Topic 19: Extreme sets

19.1 Extreme points of convex sets

Refer to Section 2.6 for the definition of extreme sets and extreme points. The following lemma is the basic result concerning the existence of extreme points.

19.1.1 Lemma *In a locally convex Hausdorff space, every compact extreme subset of a set C contains an extreme point of C.*

Proof: Let C be a subset of some locally convex Hausdorff space and let F be a compact extreme subset of C. Consider the collection of sets

$$\mathcal{F} = \{G \subset F : G \text{ is a compact extreme subset of } C\}.$$

Since $F \in \mathcal{F}$, we have $\mathcal{F} \neq \varnothing$, and $\mathcal{F}$ is partially ordered by set inclusion. The compactness of F (as expressed in terms of the finite intersection property) guarantees that every chain in $\mathcal{F}$ has a nonempty intersection. Clearly, the intersection of extreme subsets of C is an extreme subset of C if it is nonempty. Thus, Zorn's Lemma applies, and yields a minimal compact extreme subset of C included in F, call it G. We claim that G is a singleton. To see this, assume by way of contradiction that there exist $a, b \in G$ with $a \neq b$. Then there is a continuous linear functional f on X such that $f(a) > f(b)$. Let M be the maximum value of f on G. Arguing as in the proof of Lemma 2.6.6, we see that the compact set $G_0 = \{c \in G : f(c) = M\}$ is an extreme subset of G (and hence of C) and $b \notin G_0$, contrary to the minimality of G. Hence G must be a singleton. Its unique element is an extreme point of C lying in F. ∎

Since every nonempty compact subset C is itself an extreme subset of C, we have the following immediate consequence of Lemma 19.1.1.

19.1.2 Corollary *Every nonempty compact subset of a locally convex Hausdorff space has an extreme point.*

19.1.3 Theorem *Every nonempty compact subset of a locally convex Hausdorff space is included in the closed convex hull of its extreme points.*

This refers to not yet proven result.

Proof: Let C be a nonempty compact subset of a locally convex Hausdorff space X, and let B denote the closed convex hull of its extreme points. We claim that $C \subset B$. Suppose by way of contradiction that there is some $a \in C$ with $a \notin B$. By Corollary 19.1.2 the set B is nonempty. So by the Separation Corollary 8.3.2

there exists a continuous linear functional f on X with $f(a) > f(b)$ for all $b \in B$. Let A be the set of maximizers of f over C. Clearly, A is a nonempty compact extreme subset of C, and $A \subset C \setminus B$. By Lemma 19.1.1, A contains an extreme point of C. But then, $A \cap B \neq \varnothing$, a contradiction. Hence $C \subset B$, as claimed. ∎

The celebrated Krein–Milman Theorem [1] is now a consequence of the preceding result.

19.1.4 The Krein–Milman Theorem *In a locally convex Hausdorff space X each nonempty convex compact subset is the closed convex hull of its extreme points.*

If X is finite dimensional, then every nonempty convex compact subset is the convex hull of its extreme points.

Proof: Only the second part needs proof. The proof will be done by induction on the dimension n of X. For $n = 1$ a nonempty convex compact subset of $\boldsymbol{R}$ is either a point or a closed interval, in which case the conclusion is obvious. For the induction step, assume that the result is true for all nonempty convex compact subsets of finite dimensional vector spaces of dimension less than or equal to n. This implies that the result is also true for all nonempty convex compact subsets of affine subspaces of dimension less than or equal to n. Now assume that C is a nonempty convex compact subset of an $(n+1)$-dimensional vector space X and let $\mathcal{E}$ be the collection of all extreme points of C. By the "Krein–Milman" part, we have $\overline{\text{co}}\,\mathcal{E} = C$.

If the affine subspace generated by C is of dimension less that $n+1$, then the conclusion follows from our induction hypothesis. So we can assume that the affine subspace generated by C is X itself. This means that the interior of C is nonempty. In particular, $\text{co}\,\mathcal{E}$ must have a nonempty interior. Otherwise, if $\text{co}\,\mathcal{E}$ has an empty interior, then $\overline{\text{co}}\,\mathcal{E}$ has dimension less than $n+1$, contrary to $\overline{\text{co}}\,\mathcal{E} = C$, as desired.

