Caltech Division of the Humanities and Social Sciences

Ec 181 Convex Analysis and Economic Theory — KC Border, AY 2019–2020

Topic 20: When are Sums Closed?

20.1 Is a sum of closed sets closed?

Example 0.2.2 shows that the sum of two closed sets need not be closed. To state sufficient conditions for the sum to be closed we must make a fairly long digression.

20.2 Asymptotic cones

A **cone** is a nonempty subset of $\boldsymbol{R}^{\mathrm{m}}$ closed under multiplication by nonnegative scalars. That is, C is a cone if whenever $x \in C$ and $\lambda \in \boldsymbol{R}_+$, then $\lambda x \in C$. A cone is **nontrivial** if it contains a point other than zero.

20.2.1 Definition *Let $E \subset \boldsymbol{R}^{\mathrm{m}}$. The* ***asymptotic cone*** *of E, denoted $\boldsymbol{A}E$ is the set of all possible limits z of sequences of the form $(\lambda_n x_n)_n$, where each $x_n \in E$, each $\lambda_n > 0$, and $\lambda_n \to 0$. Let us call such a sequence a* ***defining sequence for z****.*

This definition is equivalent to that in Debreu [1], and generalizes the notion of the recession cone of a convex set. This form of the definition was chosen because it makes most properties of asymptotic cones trivial consequences of the definition.

The **recession cone** 0^+F of a closed convex set F is the set of all directions in which F is unbounded, that is,

$$0^+F = \{z \in \boldsymbol{R}^{\mathrm{m}} : (\forall x \in F)\ (\forall \alpha \geqslant 0)\ [x + \alpha z \in F]\}.$$

(See Rockafellar [3, Theorem 8.2].)

20.2.2 Lemma **(a)** *$\boldsymbol{A}E$ is indeed a cone.*

(b) *If $E \subset F$, then $\boldsymbol{A}E \subset \boldsymbol{A}F$.*

(c) *$\boldsymbol{A}(E + x) = \boldsymbol{A}E$ for any $x \in \boldsymbol{R}^{\mathrm{m}}$.*

(cc) *$0^+E \subset \boldsymbol{A}E$.*

(d) *$\boldsymbol{A}E_1 \subset \boldsymbol{A}(E_1 + E_2)$.*

(e) *$\boldsymbol{A}\prod_{i\in I} E_i \subset \prod_{i\in I} \boldsymbol{A}E_i$.*

(f) *$\boldsymbol{A}E$ is closed.*

(g) *If E is convex, then $\boldsymbol{A}E$ is convex.*

(h) *If E is closed and convex, then $\boldsymbol{A}E = 0^+E$. (The asymptotic cone really is a generalization of the recession cone.)*

(i) *If C is a cone, then $\boldsymbol{A}C = \overline{C}$.*

(j) *$\boldsymbol{A}\bigcap_{i \in I} E_i \subset \bigcap_{i \in I} \boldsymbol{A}E_i$. The reverse inclusion need not hold.*

(k) *If $E + F$ is convex, then $\boldsymbol{A}E + \boldsymbol{A}F \subset \boldsymbol{A}(E + F)$.*

(l) *A set $E \subset \boldsymbol{R}^{\mathrm{m}}$ is bounded if and only if $\boldsymbol{A}E = \{0\}$.*

Proof: Here are proofs of selected parts. The others are easy, and should be treated as an exercise.

(cc) $0^+E \subset \boldsymbol{A}E$.

Let $z \in 0^+E$. Then for any $n > 0$ and any $x \in E$, we have $x + nz \in E$. But $\frac{1}{n}(x + nz) \to z$, so $z \in \boldsymbol{A}E$.

(d) $\boldsymbol{A}E_1 \subset \boldsymbol{A}(E_1 + E_2)$.

For $x_2 \in E_2$, by definition $E_1 + x_2 \subset E_1 + E_2$, so by (b), $\boldsymbol{A}(E_1 + x_2) \subset \boldsymbol{A}(E_1 + E_2)$, so by (c), $\boldsymbol{A}E_1 \subset \boldsymbol{A}(E_1 + E_2)$.

(f) $\boldsymbol{A}E$ is closed.

Let x_n be a sequence in $\boldsymbol{A}E$ with $x_n \to x$. For each n there is a sequence $\lambda_{n,m}x_{n,m}$ with $\lim_m \lambda_{n,m}x_{n,m} = x_n$, $\lambda_{n,m} \to 0$ as $m \to \infty$, $x_{n,m} \in E$, and each $\lambda_{n,m} > 0$. Then for each k there is N_k such that for all $n \geqslant N_k$, $\|x_n - x\| < 1/k$, and M_k such that for all $m \geqslant M_k$, $\|\lambda_{N_k,m}x_{N_k,m} - x_{N_k}\| < 1/k$, and L_k such that for all $m \geqslant L_k$, $\lambda_{N_k,m} < 1/k$. Set $P_k = \max\{M_k, L_k\}$, $y_k = x_{N_k,P_k}$, and $\lambda_k = \lambda_{N_k,P_k}$. Then each $\lambda_k > 0$, $\lambda_k \to 0$ and $\|\lambda_k y_k - x\| < 2/k$, so $x \in \boldsymbol{A}E$.

