix whose i^{th} row is p_i and $Y = \{y_1, \dots, y_n\}$ as the $m \times n$ matrix whose j^{th} column is y^j. Then

$$C = \{Y\lambda : \lambda \in \boldsymbol{R}^{\mathrm{n}}_+\} = \{x \in \boldsymbol{R}^{\mathrm{m}} : Px \leqq 0\}.$$

Proposition 26.8.2 can be restated in matrix terms as follows. (Recall that for a matrix A its transpose is denoted A'.)

26.8.3 Proposition *The pair (P, Y) is a double description for a cone C if and only if the pair (Y', P') is a double description for the cone C^*.*

The **double description method** is an algorithm for finding Y given P. Or vice versa by using the dual cone. It is due to is due to Motzkin, Raiffa, Thompson, and Thrall [29]. The discussion here is influenced by that of Fukuda and Prodon [15].

The idea behind the algorithm is this:

Enumerate $P = \{p_1, \dots, p_k\}$. For each $t = 1, \dots, k$, define $P^t = \{p_1, \dots, p_t\}$, so that P^k is the original set P of generators of the dual cone of C. We shall show how to recursively construct sets Y^t, $t = 1, \dots, k$, so that at each stage (P^t, Y^t) is a double description pair for the cone $C_t = P^{t*}$. That is, Y^t is a set of generators for C_t. Note that each additional vector p^t imposes additional constraints on P^{t-1*}, so $C_1 \supset C_2 \supset \cdots \supset C_k = C$.

We start with $P^1 = \{p_1\}$. We want a set Y^1 such that $\operatorname{cone} Y^1$ is the half-space $\{p_1 \leqslant 0\}$. One way to do this is to first find a basis $z_1, \dots, z_{m-1}$ for the $m-1$ dimensional linear subspace $\{p_1 = 0\}$, say by using the technique in Example 25.9.4. Then the subspace $\{p_1 = 0\}$ is the finitely generated convex cone $\langle z_1, \dots, z_{m-1}, z_m \rangle$, where $z_m = -(z_1 + \cdots + z_{m-1})$ (Lemma 26.2.3). Finally, observe that the half-space $\{p_1 \leqslant 0\}$ is the finitely generated convex cone $\langle z_1, \dots, z_{m-1}, z_m, -p_1 \rangle$. Thus we may take $Y^1 = \{z_1, \dots, z_{m-1}, z_m, -p_1\}$. Note that the cardinality of Y^1 is $m + 1$, where m is the dimension of the space.

Now suppose we have constructed a double description pair (P^{t-1}, Y^{t-1}) for the cone $C_{t-1} = P^{t-1*}$. The set Y^t is constructed as follows:

Enumerate Y^{t-1} as $\{y_j : j \in J\}$. By construction, $p_i \cdot y_j \leqslant 0$ for all $j \in J$ and $i < t$. So we now compute the inner product $p_t \cdot y_j$ for each $j \in J$. Let

$$J^+ = \{j \in J : p_t \cdot y_j > 0\}, \quad J^0 = \{j \in J : p_t \cdot y_j = 0\}, \quad J^- = \{j \in J : p_t \cdot y_j < 0\}.$$

Now for $\ell \in J^+$, the ray $\langle y_\ell \rangle$ cannot belong to the cone P^{t*}, so we have to discard y_ℓ and replace it with a set of other points that lie on the hyperplane $\{p_t = 0\}$. So for each $\ell \in J^+$ and $h \in J^-$, we find a point z that is a convex combination of y_ℓ and y_h which satisfies $p_t \cdot z = 0$. (Since $p^t \cdot y_\ell > 0$ and $p^t \cdot y_h < 0$, at some point $z = (1-\alpha)y_\ell + \alpha y_h$ on the line segment joining y_ℓ and y_h the value of $p_t \cdot z$ is zero.) The z we want is given by

$$z(\ell, h) = \frac{(p_t \cdot y_\ell)y_h - (p_t \cdot y_h)y_\ell}{p_t \cdot y_\ell - p_t \cdot y_h}.$$

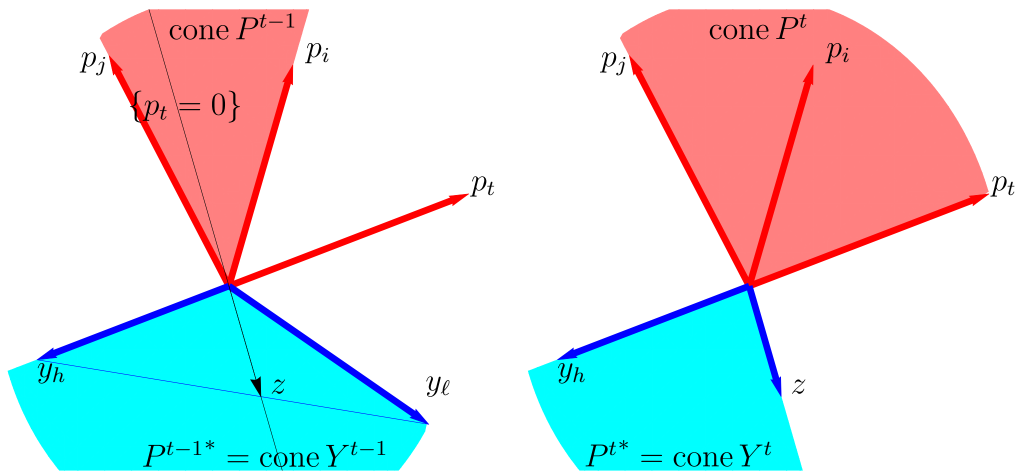

Figure 26.8.1. Finding the point $z(\ell, h)$ and the new Y^t when adding p^t.

