Caltech Division of the Humanities and Social Sciences

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Convex Analysis and Economic Theory AY 2019–2020

Topic 22: Introduction to posets and lattices

22.1 Partially ordered sets

A **partial order** or **partial ordering** is a binary relation $\succeq$ on a set X. The statement that satisfies the following properties:

1. $\succeq$ is **reflexive**, that is, for all $x \in X$,
$$x \succeq x.$$
2. $\succeq$ is **transitive**, that is, for all $x, y, z \in X$
$$x \succeq y \ \&\ y \succeq z \implies x \succeq z.$$
3. $\succeq$ is **antisymmetric**, that is, for all $x, y \in X$,
$$x \succeq y \ \&\ y \succeq x \implies x = y.$$

The expression $x \succeq y$ is read "x is greater than or equal to y." We write $x \succ y$, read x is strictly greater than y," to mean $x \succeq y$, but $x \neq y$. We may also write $y \preceq x$ to mean $x \succeq y$, or $y \prec x$ to mean $x \succ y$.

A pair $(X, \succeq)$, where X is a nonempty set and $\succeq$ is a partial order is called a **partially ordered set**, sometimes called a **poset**. Two elements x and y of a partially ordered set are **ordered** if either $x \succeq y$ or $y \succeq x$.

A partial order $\succeq$ is a **linear order** if it is **complete**, that is, every pair is ordered. A linearly ordered subset of a partially ordered set is called a **chain**. Note that a partial order (and hence a linear order) does not allow for "indifference," that is, we cannot have $x \succeq y$, $y \succeq x$, and $x \neq y$.

22.1.1 Example Here are some familiar examples of partially ordered sets.

1. The usual ordering $\geqslant$ of the real numbers $\boldsymbol{R}$ is a partial order, in fact a linear order.
2. Set inclusion $\supset$ is a partial order on the power set of a set X. (Remember that $A \subset B$ allows for $A = B$.)
3. The pointwise ordering $\geqslant$ of real-valued functions on a set X is a partial order, where $f \geqslant g$ if $f(x) \geqslant g(x)$ for all $x \in X$. This includes as a special case the coordinatewise ordering $\geqq$ on $\boldsymbol{R}^{\mathrm{n}}$, where $x \geqq y$ if $x_i \geqslant y_i$ for each $i = 1, \ldots, n$.

4. First order stochastic dominance is a partial order of probability distributions (or random variables). Ditto for riskiness.

5. The set Bernoulli utility functions is partially ordered by de Finetti–Arrow–Pratt risk aversion.

□

It is often useful for examples to describe a finite partially ordered set in terms of its minimal directed graph. In this kind of diagram, points x and y are connected by an arrow from x to y if $x \succ y$ and there is no z with $x \succ z \succ y$. (In this case we might say that x is an immediate successor of y. If there were such a z, transitivity would imply $x \succ y$, so the arrow from x to y would be redundant.) See Example 22.3.5 and Figure 22.3.1 for an example of such a diagram. This kind of diagram, minus the arrowheads, is sometimes known as a **Hasse diagram**. Without the arrowheads, the direction of the relation ($x \succ y$ or $y \succ x$) is to be inferred from their relative vertical positions.

An element x is an **upper bound** for a set A in a partially ordered set X if $x \succeq y$ for each $y \in A$. It is a **lower bound** if $y \succeq x$ for each $y \in A$. An element x is the **greatest element** of A if it belongs to A and is an upper bound for A. An element x is the **least element** of A if it belongs to A and is an lower bound for A. The element x is a **maximal element** of A if it belongs to A and there is no y belonging to A with $y \succ x$. The element x is a **minimal element** of A if it belongs to A and there is no y belonging to A with $y \prec x$.

The element x is the **least upper bound** or **supremum** of A if it is the least element of the set of upper bounds of A. The element x is the **greatest lower bound** or **infimum** of A if it is the greatest element of the set of lower bounds of A.

22.1.2 Exercise The greatest element of A, if it exists, is unique. The greatest element is the unique maximal element of A. □

22.2 Zorn's Lemma

A number of propositions are equivalent to the Axiom of Choice. One of these is Zorn's Lemma, due to M. Zorn [6]. That is, Zorn's Lemma is a theorem if the Axiom of Choice is assumed, but if Zorn's Lemma is taken as an axiom, then the Axiom of Choice becomes a theorem.

