Ec 181 KC Border
Convex Analysis and Economic Theory AY 2019–2020

Topic 13: Convex and concave functions

13.1 Talking convex analysis

> In convex analysis, convex and concave functions are defined everywhere on a vector space X, and are allowed to assume the extended values ∞ (sometimes denoted $+\infty$) and $-\infty$.

(Actually, convex analysts talk mostly about convex functions, and only occasionally about concave functions.) Recall that $\boldsymbol{R}^\sharp$ denotes the extended real numbers, $\boldsymbol{R}^\sharp = \boldsymbol{R} \cup \{\infty, -\infty\}$. Given an extended real-valued function $f \colon X \to \boldsymbol{R}^\sharp$, recall that the **hypograph** of f is the subset of $X \times \boldsymbol{R}$ defined by

$$\operatorname{hypo} f = \{(x, \alpha) \in X \times \boldsymbol{R} : \alpha \leqslant f(x)\}.$$

The **epigraph** is defined by reversing the inequality

$$\operatorname{epi} f = \{(x, \alpha) \in X \times \boldsymbol{R} : \alpha \geqslant f(x)\}.$$

Note well that the hypograph or epigraph of f is a subset of $X \times \boldsymbol{R}$, not of $X \times \boldsymbol{R}^\sharp$. That is, each α in the definition of hypograph or epigraph is a real number, not an infinity. For example, the epigraph of the constant function ∞ is the empty set.

To a convex analyst, an extended real-valued function is **convex** if its epigraph is a convex subset of $X \times \boldsymbol{R}$. Given a convex function $f \colon X \to \boldsymbol{R}^\sharp$, its **effective domain** is

$$\operatorname{dom} f = \{x \in X : f(x) < \infty\}.$$

We can extend a conventional real-valued convex function f defined on a subset C of a vector space X to an extended real-valued function $\tilde{f}$ defined on all of X by setting

$$\tilde{f}(x) = \begin{cases} f(x) & x \in C \\ \infty & x \notin C. \end{cases}$$

Note that $\tilde{f}$ is convex in the convex analyst's sense if and only if f is convex in the conventional sense, in which case we also have that $\operatorname{dom} \tilde{f} = C$.

We can similarly extend conventionally defined concave functions to X by setting them equal to $-\infty$ where conventionally undefined.

Proper functions

In the language of convex analysis, a convex function is **proper** if its effective domain is nonempty and its epigraph contains no vertical lines. A concave function is proper if its effective domain is nonempty and its hypograph contains no vertical lines. (A **vertical line** in $X \times \boldsymbol{R}$ is a set of the form $\{x\} \times \boldsymbol{R}$ for some $x \in X$.) That is, a convex f is proper if $f(x) < \infty$ for at least one x and $f(x) > -\infty$ for every x. Every proper convex function is gotten by taking a finite-valued convex function defined on some nonempty convex set and extending it to all of X as above. Clearly, a convex function f is proper if and only if $-f$ is a proper concave function. Thus:

> *Every convex function in the conventional sense is a proper convex function in the sense of convex analysis. Likewise for concave functions. For a proper function, its effective domain is the set of points where it is finite.*

As an example of a nontrivial improper convex function, consider this one taken from Rockafellar [4, p. 24].

13.1.1 Example (A nontrivial improper convex function) The function $f \colon \boldsymbol{R} \to \boldsymbol{R}^\sharp$ defined by

$$f(x) = \begin{cases} -\infty & |x| < 1 \\ 0 & |x| = 1 \\ \infty & |x| > 1 \end{cases}$$

is an improper convex function that is not constant. □

Some authors, for example, Aubin [1] or Hiriart-Urruty and Lemaréchal [3], do not permit convex functions to assume the value $-\infty$, so for them, properness is equivalent to nonemptyness of the effective domain.

Indicator functions

Another deviation from conventional terminology is the convex analysts' definition of the indicator function. The **indicator function** of the set C, denoted $\delta(\cdot \mid C)$, is defined by

$$\delta(x \mid C) = \begin{cases} 0 & x \in C \\ \infty & x \notin C. \end{cases}$$

The indicator of C is a convex function if and only if C is a convex set. It is proper if and only if C is nonempty. This is not to be confused with the probabilists' indicator function $\mathbf{1}_C$ defined by

$$\mathbf{1}_C(x) = \begin{cases} 1 & x \in C \\ 0 & x \notin C. \end{cases}$$

We now turn your attention to a few simple facts.