See Figure 26.8.1. Now we set

$$Y^t = \{y_j : j \in J^- \cup J^0\} \cup \{z(\ell, h) : (\ell, h) \in J^+ \times J^-\}.$$

Observe that every such $z(\ell, h)$ already belongs to $\operatorname{cone} Y^{t-1}$, so $\operatorname{cone} Y^{t-1} \supset \operatorname{cone} Y^t$. In general, many of these z vectors are redundant. Also note that Y^t can be much larger (in cardinality) than Y^{t-1}. The worst case is when $|J_+| = |J_-| = |Y^{t-1}|/2$, so that $|Y^t| = |Y^{t-1}|^2/4$.

We now need to show that (P^t, Y^t) is a double description pair for P^{t*}:

Lemma: $\operatorname{cone} Y^t = P^{t*}$.

Proof: By construction, if $y \in Y^t$ and $p \in P^t$, then $p \cdot y \leqslant 0$, so $\operatorname{cone} Y^t \subset P^{t*}$.

For the reverse inclusion, let x belong to P^{t*}, that is, $p \cdot x \leqslant 0$ for all $p \in P^t$. Then a fortiori, $p \cdot x \leqslant 0$ for all $p \in P^{t-1} \subset P^t$. Therefore $x \in P^{t-1*}$. By hypothesis (P^{t-1}, Y^{t-1}) is a double description pair, so $x \in \operatorname{cone} Y^{t-1}$. So write

$$x = \sum_{j \in J^+} \lambda_j y_j + \sum_{j \in J^-} \lambda_j y_j + \sum_{j \in J^0} \lambda_j y_j, \qquad \text{each } \lambda_j \geqslant 0. \tag{9}$$

If $\lambda_j = 0$ for all $j \in J^+$, then $x \in Y^t$ and we are done. If $\lambda_\ell > 0$ for some $\ell \in J^+$, we show how to eliminate y_ℓ from (9) and replace it with points from Y^t. Since $\ell \in J^+$, we have $p_t \cdot y_\ell > 0$. By assumption $p_t \cdot x \leqslant 0$, so there must be some offsetting $h \in J^-$ with $\lambda_h > 0$. Let $\alpha = p_t \cdot y_\ell - (p_t \cdot y_h$. Then $z(\ell, h) = \frac{(p_t \cdot y_\ell) y_h - (p_t \cdot y_h) y_\ell}{\alpha}$ belongs to Y^t. Adding and subtracting $\gamma z(\ell, h)$ from (9) leads to

$$x = (\lambda_\ell + \gamma \frac{p_t \cdot y_h}{\alpha}) y_\ell + (\lambda_h - \gamma \frac{p_t \cdot y_\ell}{\alpha}) y_h + \gamma z(\ell, h) + \sum_{j \in J \setminus \{\ell, h\}} \lambda_j y_j. \tag{9$'$}$$

We need to choose $\gamma > 0$ so that both $\lambda_\ell + \gamma\frac{p \cdot y_h}{\alpha} \geqslant 0$ and $\lambda_h - \gamma\frac{p \cdot y_\ell}{\alpha}$ and one of them is equal to zero. That is, set

$$\gamma = \min\left\{\frac{\alpha\lambda_\ell}{-p_t \cdot y_h}, \frac{\alpha\lambda_j}{p_t \cdot y_\ell}\right\}.$$

(This is an example of the technique noted in Remark 2.3.4.) If the minimum occurs for $\gamma = -\alpha\lambda_\ell / p_t \cdot y_h$, then the coefficient on y_ℓ in (9') is zero, and we are done. If not, then the coefficient on λ_h is zero. This may not seem helpful, but note that in this case (9') expresses x as a linear combination that depends on one fewer vector in J^-. Since by construction, $p_t \cdot z(\ell, h) = 0$, if $\lambda'_\ell = \lambda_\ell + \gamma p_t \cdot y_h > 0$, then there is still some $h' \in J^- \setminus \{h\}$ with $\lambda_{h'} > 0$. We can repeat the same argument as often as needed until the coefficient on $y_\ell = 0$. (Since $p_t \cdot x \leqslant 0$ we cannot run out of indices in J^- before the coefficient on y_ℓ is zero.)

This can be done for every $j \in J^+$, so x can be written as a nonnegative linear combination of elements of Y^t. This completes the proof that (P^t, Y^t) is a double description pair. ∎

This process is iterated until $t = k$.

This algorithm can be modified to deal with general polyhedra, not just polyhedral cones. A major problem with this algorithm is that the number of points of Y^t can grow extremely large. It also turns out that the order of the points of P can make a huge difference in the number of steps. Practical implementations use Weyl's Facet Lemma 26.5.5 to eliminate redundant generators. See Ziegler [35, Notes, pp. 47–49], Fukuda and Prodon [15] and Fukuda [13] for more on computation.

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