22.2.1 Zorn's Lemma *If every chain in a partially ordered set X has an upper bound, then X has a maximal element.*

22.3 Lattices

A **lattice** is a partially ordered set $(X, \succeq)$ where every pair x, y in X has a supremum and an infimum (in X). The supremum of a pair x, y is also called the **join** of x and y, denoted $x \vee y$. The infimum is also called the **meet**, denoted $x \wedge y$. Note that by Exercise 22.1.2, the meet and join are unique whenever they exist.

The meet and join are called **lattice operations**. It is convenient to restate the definitions as follows. For any x, y, z, to show that $z = x \wedge y$, that is, z is the greatest lower bound of the set $\{x, y\}$, we need to show three things: $x \succeq z$, $y \succeq z$, and

$$(x \succeq u \;\&\; y \succeq u) \quad \Longrightarrow \quad z \succeq u.$$

Likewise, to show that $z = x \vee y$, that is, z is the least upper bound of the set $\{x, y\}$, we need to show: $z \succeq x$, $z \succeq y$, and

$$(u \succeq x \;\&\; u \succeq y) \quad \Longrightarrow \quad u \succeq z.$$

The following facts are now obvious (given the reflexivity of $\succeq$) and will be used over and over without any special mention. Let X be a lattice. For every $x, y, z \in X$, Draw [ictures,

1. $x \vee y \succeq x \succeq x \wedge y$. [1]
2. If $x \succeq y$, then $x = x \vee y$ and $y = x \wedge y$.
3. $x = x \wedge x = x \vee x$.
4. $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$.
5. $z \succeq x \vee y$ if and only if $(z \succeq x \;\&\; z \succeq y)$. Likewise $z \preceq x \wedge y$ if and only if $(z \preceq x \;\&\; z \preceq y)$.

The next facts are only a little less obvious.

22.3.1 Exercise (Associativity) Let X be a lattice. For every $x, y, z \in X$,

$$x \wedge (y \wedge z) = (x \wedge y) \wedge z \quad \text{and} \quad x \vee (y \vee z) = (x \vee y) \vee z.$$

□ Write out an answer.

22.3.2 Exercise Every nonempty finite lattice has a greatest and a least element. Every linearly ordered set is a lattice. □

22.3.3 Example Here are some familiar examples of lattices and partially ordered sets that are or are not lattices.

[1] A word on precedence: $x \vee y \succeq x$ means $(x \vee y) \succeq x$, which should be apparent from the fact that $x \vee (y \succeq x)$ is meaningless.

1. The numbers $\boldsymbol{R}$ with the usual ordering $\geqslant$ is a lattice. (This is obvious, but it also follows from Exercise 22.3.2, as it is a linearly ordered set.)
2. The power set is a lattice under set inclusion $\supset$. Indeed $A \vee B = A \cup B$ and $A \wedge B = \cap B$.
3. The set $\boldsymbol{R}^{\mathrm{n}}$ with the coordinatewise order $\geqq$ is a lattice, where $x \wedge y = (x_1 \wedge y_1, \dots, x_n \wedge y_n)$ and $x \vee y = (x_1 \vee y_1, \dots, x_n \vee y_n)$.

   But note that under the order $\succeq$, defined by $x \succeq y$ if $x = y$ or $x_i > y_i$ for $i = 1, \dots, n$, the set $\boldsymbol{R}^{\mathrm{n}}$ is not a lattice.
4. The set of linear subspaces of a linear space X is a lattice under set inclusion, but here $M \vee N = \operatorname{span} M \cup N$.
5. The set of continuous real-valued functions on a topological space is a lattice under the pointwise order, and $(f \vee g)(x) = f(x) \vee g(x)$ and $(f \wedge g)(x) = f(x) \wedge g(x)$ for each x. I leave it to you to prove that $f \vee g$ and $f \wedge g$ are continuous.
6. The set of differentiable functions on a real interval is *not* a lattice under the pointwise order. To see this, let $f(x) = x$ and $g(x) = -x$, and ask yourself what $f \vee g$ would have to be.

□

22.3.4 Definition *A lattice is* ***distributive*** *if if for all x, y, and z we have*

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

and

$$x \vee (y \wedge z) = (x \wedge y) \vee (x \wedge z).$$

If you are like me, you might have guessed that every lattice is distributive, and you would be wrong.