Now let x belong to C. If $x \in \text{int}\, C$, then $x \in \text{int}\, C = \text{int}\,(\overline{\text{co}}\,\mathcal{E}) = \text{int}\,(\text{co}\,\mathcal{E}) \subset \text{co}\,\mathcal{E}$. On the other hand, if $x \in \partial C$, then there exists a nonzero $f \in X^*$ supporting C at x, say $f(x) \leq f(a)$ for all $a \in C$. If we let $F = \{a \in C : f(a) = f(x)\} = C \cap \{f = f(x)\}$, then F is a compact face of C that lies in the n-dimensional flat $\{f = f(x)\}$. By the induction hypothesis x is a convex combination of extreme points of F. Now notice that every extreme point of F is an extreme point of C, and from this we get $x \in \text{co}\,\mathcal{E}$. Thus, $C \subset \text{co}\,\mathcal{E}$, so $C = \text{co}\,\mathcal{E}$. ∎

Pay careful attention to the statement of the Krein–Milman Theorem. It does *not* state that the closed convex hull of a compact set is compact. Indeed, that is not necessarily true in infinite-dimensional spaces, see Example 19.1.7. Rather it says that if a convex set is compact, then it is the closed convex hull of its extreme points. Furthermore, the hypothesis of local convexity cannot be dispensed with. Roberts [2] gives an example of a compact convex subset of the completely metrizable tvs $L_{\frac{1}{2}}[0,1]$ that has no extreme points.

We know that continuous functions always achieve their maxima and minima over nonempty compact sets. In a topological vector space we can say more. A continuous convex function on a nonempty compact convex set will always have at least one maximizer that is an extreme point of the set. This result is known as the **B**auer Maximum Principle. Note that this result does not claim that all maximizers are extreme points.

19.1.5 Bauer Maximum Principle *If C is a compact convex subset of a locally convex Hausdorff space, then every upper semicontinuous convex function on C has a maximizer that is an extreme point.*

Proof: Let f be an upper semicontinuous convex function on the nonempty, compact, and convex set. Now the set F of maximizers of f is nonempty and compact. By Lemma 2.6.6 it is an extreme set. But then Lemma 19.1.1 implies that F contains an extreme point of C. ■

The following corollary gives two immediate consequences of the Bauer Maximum Principle.

19.1.6 Corollary *If C is a nonempty compact convex subset of a locally convex Hausdorff space, then:*

1. *Every lower semicontinuous concave function on C has a minimizer that is an extreme point of C.*
2. *Every continuous linear functional has a maximizer and a minimizer that are extreme points of C.*

The convex hull of a compact subset of an infinite dimensional topological vector space need not be a compact set.

19.1.7 Example (Noncompact convex hull) Consider ℓ_2, the space of all square summable sequences. For each n let $u_n = \big(\underbrace{0,\dots,0}_{n-1}, \frac{1}{n}, 0, 0, \dots\big)$. Observe that $\|u_n\|_2 = \frac{1}{n}$, so $u_n \to 0$. Consequently,

$$A = \Big\{u_1, u_2, u_3, \dots\Big\} \cup \Big\{0\Big\}$$

is a norm compact subset of ℓ_2. Since $0 \in A$, it is easy to see that

$$\operatorname{co} A = \Big\{\sum_{i=1}^{k} \alpha_i u_i : \alpha_i \geq 0 \text{ for each } i \text{ and } \sum_{i=1}^{k} \alpha_i \leqslant 1\Big\}.$$

In particular, each vector of $\operatorname{co} A$ has only finitely many nonzero components. We claim that $\operatorname{co} A$ is not norm compact. To see this, set

$$x_n = \Big(\tfrac{1}{2}, \tfrac{1}{2}\cdot\tfrac{1}{2^2}, \tfrac{1}{3}\cdot\tfrac{1}{2^3}, \dots, \tfrac{1}{n}\cdot\tfrac{1}{2^n}, 0, 0, \dots\Big) = \sum_{i=1}^{n} \tfrac{1}{2^i} u_i,$$

so $x_n \in \operatorname{co} A$. Now $x_n \xrightarrow{\|\cdot\|_2} x = \left(\frac{1}{2}, \frac{1}{2}\frac{1}{2^2}, \frac{1}{3}\cdot\frac{1}{2^3}, \dots, \frac{1}{n}\cdot\frac{1}{2^n}, \frac{1}{n+1}\cdot\frac{1}{2^{n+1}}, \dots\right)$ in ℓ_2. But $x \notin \operatorname{co} A$, so $\operatorname{co} A$ is not even closed, let alone compact.

In this example, the convex hull of a compact set failed to be closed. The question remains as to whether the closure of the convex hull is compact. In general, the answer is no. To see this, let X be the space of sequences that are eventually zero, equipped with the ℓ_2-norm. Let A be as above, and note that $\overline{\operatorname{co}}\, A$ (where the closure is taken in X, not ℓ_2) is not compact either. To see this, observe that the sequence $\{x_n\}$ defined above has no convergent subsequence (in X). □

However there are three important cases when the closed convex hull of a compact set is compact. The first is when the compact set is a finite union of compact convex sets. This is just Lemma 2.1.6. The second is when the space is completely metrizable and locally convex. This includes the case of all Banach spaces with their norm topologies. Failure of completeness is where the last part of Example 19.1.7 goes awry. The third case is a compact set in the weak topology on a Banach space; this is the Krein–Šmulian Theorem.

References

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