(g) If E is convex, then $\boldsymbol{A}E$ is convex.

Let $x, y \in \boldsymbol{A}E$ and $\alpha \in [0, 1]$. Since $\boldsymbol{A}E$ is a cone, $\alpha x \in \boldsymbol{A}E$ and $(1-\alpha)y \in \boldsymbol{A}E$. Thus there are defining sequences $\lambda_n x_n \to \alpha x$ and $\gamma_n y_n \to (1-\alpha)y$. Since E is convex, $z_n = \frac{\lambda_n}{\gamma_n + \lambda_n}x_n + \frac{\gamma_n}{\gamma_n + \lambda_n}y_n \in E$ for each n. Set $\delta_n = \gamma_n + \lambda_n > 0$. Then $\delta_n \to 0$ and $\delta_n z_n = \lambda_n x_n + \gamma_n y_n \to \alpha x + (1-\alpha)y$. Thus $\alpha x + (1-\alpha)y \in \boldsymbol{A}E$.

(h) If E is closed and convex, then $\boldsymbol{A}E = 0^+E$.

In light of (cc), it suffices to prove that $\boldsymbol{A}E \subset 0^+E$, so let $z \in \boldsymbol{A}E$, $x \in E$, and $\alpha \geqslant 0$. We wish to show that $x + \alpha z \in E$. By definition of $\boldsymbol{A}E$ there is a sequence $\lambda_n z_n \to z$ with $z_n \in E$, $\lambda_n > 0$, and $\lambda_n \to 0$. Then for n

large enough $0 \leqslant \alpha\lambda_n < 1$, so $(1-\alpha\lambda_n)x + \alpha\lambda_n z_n \in E$ as E is convex. But $(1-\alpha\lambda_n)x + \alpha\lambda_n z_n \to x + \alpha z$. Since E is closed, $x + \alpha z \in E$.

(i) If C is a cone, then $\boldsymbol{A}C = \overline{C}$.

It is easy to show that $C \subset \boldsymbol{A}C$, as $\frac{1}{n}nx \to x$ is a defining sequence. Since $\boldsymbol{A}C$ is closed by (f), we have $\overline{C} \subset \boldsymbol{A}C$. On the other hand if $\lambda_n \geqslant 0$ and $x_n \in C$, then $\lambda_n x_n \in C$, as C is a cone, so $\boldsymbol{A}C \subset \overline{C}$.

(j) $\boldsymbol{A}\bigcap_{i\in I} E_i \subset \bigcap_{i\in I} \boldsymbol{A}E_i$. The reverse inclusion need not hold.

By (b), $\boldsymbol{A}\bigcap_{i\in I} E_i \subset \boldsymbol{A}E_j$ for each j, so $\boldsymbol{A}\bigcap_{i\in I} E_i \subset \bigcap_{i\in I} \boldsymbol{A}E_i$.

For a failure of the reverse inclusion, consider the even nonnegative integers $E_1 = \{0, 2, 4, \ldots\}$ and the odd nonnegative integers $E_2 = \{1, 3, 5, \ldots\}$. Then $E_1 \cap E_2 = \varnothing$, so $\boldsymbol{A}(E_1 \cap E_2) = \varnothing$, but $\boldsymbol{A}E_1 = \boldsymbol{A}E_2 = \boldsymbol{A}E_1 \cap \boldsymbol{A}E_2 = \boldsymbol{R}_+$.

But what if each E_i is convex?

(k) If $E + F$ is convex, then $\boldsymbol{A}E + \boldsymbol{A}F \subset \boldsymbol{A}(E+F)$.

Let z belong to $\boldsymbol{A}E + \boldsymbol{A}F$. Then there exist defining sequences $(\lambda_n x_n) \subset E$ and $(\alpha_n y_n) \subset F$ with $\lambda_n x_n + \alpha_n y_n \to z$. Let $x' \in E$ and $y' \in F$. (If either E or F is empty, the result is trivial.) Then $\big(\lambda_n(x_n + y')\big) \subset E + F$ and $\big(\alpha_n(x' + y_n)\big) \subset E + F$, so

$$(\lambda_n + \alpha_n)\left(\frac{\lambda_n}{\lambda_n + \alpha_n}(x_n + y') + \frac{\alpha_n}{\lambda_n + \alpha_n}(x' + y_n)\right) \to z,$$

is a defining sequence for z in $E + F$.