22.3.5 Example (A nondistributive lattice) Let $X = \{u, v, x, y, z\}$ and define $\succeq$ by $u \succeq x \succeq v$, $u \succeq y \succeq v$, and $u \succeq z \succeq v$. See Figure 22.3.1. Then $(X, \succeq)$ is a lattice. But

$$x \wedge (y \vee z) = x \wedge u = x \quad \text{and} \quad (x \wedge y) \vee (x \wedge z) = v \vee v = v,$$

and

$$x \vee (y \wedge z) = x \vee v = x \quad \text{and} \quad (x \vee y) \wedge (x \vee z) = u \vee u = u.$$

Thus no distributive law holds. □

A **sublattice** of $(X, \succeq)$ is a subset A of X that contains the meet and join (in X) of each pair of elements of A. This is not the same as A being a lattice in its own right. It is possible that A (ordered by $\succeq$) is a lattice, but the meet and join of x and y may be different in A than in X, so be careful. See Section 22.5 for an important example.

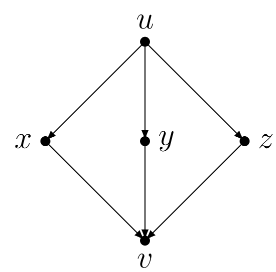


Figure 22.3.1. A non-distributive lattice.

22.4 Lattice homomorphisms

A **lattice homomorphism** between lattices $(X, \succeq_X)$ and $(Y, \succeq_Y)$ is a function $\varphi \colon X \to Y$ satisfying

$$\varphi(x \wedge_X z) = \varphi(x) \wedge_Y \varphi(z), \quad \text{and} \quad \varphi(x \vee_X z) = \varphi(x) \vee_Y \varphi(z),$$

A **lattice isomorphism** is one-to-one lattice homomorphism.

22.5 The lattice $\mathcal{K}$ of compact convex sets

Let $\mathcal{K}$ denote the collection of compact convex subsets of $\boldsymbol{R}^{\mathrm{m}}$. It is a lattice under the usual order of set inclusion,

$$K \succeq C \iff K \supset C.$$

The lattice operations are

$$K \wedge C = K \cap C, \qquad K \vee C = \operatorname{co}(K \cup C).$$

The latter equivalence follows from the fact that the convex hull of the union of two compact sets is compact (Lemma 2.1.6). This is an example where $\mathcal{K}$ has the same partial order as the power set, and is itself a lattice, but is not a sublattice of the power set since $K \cup C$ is not generally convex.

22.6 The lattice $\mathcal{S}$ of continuous sublinear functions

Recall that the profit function π_A of a subset A of $\boldsymbol{R}^{\mathrm{m}}$ is defined by $\pi_A(p) = \sup_{x \in A} p \cdot x$. When A is nonempty, compact, and convex, then π_A is a continuous, finite-valued, sublinear function on $\boldsymbol{R}^{\mathrm{m}}$. Conversely, every continuous, finite-valued sublinear function π on $\boldsymbol{R}^{\mathrm{m}}$ is the profit function of some nonempty compact convex set. Cite the theorems.

We can put the usual pointwise partial order on the set $\mathcal{S}$ of continuous sublinear functions. This set is a lattice where $f \vee g$ is just the usual pointwise maximum of f and g. The meet however is more subtle. The pointwise minimum need not be convex, so we need to define the meet to be the affine envelope of the pointwise minimum.

22.7 The lattice isomorphism of $\mathcal{K}$ and $\mathcal{S}$

I now assert that the mapping

$$K \mapsto \pi_K$$

from $\mathcal{K}$ to $\mathcal{S}$ is a lattice isomorphism. That is,

$$K \supset C \iff \pi_K \geqq \pi_C,$$
$$\pi_{K \vee C} = \pi_{\operatorname{co}(K \cup C)} = \max\{\pi_K, \pi_C\} = \pi_K \vee \pi_C,$$
$$\pi_{K \wedge C} = \pi_{K \cap C} = \text{affine envelope } \min\{\pi_K, \pi_C\} = \pi_K \wedge \pi_C.$$

If $K \cap C = \varnothing$, $\pi_\varnothing$ is an improper convex function, the constant $-\infty$, and we shall agree to make it an "honorary" sublinear function. It is also the affine envelope of $\min\{\pi_K, \pi_C\}$. The details may be found in Aliprantis and Border [1, Section 7.10, pp. 288–292].

Even more is true:

$$\pi_{K+C} = \pi_K + \pi_C,$$
$$\pi_{\alpha K} = \alpha \pi_K \text{ for } \alpha > 0,$$
$$K_n \downarrow K \iff \pi_{K_n} \downarrow \pi_K.$$

22.8 Aside: More on profit functions

Recall that the profit function $\pi_C \colon X \to \boldsymbol{R}^\sharp$ is defined by

$$\pi_C(p) = \sup\Big\{p \cdot x : x \in C\Big\}.$$

Note that this supremum may be ∞ if C is not compact. Given an extended real-valued sublinear function $h \colon X \to (-\infty, \infty]$, define

$$C_h = \Big\{p \in X' : p \cdot x \leqslant h(x) \text{ for all } x \in X\Big\}.$$

That is, C_h is the set of linear functionals that are dominated by h.