- Affine functions are both concave and convex.
- If f is continuous on $\operatorname{dom} f$ and $\operatorname{dom} f$ is closed in X, then f is lower semicontinuous as an extended real-valued function on X.
- Note well that the epigraph of an extended-real valued function is a subset of $X \times \boldsymbol{R}$, not a subset of $X \times \boldsymbol{R}^\sharp$. As a result, for a convex function f,
$$x \in \operatorname{dom} f \quad \iff \quad \big(x, f(x)\big) \in \operatorname{epi} f.$$
In other words, the effective domain of f is the projection on X of its epigraph. For concave functions replace epigraph by hypograph.
- The effective domain of a convex or concave function is a convex set.
- The constant function $f = -\infty$ is convex (its epigraph is $X \times \boldsymbol{R}$), but not proper, and the constant function $g = \infty$ is also convex (its epigraph is the empty set, which is convex), but not proper. These functions are also concave.
- If a convex function is proper, then its epigraph is a nonempty proper subset of $X \times \boldsymbol{R}$. If a concave function is proper, then its hypograph is a nonempty proper subset of $X \times \boldsymbol{R}$.
- Let f be an extended real-valued function on a tvs X. *If is f finite at x and continuous at x, then in fact x belongs to the interior of the effective domain of f.* Why?
- A convex function need not be finite at all points of continuity. The proper convex function f on $\boldsymbol{R}$ defined by $f(x) = 1/x$ for $x > 0$, and $f(x) = \infty$ for $x \leqslant 0$ is continuous everywhere, even at zero.

13.1.2 Exercise (Rockafellar [4, Lemma 7.3, p. 54]) Prove the following. For any concave function f on $\boldsymbol{R}^{\mathrm{m}}$,

$$\operatorname{ri} \operatorname{hypo} f = \{(x, \alpha) \in \boldsymbol{R}^{\mathrm{m}} \times \boldsymbol{R} : x \in \operatorname{ri} \operatorname{dom} f,\ \alpha < f(x)\}.$$

For a convex function f the corresponding result is

$$\operatorname{ri} \operatorname{epi} f = \{(x, \alpha) \in \boldsymbol{R}^{\mathrm{m}} \times \boldsymbol{R} : x \in \operatorname{ri} \operatorname{dom} f,\ \alpha > f(x)\}.$$

□

13.2 Hyperplanes in $\boldsymbol{X} \times \boldsymbol{R}$ and affine functions on $\boldsymbol{X}$

I will refer to a typical element in $X \times \boldsymbol{R}$ as a point (x, α) where $x \in X$ and $\alpha \in \boldsymbol{R}$. I may call x the "vector component" and α the "real component," even when $X = \boldsymbol{R}$. A hyperplane in $X \times \boldsymbol{R}$ is defined in terms of its "normal vector" (p, λ), which belongs to the dual space $(X \times \boldsymbol{R})^* \equiv X^* \times \boldsymbol{R}$. (That is, every (continuous) linear functional ℓ on $X \times \boldsymbol{R}$ is of the form $\ell(x, \alpha) = p(x) + \lambda\alpha$, where p is a (continuous) linear functional on X. When X is a Euclidean space, then $p(x) = p \cdot x$ and (p, λ) is indeed the normal vector to the hyperplane.) If the real component $\lambda = 0$, we say the hyperplane is **vertical**. If the hyperplane is not vertical, by homogeneity we can arrange for λ to be -1 (you will see why in just a moment). Here is an obvious fact about vertical hyperplanes that is just begging for a name:

13.2.1 Verticality *If H is a vertical hyperplane in $X \times \boldsymbol{R}$ and if $(x, \alpha) \in H$, then for every $\beta \in \boldsymbol{R}$, we have $(x, \beta) \in H$. Consequently a vertical hyperplane can never properly separate (x, α) and (x, β).*

13.2.2 Proposition *Non-vertical hyperplanes in $X \times \boldsymbol{R}$ are precisely the graphs of affine functions on X. That is,*

$$\operatorname{gr}\Big(x \mapsto p(x) - \beta\Big) \text{ is the non-vertical hyperplane } \Big\{(x, \alpha) \in X \times \boldsymbol{R} : (p, -1) \cdot (x, \alpha) = \beta\Big\}.$$

And the non-vertical hyperplane

$$\Big\{(x, \alpha) \in X \times \boldsymbol{R} : (p, \lambda) \cdot (x, \alpha) = \beta\Big\} \text{ is } \operatorname{gr}\Big(x \mapsto (-1/\lambda)p(x) + \beta/\lambda\Big).$$

See Figure 13.2.1.

13.3 Lower semicontinuous convex functions

Recall that an extended real-valued function on a topological space X is lower semicontinuous if and only if its epigraph is closed (Theorem 13.4.3 in the Appendix). Recall that a locally convex space is a tvs where each neighborhood of a point includes a convex neighborhood of the point. It is for these spaces that the strong separating hyperplane theorem for continuous linear functionals (topologically closed half-spaces) holds.

Taking a page from Barry Simon's book [5], let's make the following definition.

