Topic 21: Rockafellar's Closed Functions

21.1 ⋆ Closed convex functions

Convex analysts often refer to closed functions and the closure of a function. For proper functions closedness and semicontinuity agree. There are differences for improper functions

21.1.1 Definition *Let C be a closed convex subset of a tvs X. (It may well be that $C = X$). Let $f\colon C \to \boldsymbol{R}^\sharp$ be an extended real-valued function on C. Define the **concave envelope** $\hat{f}$ of f on C by*

$$\hat{f}(x) = \inf\{g(x) : g \geqslant f \text{ on } C \text{ and } g \text{ is affine and continuous}\}$$

*and the and the **convex envelope** $\check{f}$ of f by*

$$\check{f}(x) = \sup\{g(x) : g \leqslant f \text{ and } g \text{ is affine and continuous}\}$$

(where, as you may recall, $\sup\varnothing = -\infty$ and $\inf\varnothing = \infty$).

21.1.2 Lemma *The convex envelope $\check{f}$ of a function f is a lower semicontinuous convex function. The concave envelope $\hat{f}$ of f is an upper semicontinuous concave function. Moreover*

$$\check{f} \leqslant f \leqslant \hat{f}.$$

Proof: Clearly $\check{f} \leqslant f \leqslant \hat{f}$. Moreover, every affine functions is both convex and concave. Thus $\check{f}$, being the supremum of a family of lower semicontinuous convex functions is convex and lower semicontinuous (Exercise 1.3.3 and Proposition 13.4.5). Similarly $\hat{f}$, being the infimum of a family of upper semicontinuous concave functions is concave and upper semicontinuous. ∎

Warning! The definition I am about to give is *not* standard. However, it is equivalent to the standard definition (see Section 13.5 ⋆ below), and in my view more natural.

21.1.3 Definition (Closed functions) *A proper convex function f is **closed** if it is equal to its convex envelope, $f = \check{f}$. A proper concave function is **closed** if it is equal to its concave envelope $f = \hat{f}$.*

- Since every affine function on $\boldsymbol{R}^{\mathrm{m}}$ is continuous, we may omit that requirement from the definition for finite dimensional spaces.

- This is not really subtle, but I should point out that a function can be closed without having a closed effective domain. For example, the logarithm function (extended to be an extended real-valued concave function) is closed, but has $(0, \infty)$ as its effective domain.

Closedness is closely related to semicontinuity.

21.1.4 Proposition *A concave function on a locally convex Hausdorff space X is closed if and only if one of the following conditions holds.*

1. *The function is identically $+\infty$ (and hence improper).*
2. *The function is identically $-\infty$ (and hence improper).*
3. *The function is proper and upper semicontinuous.*

A convex function is closed if and only if (1)*, or* (2)*, or*

3′. *The function is proper and lower semicontinuous.*

Proof: For proper functions the conclusions follow from Theorem 13.3.3 and Lemma 21.1.2.

Now consider the case of an improper convex function f. There a two ways f can fail to be proper. The first is that $f(x) = -\infty$ at some point x. In this case there is no affine function that f dominates, so $\check{f}(y) = \inf \varnothing = \infty$ for all y. The second way f can fail to be proper is if $\operatorname{dom} f = \varnothing$, that is, $f(y) = \infty$ for all y. ∎

21.2 ⋆ The difference between closedness and semicontinuity

The next example clarifies the difference between closed and semicontinuous improper functions.

21.2.1 Example (Closedness vs. semicontinuity: improper functions) Define the improper concave function f by

$$f(x) = \begin{cases} +\infty & x \in C \\ -\infty & x \notin C \end{cases}$$

where C is a nonempty closed convex set in $\boldsymbol{R}^{\mathrm{m}}$. Then the hypograph of f is a nonempty closed convex set, but f is not a closed function (unless $C = \boldsymbol{R}^{\mathrm{m}}$). □

In fact this is the only kind of upper semicontinuous improper function on $\boldsymbol{R}^{\mathrm{m}}$.

21.2.2 Theorem *If f is an improper concave function on $\boldsymbol{R}^{\mathrm{m}}$, then $f(x) = \infty$ for every $x \in \operatorname{ri} \operatorname{dom} f$. If f is an improper convex function, then $f(x) = -\infty$ for every $x \in \operatorname{ri} \operatorname{dom} f$.*

Proof: There are two ways a concave function f can be improper. The first is that $\operatorname{dom} f$ is empty, in which case, the conclusion holds vacuously. The second case is that $f(x) = +\infty$ for some $x \in \operatorname{dom} f$. Let y belong to $\operatorname{ri}\operatorname{dom} f$. Then by Proposition 5.2.6, y is proper convex combination $\lambda x + (1-\lambda)z$ $(0 < \lambda < 1)$, where $z = x + \varepsilon(y-x)$ for some $\varepsilon < 0$ and $z \in \operatorname{dom} f$ (so that $f(z) > -\infty$). Then $f(y) \geqslant \lambda\infty + (1-\lambda)f(z) = +\infty$. ∎

21.2.3 Proposition *An upper semicontinuous improper concave function has no finite values. Ditto for a lower semicontinuous improper convex function.*

Proof: By Theorem 21.2.2, if a concave f has $f(x) = +\infty$, then $f(y) = +\infty$ for all $\in \operatorname{ri}\operatorname{dom} f$. By upper semicontinuity, $f(y) = +\infty$ for all $y \in \overline{\operatorname{ri}\operatorname{dom} f} \supset \operatorname{dom} f$. By definition of the effective domain, $f(y) = -\infty$ for $y \notin \operatorname{dom} f$. (This shows that $\operatorname{dom} f$ is closed.) ∎