Also recall that the profit function of a nonempty closed convex subset is a proper sublinear and lower semicontinuous functional.

Conversely, if $h \colon X \to (-\infty, \infty]$ is a proper lower semicontinuous sublinear function, then C_h is a nonempty closed convex subset of X'.

Furthermore, we have the duality $C = C_{\pi_C}$ and $h = \pi_{C_h}$.

If, in addition, the set C is compact, we can say more, namely that its support functional is finite and continuous.

22.8.1 Theorem *Let K be a nonempty compact convex subset of $\boldsymbol{R}^{\mathrm{n}}$. Then the profit functional π_K is a proper continuous sublinear function on $\boldsymbol{R}^{\mathrm{n}}$.*

Conversely, if $h\colon X \to \boldsymbol{R}$ is a continuous sublinear function, then K_h is a nonempty compact convex subset of $\boldsymbol{R}^{\mathrm{n}}$.

Furthermore, we have the duality $K = K_{\pi_K}$ and $h = \pi_{K_h}$.

We take this opportunity to point out the following simple results.

22.8.2 Lemma *For a dual pair $\langle X, X'\rangle$ we have the following.*

1. *The support functional of a singleton $\{p\}$ is p itself.*
2. *The profit functional of the sum of two nonempty sets F and C satisfies $\pi_{F+C} = \pi_F + \pi_C$.*
3. *Let $\{K_n\}$ be a decreasing sequence of nonempty compact sets. If $K = \bigcap_{n=1}^{\infty} K_n$, then $K \neq \varnothing$ and the sequence $\{\pi_{K_n}\}$ of support functionals satisfies $h_{\pi_n}(x) \downarrow h\pi_K(x)$ for each x.*

22.8.3 Lemma *Let C be a closed convex set with profit functional π_C. If $g(x) = p \cdot x + c$ is a continuous affine function satisfying $g \leqslant \pi_C$, then $p \in C$ and $c \leqslant 0$.*

We can now describe the support functional of the intersection of two closed convex sets.

22.8.4 Theorem *Let A and B be closed convex sets with $A \cap B \neq \varnothing$. Then the profit functional of $A \cap B$ is the convex envelope of $\min\{\pi_A, \pi_B\}$.*

We now point out that the family of compact convex sets partially ordered by inclusion is a lattice. (That is, every pair of sets has both an infimum and a supremum.) The infimum of A and B, $A \wedge B$, is just $A \cap B$, and the supremum $A \vee B$ is $\operatorname{co}(A \cup B)$. (See [1, Lemma 5.29, p. 183] for a proof that $\operatorname{co}(A \cup B)$ is compact.) Likewise, the collection of continuous sublinear functions on X under the pointwise ordering is a lattice with $f \vee g = \max\{f, g\}$, and $f \wedge g$ is the convex envelope of $\min\{f, g\}$. (Here we include the constant $-\infty$ as an honorary member of the family.) Now consider the surjective one-to-one mapping $A \mapsto \pi_A$ between these two lattices. It follows from Lemma 22.8.2 and Theorem 22.8.4 that this mapping preserves the algebraic and lattice operations in the following sense:

Prove that $\operatorname{co}(A \cup B)$ is compact.

- $\pi_{A \vee B} = \pi_A \vee \pi_B$ and $\pi_{A \wedge B} = \pi_A \wedge \pi_B$.
- $A \subset B$ implies $\pi_A \leqslant \pi_B$.
- $\pi_{A+B} = \pi_A + \pi_B$ and $\pi_{\lambda A} = \lambda \pi_A$ for $\lambda > 0$.