13.3.1 Definition (Regular convex functions) *A* ***regular convex function*** *on a topological vector space is a lower semicontinuous proper convex function.*

A ***regular concave function*** *on a topological vector space is an upper semicontinuous proper concave function.*

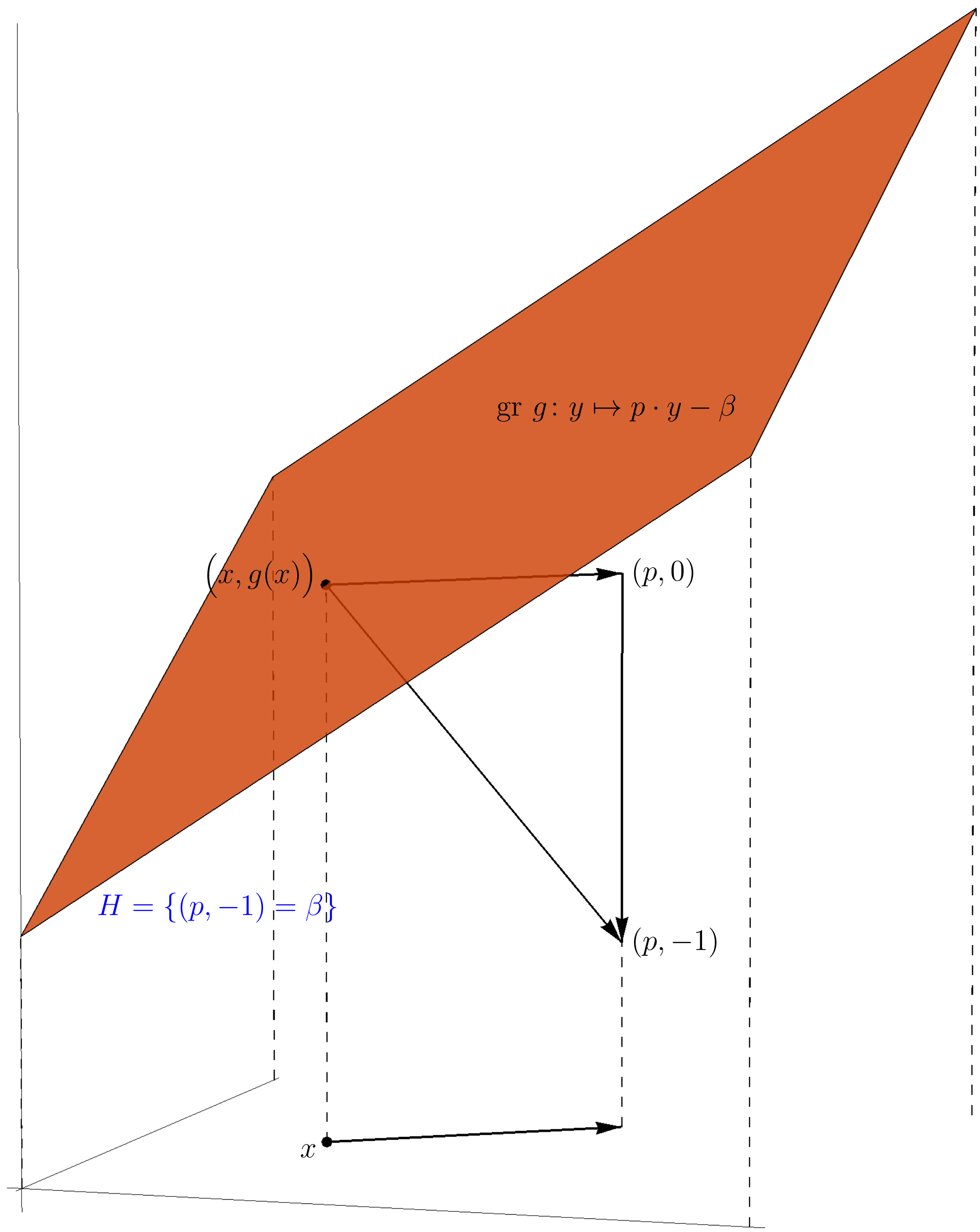

Figure 13.2.1. The graph of $g\colon y \mapsto p \cdot y - \beta$ is the hyperplane $H = \{(x, \alpha) \in X \times \boldsymbol{R} : (p, -1) \cdot (x, \alpha) = \beta\}$.

It is the class of regular convex functions that are most useful and the nicest to work with.

13.3.2 Lemma *Let X be a locally convex Hausdorff space (such as $\boldsymbol{R}^{\mathrm{m}}$), and let $f\colon X \to \boldsymbol{R}^\sharp$ be a regular convex function. If x belongs to the effective domain of f and $\alpha \in \boldsymbol{R}$ satisfies $\alpha < f(x)$, then there exists a continuous affine function g satisfying*

$$g(x) = \alpha \quad \text{and} \quad g \ll f,$$

where $g \ll f$ means $g(y) < f(y)$ for all $y \in X$.