(l) A set $E \subset \boldsymbol{R}^{\mathrm{m}}$ is bounded if and only if $\boldsymbol{A}E = \{0\}$.

If E is bounded, clearly $\boldsymbol{A}E = \{0\}$. If E is not bounded, let $\{x_n\}$ be an unbounded sequence in E. Then $\lambda_n = \|x_n\|^{-1} \to 0$ and $(\lambda_n x_n)$ is a sequence on the unit sphere, which is compact. Thus there is a subsequence converging to some x in the unit sphere. Such an x is a nonzero member of $\boldsymbol{A}E$.

∎

20.2.3 Example The asymptotic cone of a non-convex set need not be convex. Let $E = \{(x,y) \in \boldsymbol{R}^2 : y = \frac{1}{x},\ x > 0\}$. This hyperbola is not convex and its asymptotic cone is the union of the nonnegative x- and y-axes. But the asymptotic cone of a non-convex set may be convex. Just think of the integers in $\boldsymbol{R}^1$. □

20.2.4 Example It need not be the case that $\boldsymbol{A}(E+F) \subset \boldsymbol{A}E + \boldsymbol{A}F$, even if E and F are closed and convex. For instance, let E be the set of points lying above a standard parabola:

$$E = \{(x,y) : y \geqslant x^2\}.$$

The asymptotic cone of E, which is the same as its recession cone, is just the positive y-axis:

$$\boldsymbol{A}E = \{(0, y) : y \geqslant 0\}.$$

So $\boldsymbol{A}E + \boldsymbol{A}(-E)$ is just the y-axis. Now observe that $E + (-E) = \boldsymbol{R}^2$, so $\boldsymbol{A}\big(E + (-E)\big) = \boldsymbol{R}^2$. Thus

$$\boldsymbol{A}E + \boldsymbol{A}(-E) \subsetneq \boldsymbol{A}\big(E + (-E)\big).$$

□

20.3 When a sum of closed sets is closed

We now turn to the question of when a sum of closed sets is closed. The following definition may be found in Debreu [1, 1.9. m., p. 22].

20.3.1 Definition *Let $C_1, \dots, C_n$ be cones in $\boldsymbol{R}^{\mathrm{m}}$. We say that they are* ***positively semi-independent*** *if whenever $x_i \in C_i$ for each $i = 1, \dots, n$,*

$$x_1 + \cdots + x_n = 0 \quad \Longrightarrow \quad x_1 = \cdots = x_n = 0.$$

Clearly, any subcollection of a collection of semi-independent cones is also semi-independent. Note that in Example 20.2.4, $\boldsymbol{A}(-E) = -\boldsymbol{A}(E)$, so these nontrivial asymptotic cones are not positively semi-independent.

20.3.2 Theorem (Closure of the sum of sets) *Let $E, F \subset \boldsymbol{R}^{\mathrm{m}}$ be closed and nonempty. Suppose that $\boldsymbol{A}E$ and $\boldsymbol{A}F$ are positively semi-independent. (That is, $x \in \boldsymbol{A}E$, $y \in \boldsymbol{A}F$ and $x + y = 0$ together imply that $x = y = 0$.) Then $E + F$ is closed, and $\boldsymbol{A}(E + F) \subset \boldsymbol{A}E + \boldsymbol{A}F$.*

The proof relies on the following simple lemma, which is closely related to Lemma 1 in Gale and Rockwell [2].

20.3.3 Lemma *Under the hypotheses of Theorem 20.3.2, if (λ_n) is a bounded sequence of real numbers with each $\lambda_n > 0$, (x_n) is a sequence in E, and (y_n) is a sequence in F, and if $\lambda_n(x_n + y_n)$ converges to some point, then there is a common subsequence along which both $(\lambda_k x_k)$ and $(\lambda_k y_k)$ converge.*

Proof: It suffices to prove that both $(\lambda_n x_n)$ and $(\lambda_n y_n)$ are bounded sequences. Suppose by way of contradiction that $\lambda_n(x_n + y_n)$ converges to some point, but say $(\lambda_n x_n)$ is unbounded. Since (λ_n) is bounded, it must be the case that both $\|\lambda_n x_n\| \to \infty$ and $\|x_n\| \to \infty$, so for large enough n we have $\|\lambda x_n\| > 0$. Thus for large n we may divide by $\|\lambda_n x_n\|$ and define

$$\hat{x}_n = \frac{\lambda_n}{\|\lambda_n x_n\|} x_n, \quad \hat{y}_n = \frac{\lambda_n}{\|\lambda_n x_n\|} y_n, \quad \hat{z}_n = \frac{\lambda_n}{\|\lambda_n x_n\|} (x_n + y_n),$$