There are some subtleties in dealing with closed and semicontinuous functions, particularly if you are used to the conventional approach. For instance, if C is nonempty convex subset of $\boldsymbol{R}^{\mathrm{m}}$ that is not closed, the conventional approach allows us to define a concave or convex function with domain C, and undefined elsewhere. Consider the constant function zero on C. Its hypograph is closed in $C \times \boldsymbol{R}$, but not closed in $\boldsymbol{R}^{\mathrm{m}} \times \boldsymbol{R}$. Regarded as a conventional function on C, it is both upper and lower semicontinuous on C, but regarded as a concave extended real-valued function on $\boldsymbol{R}^{\mathrm{m}}$, it is not upper semicontinuous. This is because at a boundary point x, we have $\limsup_{y\to x} = 0 > f(x) = -\infty$.

21.3 The closure of a function

Warning: The following definition is not standard, but it is equivalent to the standard definition. (Again see Section 13.5$\star$.)

21.3.1 Definition *The closure of a convex function is its convex envelope. The closure of a concave function is its concave envelope.*

Every continuous affine function is its own closure. (On infinite dimensional spaces there are discontinuous affine functions. What about them? Can an affine function be semicontinuous without being continuous? What continuous affine functions can a semicontinuous affine function dominate?)

According to Rockafellar [2, p. 53], "the closure operation is a reasonable normalization which makes convex functions more regular by redefining their values at certain points where there are unnatural discontinuities. This is the secret of the great usefulness of the operation in theory and in applications." Implicit in Rockafellar's remark is that the closure operation only deals with bad behavior on the boundary of the domain. Indeed we have the following result.

21.3.2 Theorem *Let f be a proper concave function on $\boldsymbol{R}^{\mathrm{m}}$. Then $\hat{f}$ is a proper closed concave function, and f and $\hat{f}$ agree on* $\operatorname{ri}\operatorname{dom} f$.

See [2, Theorem 7.4, p. 56].

21.3.3 Corollary *If f is a proper concave or convex function and* $\operatorname{dom} f$ *is affine, then f is closed.*

This is because an affine set is its own relative interior. See [2, Corollary 7.4.2, p. 56].

21.4 Closed sublinear functions

The next result refines Theorem 9.1.3, which asserts that every proper, lower semicontinuous, sublinear function is a profit function. If the function is not lower semicontinuous or proper we have the following.

21.4.1 Theorem *Let π be a positively homogeneous convex function on $\boldsymbol{R}^{\mathrm{m}}$. Define the closed convex set*

$$C = \operatorname{cl}\{x \in \boldsymbol{R}^{\mathrm{m}} : (\forall p \in \boldsymbol{R}^{\mathrm{m}})\ [\, p \cdot x \leqslant \pi(p)\,]\}.$$

Then the convex envelope $\breve{\pi}$ of π is the profit function π_C of C. That is,

$$\breve{\pi}(p) = \sup\{p \cdot x : x \in C\}.$$

Proof: There are three cases: Cases (i) and (ii) cover the two ways π could fail to be proper, and case (iii) is that π is proper.

(i) If π is improper by way of having $\pi(p) = -\infty$ for some p, then $C = \varnothing$, so $\pi_C(p) = -\infty = \breve{\pi}(p)$ for all p.

(ii) If π is improper by way of being identically ∞, then $C = \boldsymbol{R}^{\mathrm{m}}$, so $\pi_C(p) = \infty = \breve{\pi}(p)$ for all p.

(iii) The remaining case is that π is proper, so $\pi(p) > -\infty$ for all p and $\operatorname{dom}\pi$ is nonempty.

Let ℓ_x denote the linear function defined by $\ell_x(p) = p \cdot x$. Every affine function is thus of the form $\ell_x + \alpha$, so let $g(p)\colon p \mapsto \ell_x(p) + \alpha$ be an affine function satisfying $g \ll \pi$. Since π is positively homogeneous and proper, we have $\pi(0) = 0$, so $g \ll \pi$ implies that $\alpha < 0$. Also, for every p and every $\lambda > 0$ we have

$$\lambda x \cdot p + \alpha = \ell_x(\lambda p) + \alpha = g(\lambda p) < \pi(\lambda p) = \lambda \pi(p).$$

Dividing by λ we see that for every p and $\lambda > 0$,

$$x \cdot p. + \alpha/\lambda < \pi(p).$$

Letting $\lambda \to \infty$ we see that $g = \ell_x + \alpha$ is affine and satisfies $g \ll \pi$ if and only if $\ell_x \leqq \pi$ and $\alpha < 0$. But $\ell_x \leqq \pi$ if and only if $x \in C$. Thus

$$\begin{aligned}\breve{\pi}(p) &= \sup\{g(p) : g \ll \pi,\ g \text{ is affine}\} \\ &= \sup\{\ell_x(p) : \ell_x \leqq \pi\} = \sup\{p \cdot x : x \in C\} = \pi_C(p).\end{aligned}$$

■

References

[1] J. M. Borwein and D. Noll. 1994. Second order differentiability of convex functions in Banach spaces. *Transactions of the American Mathematical Society* 342(1):43–81. http://www.jstor.org/stable/2154684

[2] R. T. Rockafellar. 1970. *Convex analysis*. Number 28 in Princeton Mathematical Series. Princeton: Princeton University Press.