Proof: First note that the epigraph of f is a nonempty closed convex subset of $X \times \boldsymbol{R}$, and by hypothesis (x, α) does not belong to $\operatorname{epi} f$. Thus by the Strong Separating Hyperplane Theorem 8.3.2 there is a closed hyperplane H that strongly separates (x, α) from $\operatorname{epi} f$. The hyperplane H cannot be vertical, for by Verticality 13.2.1 a vertical hyperplane cannot properly separate (x, α) from $\big(x, f(x)\big)$. Therefore H is the graph of a continuous affine function g (Proposition 13.2.2). Moreover, $g \ll f$, for if $g(y) \geqslant f(y)$, then $\big(y, g(y)\big) \in \operatorname{gr} g \cap \operatorname{epi} f$, which is ruled out by strong separation. See Figure 13.3.1. ∎

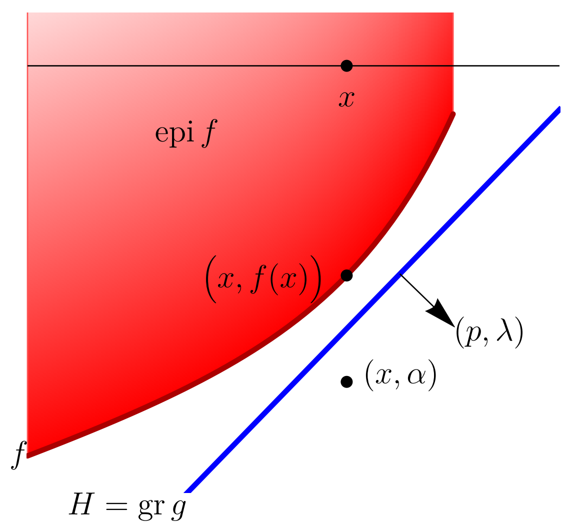

Figure 13.3.1. The linear function (p, λ) (or, equivalently, the non-vertical hyperplane $H = \operatorname{gr} g$) strongly separates (x, α) from $\operatorname{epi} f$.

13.3.3 Theorem *Let X be a locally convex Hausdorff space, and let $f\colon X \to \boldsymbol{R}^\sharp$ be a regular convex function. Then for each x we have*

$$f(x) = \sup\{g(x) : g \ll f \ \text{ and } \ g \text{ is affine and continuous}\}.$$

Proof: Fix x and let $\alpha \in \boldsymbol{R}$ satisfy $\alpha < f(x)$. (Since f is proper, we cannot have $f(x) = -\infty$, so such a finite real number α exists.) We need to show that there is a continuous affine function g with $g \ll f$ and $g(x) \geqslant \alpha$.

There are two cases to consider. The first is that x belongs to the effective domain of f. This is covered by Lemma 13.3.2 directly.

In case x is not in the effective domain, we may still proceed as in the proof of Lemma 13.3.2 to show that there exists a hyperplane H defined by a nonzero continuous linear functional (p, λ) that strongly separates (x, α) from $\operatorname{epi} f$. But we cannot use the previous argument to conclude that the hyperplane is non-vertical! So suppose that $\lambda = 0$. Then strong separation can be written as

$$p(y) < p(x) - \varepsilon \text{ for every } y \in \operatorname{dom} f$$

for some $\varepsilon > 0$. See Figure 13.3.2. Define the affine function h by

$$h(z) = p(z) - p(x) + \varepsilon/2$$

and observe that

$$h(x) = \varepsilon/2 > 0 \quad \text{and for } y \in \operatorname{dom} f, \quad h(y) < -\varepsilon/2 < 0.$$

Next pick some $\bar{y} \in \operatorname{dom} f$, and use Lemma 13.3.2 to find an affine function $\bar{g}$ satisfying

$$\bar{g} \ll f$$

(and $\bar{g}(\bar{y}) = f(\bar{y}) - 1$, which is irrelevant for our purpose). Now consider the affine functions g_γ of the form

$$g_\gamma(z) = \gamma h(z) + \bar{g}(z), \qquad \gamma > 0.$$

For $y \in \operatorname{dom} f$ we have $h(y) < 0$ so $g_\gamma(y) < \bar{g}(y) < f(y)$. For $y \notin \operatorname{dom} f$, we have $f(y) = \infty$. Thus for any $\gamma > 0$ and any y, we have

$$g_\gamma(y) < f(y).$$

But $h(x) > 0$, so for γ large enough,

$$g_\gamma(x) > \alpha,$$

as desired. ∎

A remark is in order. We know that the epigraph of a regular convex function is a proper closed convex subset of $X \times \boldsymbol{R}$. Therefore it is the intersection of all the closed half-spaces that include it. The theorem refines this to the intersection of all the closed non-vertical half spaces that include it.