22.9 Aside: Support functionals and the Hausdorff metric

The **operator norm** of a linear function p on a normed space is defined by

$$\|p\| = \sup\{p(x) : \|x\| \leqslant 1\}.$$

22.9.1 Proposition (Hausdorff metric on convex sets) *Let A and B be nonempty closed bounded convex subsets of a normed space. Then the Hausdorff metric ρ satisfies*

$$\rho(A, B) = \sup_{p:\|p\|\leqslant 1} |\pi_A(p) - \pi_B(p)|.$$

22.10 Aside: the Hausdorff metric

22.10.1 Definition *Let (X, d) be a metric space. For each pair of nonempty subsets A and B of X, define*

$$\mathsf{h}_d(A, B) = \max\Big\{\sup_{a\in A} d(a, B),\ \sup_{b\in B} d(b, A)\Big\}.$$

The extended real number $\mathsf{h}_d(A, B)$ is the ***Hausdorff distance*** *between A and B relative to the semimetric d. The function h_d is the* ***Hausdorff semimetric*** *induced by d. By convention, $\mathsf{h}_d(\varnothing, \varnothing) = 0$ and $\mathsf{h}_d(A, \varnothing) = \infty$ for $A \neq \varnothing$.*

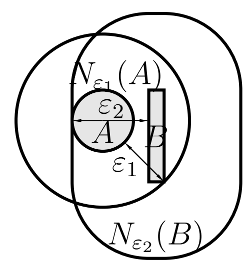
While h_d depends on d, we may omit the subscript when d is clear from the context.

We can also define the Hausdorff distance in terms of neighborhoods of sets. Recall that the ε-neighborhood of a nonempty subset A of the semimetric space (X, d) is the set

$$N_\varepsilon(A) = \Big\{x \in X : d(x, A) < \varepsilon\Big\}.$$

Recall that $\bigcap_{\varepsilon>0} N_\varepsilon(A) = \overline{A}$ and note that

$$N_\varepsilon\Big(\bigcup_{i\in I} A_i\Big) = \bigcup_{i\in I} N_\varepsilon(A_i).$$



$\varepsilon_1 = \sup_{b\in B} d(b, A)$,
$\varepsilon_2 = \sup_{a\in A} d(a, B)$

Figure 22.10.1.

22.10.2 Lemma *If A and B are nonempty subsets of a semimetric space (X, d), then*

$$\mathsf{h}(A, B) = \inf\Big\{\varepsilon > 0 : A \subset N_\varepsilon(B) \text{ and } B \subset N_\varepsilon(A)\Big\}.$$

Proof: If $\{\varepsilon > 0 : A \subset N_\varepsilon(B) \text{ and } B \subset N_\varepsilon(A)\} = \varnothing$, then for each $\varepsilon > 0$, either there is some $a \in A$ with $d(a, B) \geq \varepsilon$ or there is some $b \in B$ with $d(b, A) \geq \varepsilon$. This implies $\mathsf{h}(A, B) \geq \varepsilon$ for each $\varepsilon > 0$, so $\mathsf{h}(A, B) = \infty$. (Recall that $\inf \varnothing = \infty$.)

Now suppose $\delta = \inf\Big\{\varepsilon > 0 : A \subset N_\varepsilon(B) \text{ and } B \subset N_\varepsilon(A)\Big\} < \infty$. If ε satisfies $A \subset N_\varepsilon(B)$ and $B \subset N_\varepsilon(A)$, then $d(a, B) < \varepsilon$ for all $a \in A$ and $d(b, A) < \varepsilon$ for each $b \in B$, so $\mathsf{h}(A, B) \leq \varepsilon$. Thus $\mathsf{h}(A, B) \leq \delta$. On the other hand, if $\varepsilon > \mathsf{h}(A, B)$, then obviously $A \subset N_\varepsilon(B)$ and $B \subset N_\varepsilon(A)$, so indeed $\mathsf{h}(A, B) = \delta$. (See Figure 22.10.1.) ∎

22.11 Aside: Strassen's integrability theorem

The next result is due to Strassen [4, Theorem 1]. For a more rigorous statement and proof (in a much more general setting) see [1, pp. 615–620].

Relate this to the Summation Principle.

22.11.1 Strassen's Theorem *Let (S, Σ, μ) be a probability space, and let X be a separable Banach space (think $\boldsymbol{R}^n$ if convenient). Let $h\colon S \times X \to \boldsymbol{R}$ satisfy the following properties.*

1. *$x \mapsto h(s, x)$ is a continuous sublinear function on X for each $s \in S$. Thus $x \mapsto h(s, x)$ is the profit function of a compact convex set $\varphi(s)$.*

2. *For each $x \in X$, the function $s \mapsto h(s, x)$ is integrable.*

Define $h\colon X \to \boldsymbol{R}$ by the integral

$$h(x) = \int_S h(s, x)\, d\mu(s).$$

Then h is continuous and sublinear, and is thus the profit function of a compact convex set C. Moreover

$$C = \left\{ \int_S f(s)\, d\mu(s) : f \text{ is integrable and } (\forall s)\ [\, f(s) \in \varphi(s)\,] \right\}.$$

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