and observe that

$$\hat{z}_n = \hat{x}_n + \hat{y}_n.$$

But $\big(\lambda_n(x_n + y_n)\big)$ is convergent, and hence bounded, so $\hat{z}_n \to 0$. In addition the sequence $(\hat{x}_n)$ lies on the unit sphere, so it has a convergent subsequence, say $\hat{x}_k \to \hat{x}$, where $\|\hat{x}\| = 1$. Then

$$\hat{y}_k = \hat{z}_k - \hat{x}_k \to -\hat{x}.$$

But $\hat{y}_k = (\lambda_k/\|\lambda_k x_k\|)y_k$, and $\lambda_k/\|\lambda_k x_k\| \to 0$, so $(\lambda_k/\|\lambda_k x_k\|)y_k$ is a defining sequence that puts $-\hat{x} \in \boldsymbol{A}F$. But a similar argument shows that $\hat{x} \in \boldsymbol{A}E$. Since $\boldsymbol{A}E$ and $\boldsymbol{A}F$ are positively semi-independent, it follows that $\hat{x} = 0$, contradicting $\|\hat{x}\| = 1$.

Thus $(\lambda_n x_n)$, is a bounded sequence, and by a similar argument so is $(\lambda_n y_n)$, so they have common subsequence on which they both converge. ∎

Proof of Theorem 20.3.2: First, $E + F$ is closed: Let $x_n + y_n \to z$ with $\{x_n\} \subset E$, $\{y_n\} \subset F$. By Lemma 20.3.3 (with $\lambda_n = 1$ for all n) there is a common subsequence with $x_k \to x$ and $y_k \to y$. Since E and F are closed, $x \in E$ and $y \in F$. Therefore $z = x + y \in E + F$, so $E + F$ is closed.

To see that $\boldsymbol{A}(E + F) \subset \boldsymbol{A}E + \boldsymbol{A}F$, let $z \in \boldsymbol{A}(E + F)$. That is, z is the limit of a defining sequence $\big(\lambda_n(x_n + y_n)\big)$, where $x_n \in E$ and $y_n \in F$. Since $\lambda_n \to 0$, it is a bounded sequence. Thus by Lemma 20.3.3 there is a common convergent subsequence, and by definition $\lim_k \lambda_k x_k \in \boldsymbol{A}E$ and $\lim_k \lambda_k y_k \in \boldsymbol{A}F$, so $z \in \boldsymbol{A}E + \boldsymbol{A}F$. ∎

20.3.4 Corollary *Let $E_i \subset \boldsymbol{R}^{\mathrm{m}}$, $i = 1, \dots, n$, be closed and nonempty. If $\boldsymbol{A}E_i$, $i = 1, \dots, n$, are positively semi-independent, then $\sum_{i=1}^n E_i$ is closed, and $\boldsymbol{A}\sum_{i=1}^n E_i \subset \sum_{i=1}^n \boldsymbol{A}E_i$.*

Proof: This follows from Theorem 20.3.2 by induction on n. ∎

20.3.5 Corollary *Let $E, F \subset \boldsymbol{R}^{\mathrm{m}}$ be closed and let F be compact. Then $E + F$ is closed.*

Proof: A compact set is bounded, so by Lemma 20.2.2(l) its asymptotic cone is $\{0\}$. Apply Theorem 20.3.2. ∎

20.4 When is an intersection of closed sets bounded?

20.4.1 Proposition *Let $E_i \subset \boldsymbol{R}^{\mathrm{m}}$, $i = 1, \dots, n$, be nonempty. If $\bigcap_{i=1}^n \boldsymbol{A}E_i = \{0\}$, then $\bigcap_{i=1}^n E_i$ is bounded.*

Proof: By Lemma 20.2.2(l), $\bigcap_{i=1}^n E_i$ is bounded if and only if $\boldsymbol{A}\left(\bigcap_{i=1}^n E_i\right) = \{0\}$. But by Lemma 20.2.2(j), $\boldsymbol{A}\left(\bigcap_{i=1}^n E_i\right) \subset \bigcap_{i=1}^n \boldsymbol{A}E_i$, and the proposition follows. ∎

References

[1] G. Debreu. 1959. *Theory of value: An axiomatic analysis of economic equilibrium*. Number 17 in Cowles Foundation Monographs. New Haven: Yale University Press. http://cowles.econ.yale.edu/P/cm/m17/m17-all.pdf

[2] D. Gale and R. Rockwell. 1976. The Malinvaud eigenvalue lemma: Correction and amplification. *Econometrica* 44(6):1323–1324. http://www.jstor.org/stable/1914264

[3] R. T. Rockafellar. 1970. *Convex analysis*. Number 28 in Princeton Mathematical Series. Princeton: Princeton University Press.