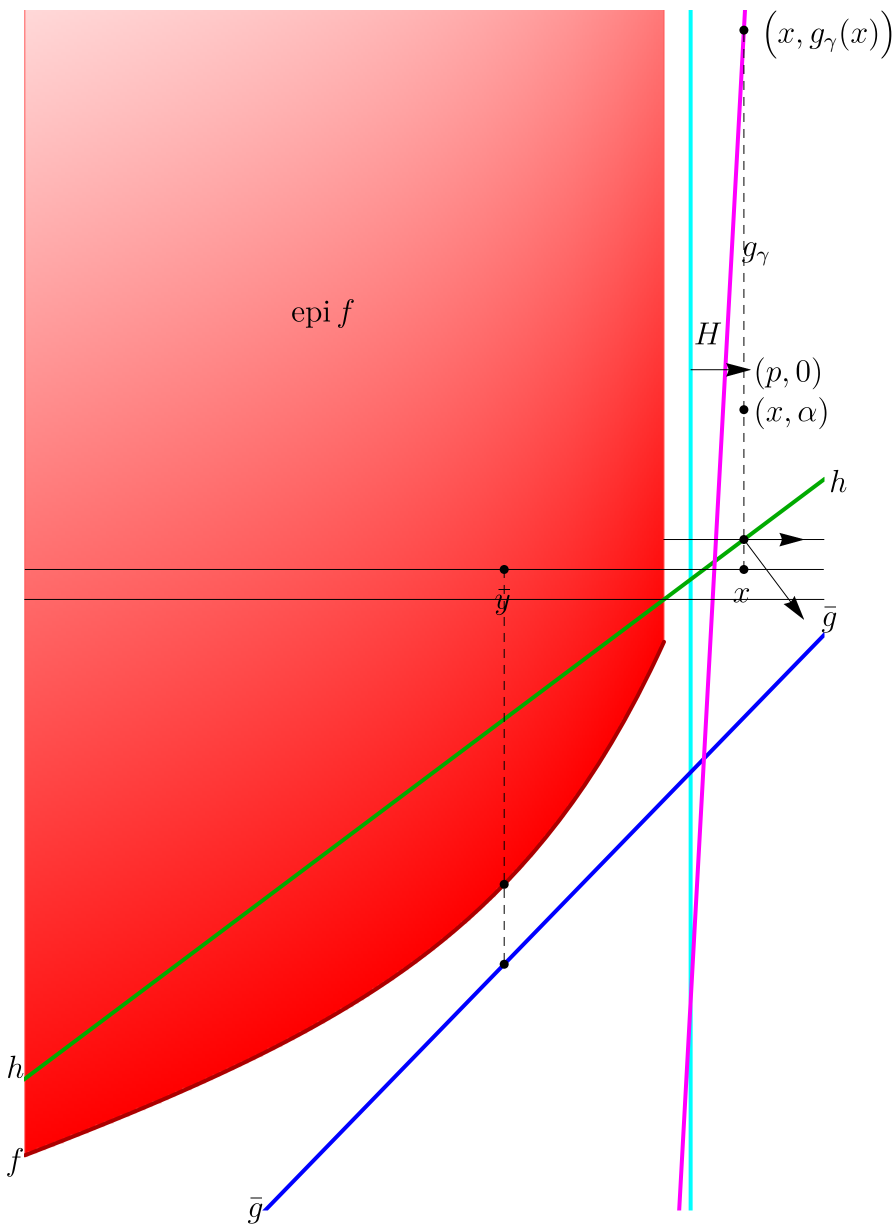

Figure 13.3.2. The proof of Theorem 13.3.3: Here $x \notin \operatorname{dom} f$, but $\bar y \in \operatorname{dom} f$. The linear function $(p, 0)$ (or the vertical hyperplane H) strongly separates (x, α) from epi f by ε. (That is, $p(y) < p(x) - \varepsilon$ for all $y \in \operatorname{dom} f$.) The affine function $\bar g$ satisfies $\bar g \ll f$, and the affine function

$$h \colon y \mapsto p(y) - p(x) + \varepsilon/2$$

satisfies $h(x) = \varepsilon/2 > 0$ and $h(y) < -\varepsilon/2 < 0$ for $y \in \operatorname{dom} f$. The affine function $g_\gamma = \gamma h + \bar g$ satisfies $g_\gamma \ll f$ for $\gamma > 0$, and for γ large enough $g_\gamma(x) > \alpha$.

13.4 Appendix: Semicontinuous functions

Combine this with Appendix A.8 and move to a self-contained location.

The real-valued function $f\colon X \to \boldsymbol{R}$ is **upper semicontinuous on $\boldsymbol{X}$** if for each $\alpha \in \boldsymbol{R}$, the superlevel set $\{f \geqslant \alpha\}$ is closed, or equivalently, the strict sublevel set $\{f < \alpha\}$ is open. It is **lower semicontinuous** if every sublevel set $\{f \leqslant \alpha\}$ is closed, or equivalently, the strict superlevel set $\{f > \alpha\}$ is open.

The extended real valued function f is **upper semicontinuous at the point $\boldsymbol{x}$** if $f(x) < \infty$ and

$$(\forall \varepsilon > 0)\ (\exists \delta > 0)\ [\, d(y,x) < \delta \implies f(y) < f(x) + \varepsilon \,].$$

Similarly, f is **lower semicontinuous at the point $\boldsymbol{x}$** if $f(x) > -\infty$ and

$$(\forall \varepsilon > 0)\ (\exists \delta > 0)\ [\, d(y,x) < \delta \implies f(y) > f(x) - \varepsilon \,].$$

Equivalently, f is upper semicontinuous at x if $f(x) < \infty$ and

$$f(x) \geqslant \limsup_{y \to x} f(y) = \inf_{\varepsilon > 0} \sup_{0 < d(y,x) < \varepsilon} f(y).$$

Similarly, f is lower semicontinuous at x if $f(x) > -\infty$ and

$$f(x) \leqslant \liminf_{y \to x} f(y) = \sup \varepsilon > 0 \inf_{0 < d(y,x) < \varepsilon} f(y).$$

Note that f is upper semicontinuous if and only if $-f$ is lower semicontinuous.

13.4.1 Lemma *A real valued function $f\colon X \to \boldsymbol{R}$ is upper semicontinuous on X if and only if it is upper semicontinuous at each point of X. It is lower semicontinuous on X if and only if it is lower semicontinuous at each point of X.*

Proof: I'll prove the result for upper semicontinuity. Assume that f is upper semicontinuous on X. For any real number α, if $f(x) < \beta < \alpha$, then $\{y \in X : f(y) < \beta\}$ is an open neighborhood of x. Thus for $\varepsilon > 0$ small enough $d(y,x) < \varepsilon$ implies $f(y) < \beta$. Therefore $\limsup_{y \to x} f(y) \leqslant \beta < \alpha$. Setting $\alpha = \limsup_{y \to x} f(y)$, we see that it cannot be the case that $f(x) < \limsup_{y \to x} f(y)$, for then $f(x) < \alpha = \limsup_{y \to x} f(y) < \alpha$, a contradiction. That is, f is upper semicontinuous at x.

For the converse, assume that f is upper semicontinuous at each x. Fix a real number α, and let $f(x) < \alpha$. Since $f(x) \geqslant \limsup_{y \to x} f(y)$, there is $\varepsilon > 0$ small enough so that $\sup_{y : 0 < d(y,x) < \varepsilon} f(y) < \alpha$, but this implies $\{x \in X : f(x) < \alpha\}$ is open, so f is upper semicontinuous on X. ∎

13.4.2 Corollary *A real-valued function is continuous if and only if it is both upper and lower semicontinuous.*

13.4.3 Theorem *An extended real-valued function f is upper semicontinuous on X if and only if its hypograph is closed. It is lower semicontinuous on X if and only if its epigraph is closed.*

Proof: Assume f is upper semicontinuous, and let (x_n, α_n) be a sequence in its hypograph, that is, $f(x_n) \geqslant \alpha_n$ for all n. Therefore $\limsup_n f(x_n) \geqslant \limsup_n \alpha_n$. If $(x_n, \alpha_n) \to (x, \alpha)$, since f is upper semicontinuous at x, we have $\alpha = \lim_n \alpha_n \leqslant \limsup_n f(x_n) \leqslant f(x)$. Thus $\alpha \leqslant f(x)$, or (x, α) belong to the hypograph of f. Therefore the hypograph is closed.

Assume now that the hypograph is closed. Pick x and let $\alpha = \limsup_{y \to x} f(y)$. Then there is a sequence $x_n \to x$ with $f(x_n) \uparrow \alpha$. Since $\big(x_n, f(x_n)\big)$ belongs to the hypograph for each n, so does its limit (x, α). That is, $\limsup_{y \to x} f(y) = \alpha \leqslant f(x)$, so f is upper semicontinuous at x. ∎

13.4.4 Exercise Prove that if both the epigraph and hypograph of a function are closed, then the graph is closed. Give an example to show that the converse is not true. □

13.4.5 Proposition *The infimum of a family of upper semicontinuous functions is upper semicontinuous. The supremum of a family of lower semicontinuous functions is lower semicontinuous.*

Proof: Let $\{f_\nu\}_\nu$ be a family of upper semicontinuous functions, and let $f(x) = \inf_\nu f_\nu(x)$. Then $\{f \geqslant \alpha\} = \cap_\nu \{f_\nu \geqslant \alpha\}$, which is closed. Lower semicontinuity is dealt with *mutatis mutandis*. ∎

13.4.6 Definition *Given an extended real-valued function f on the metric space X, we define the* ***upper envelope*** $\overline{f}$ *of f by*

$$\overline{f}(x) = \max\{f(x), \limsup_{y \to x} f(y)\} = \inf_{\varepsilon > 0} \sup_{d(y,x) < \varepsilon} f(y),$$

and the ***lower envelope*** $\underline{f}$ *of f by*

$$\underline{f}(x) = \min\{f(y), \liminf_{y \to x} f(y)\}, = \sup_{\varepsilon > 0} \inf_{d(y,x) < \varepsilon} f(y).$$

Clearly if f is upper semicontinuous at x, then $f(x) = \overline{f}(x)$, and if f is lower semicontinuous at x, then $f(x) = \underline{f}(x)$. Consequently, f is upper semicontinuous if and only $f = \overline{f}$, and f is lower semicontinuous if and only $f = \underline{f}$.

We say that the real-valued function g **dominates** the real-valued function f on X if for every $x \in X$ we have $g(x) \geqslant f(x)$.

13.4.7 Theorem *The upper envelope $\overline{f}$ is the smallest upper semicontinuous function that dominates f and the lower envelope $\underline{f}$ is the greatest lower semicontinuous function that f dominates.*

Moreover,

$$\operatorname{hypo} \overline{f} = \overline{\operatorname{hypo} f}.$$

and

$$\mathit{epi}\, \underline{f} = \overline{\mathit{epi}\, f}.$$

Proof: Clearly, $\overline{f}$ dominates f and f dominates $\underline{f}$.

Now suppose g is upper semicontinuous and dominates f. Then for any x, we have $g(x) \geqslant \limsup_{y \to x} g(y) \geqslant \limsup_{y \to x}$, so $g(x) \geqslant \overline{f}(x)$. That is, g dominates $\overline{f}$.

Similarly if g is lower semicontinuous and f dominates g, then $\underline{f}$ dominates g.

It remains to show that $\overline{f}$ is upper semicontinuous. It suffices to prove that the hypograph of $\overline{f}$ is closed. We prove the stronger result that

$$\operatorname{hypo} \overline{f} = \overline{\operatorname{hypo} f}.$$

Let (x_n, α_n) be a sequence in the hypograph of f, and assume it converges to a point (x, α). Since $\alpha_n \leqslant f(x_n)$, we must have $\alpha \leqslant \limsup_{y \to x} f(y)$, so $\alpha \leqslant \overline{f}$. That is, $\overline{\operatorname{hypo} f} \subset \operatorname{hypo} \overline{f}$. For the opposite inclusion, suppose by way of contradiction that (x, α) belongs to the hypograph of $\overline{f}$, but not to $\overline{\operatorname{hypo} f}$. Then there is a neighborhood $B_\varepsilon(x) \times B_\varepsilon(\alpha)$ disjoint from $\overline{\operatorname{hypo} f}$. In particular, if $d(y, x) < \varepsilon$, then $f(y) < \alpha \leqslant \overline{f}(x)$, which implies $\overline{f}(x) > \max\{f(x), \limsup -y \to x f(y)\}$, a contradiction. Therefore $\overline{\operatorname{hypo} f} \supset \operatorname{hypo} \overline{f}$.

The case of $\underline{f}$ is similar. ∎

13.5 ⋆ Appendix: Closed functions revisited

Rockafellar [4, p 52, pp. 307–308] makes the following definition. Recall from Definition 13.4.6 that the upper envelope of f is defined by

$$\overline{f}(x) = \inf_{\varepsilon > 0} \sup_{d(y,x) < \varepsilon} f(y),$$

and that the upper envelope is real-valued if f is locally bounded, and is upper semicontinuous.

13.5.1 Definition *The **closure** $\operatorname{cl} f$ of a convex function f on $\boldsymbol{R}^{\mathrm{m}}$ is defined by*

1. $\operatorname{cl} f(x) = -\infty$ *for all* $x \in \boldsymbol{R}^{\mathrm{m}}$ *if* $f(y) = -\infty$ *for some* y.
2. $\operatorname{cl} f(x) = +\infty$ *for all* $x \in \boldsymbol{R}^{\mathrm{m}}$ *if* $f(x) = +\infty$ *for all* x.
3. $\operatorname{cl} f$ *is the lower envelope of* f *if* f *is a proper convex function.*

*The **closure** $\operatorname{cl} f$ of a concave function f on $\boldsymbol{R}^{\mathrm{m}}$ is defined by*

1'. $\operatorname{cl} f(x) = +\infty$ *for all* $x \in \boldsymbol{R}^{\mathrm{m}}$ *if* $f(y) = +\infty$ *for some* y.

2'. $\operatorname{cl} f(x) = -\infty$ *for all* $x \in \boldsymbol{R}^{\mathrm{m}}$ *if* $f(x) = -\infty$ *for all* x.

3'. $\operatorname{cl} f$ *is the upper envelope of* f *if* f *is a proper concave function.*

13.5.2 Definition (Closed functions à la Rockafellar) *A convex (or concave) function is **closed** if and only if* $f = \operatorname{cl} f$.

13.5.3 Proposition *If $f\colon \boldsymbol{R}^{\mathrm{m}} \to \boldsymbol{R}^{\sharp}$ is convex, then* $\operatorname{cl} f$ *is convex. If f is concave, then* $\operatorname{cl} f$ *is concave.*

Proof: I shall just prove the concave case. If f is concave and does not assume the value $+\infty$, Theorem 13.4.7 asserts that the hypograph of the closure of f is the closure of the hypograph of f, which is convex. If f does assume the value $+\infty$, then $\operatorname{cl} f$ is identically $+\infty$, so its hypograph is $\boldsymbol{R}^{\mathrm{m}} \times \boldsymbol{R}$, which is convex. Either way, the hypograph of $\operatorname{cl} f$ is convex. ∎

The next result is that my Definition 21.1.3 and Rockafellar's Definition 13.5.1 agree.

13.5.4 Theorem *Let $f\colon \boldsymbol{R}^{\mathrm{m}} \to \boldsymbol{R}^{\sharp}$ be concave. Then for every $x \in \boldsymbol{R}^{\mathrm{m}}$,*

$$\operatorname{cl} f(x) = \inf\{h(x) : h \geqslant f \text{ and } h \text{ is affine and continuous}\} = \hat{f}.$$

If $f\colon \boldsymbol{R}^{\mathrm{m}} \to \boldsymbol{R}^{\sharp}$ is convex, then for every $x \in \boldsymbol{R}^{\mathrm{m}}$,

$$\operatorname{cl} f(x) = \sup\{h(x) : f \geqslant h \text{ and } h \text{ is affine and continuous}\} = \check{f}.$$

Proof: I shall prove the concave case. There are three subcases. If f is improper and assumes the value $+\infty$, then by definition $\operatorname{cl} f$ is the constant function $+\infty$. In this case, no affine function, which is (finite) real-valued, dominates f so the infimum is over the empty set, and thus $+\infty$. The second subcase is that f is the improper constant function $-\infty$. In this case every affine function dominates f, so the infimum is $-\infty$.

So assume we are in the third subcase, namely that f is proper. That is, $f(x) < \infty$ for all $x \in \boldsymbol{R}^{\mathrm{m}}$, and $\operatorname{dom} f$ is nonempty. Then by definition $\operatorname{cl} f$ is the upper envelope of f. That is,

$$\operatorname{cl} f(x) = \inf_{\varepsilon>0} \sup_{d(y,x)<\varepsilon} f(y).$$

Define $g(x) = \inf\{h(x) : h \geqslant f$ and h is affine and continuous$\}$. If h is affine, continuous, and dominates f, then by Theorem 13.4.7, h dominates $\operatorname{cl} f$, so g dominates $\operatorname{cl} f$.

We now show that $\operatorname{cl} f \geqslant g$. It suffices to show that for any (x,α) with $\alpha > \operatorname{cl} f(x)$, there is an affine function h dominating f with $h(x) \leqslant \alpha$. Now $\alpha > \operatorname{cl} f(x) = \limsup_{y\to x} f(y)$ implies that (x,α) does not belong to the closure of the hypograph of f.

There are two cases to consider. The simpler case is that x belongs to $\operatorname{dom} f$. So assume now that $x \in \operatorname{dom} f$. Since f is concave, its hypograph and the closure thereof are convex, and since f is proper, its hypograph is nonempty. So by Corollary 8.3.2 there is a nonzero $(p,\lambda) \in \boldsymbol{R}^{\mathrm{m}} \times \boldsymbol{R}$ strongly separating (x,α) from the closure of the hypograph of f. In particular, for each $y \in \operatorname{dom} f$ the point

$\big(y, f(y)\big)$ belongs to the hypograph of f. Thus strong separation implies that for some $\varepsilon > 0$, for any $y \in \operatorname{dom} f$,

$$p \cdot x + \lambda\alpha > p \cdot y + \lambda f(y) + \varepsilon. \tag{1}$$

The same argument as that in the proof of Lemma 13.3.2 shows that $\lambda \geqslant 0$. Moreover, taking $y = x$ (since $x \in \operatorname{dom} f$) shows that $\lambda \neq 0$. So dividing by λ gives

$$(1/\lambda)p \cdot (x - y) + \alpha > f(y) + (\varepsilon/\lambda)$$

for all $y \in \operatorname{dom} f$. Define

$$h(y) = (1/\lambda)p \cdot (x - y) + \alpha.$$

Then h is a continuous affine function satisfying

$$h(y) > f(y) + \eta \quad \text{for all} y \in \operatorname{dom} f,$$

where $\eta = (\varepsilon/\lambda) > 0$ and $h(x) = \alpha$, as desired.

The case where (x, α) satisfies $\alpha > \operatorname{cl} f(x)$, but $x \notin \operatorname{dom} f$ is more subtle. The reason the above argument does not work is that the hyperplane may be vertical ($\lambda = 0$), and hence not the graph of any affine function. So assume that $\lambda = 0$. Then (1) becomes

$$p \cdot x > p \cdot y + \varepsilon$$

for all $y \in \operatorname{dom} f$. Define the continuous affine function g by

$$g(y) = p \cdot (x - y) - \varepsilon/2,$$

and note that $g(x) < 0$, and $g(y) > 0$ for all $y \in \operatorname{dom} f$.

But we still have (from the above argument) a continuous affine function h satisfying

$$h(y) > f(y) \quad \text{for all} y \in \operatorname{dom} f.$$

Now for any $\gamma > 0$, we have

$$\gamma g(y) + h(y) > f(y) \quad \text{for all} y \in \operatorname{dom} f,$$

and for $y \notin \operatorname{dom} f$, $f(y) = -\infty$, so the inequality holds for all y in $\boldsymbol{R}^{\mathrm{m}}$. But since $g(x) < 0$, for γ large enough, $\gamma g(x) + h(x) < \alpha$, so this is the affine function we wanted.

I think that covers all the bases (and cases).

The case of a convex function is dealt with by replacing the epigraph with the hypograph and reversing inequalities. ∎

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