Ec 181 KC Border
Convex Analysis and Economic Theory AY 2019–2020

Topic 12: The welfare theorems and the core of an economy

12.1 The First Welfare Theorem

We now come to what are known as the two Fundamental Theorems of Welfare Economics.[1] The first welfare theorem is that Walrasian equilibria are Pareto efficient, and the second is a kind of converse.

12.1.1 Theorem *If all utilities are locally nonsatiated, a Walrasian equilibrium allocation is Pareto efficient.*

Proof: Let $(\bar{x}^1, \dots, \bar{x}^m, \bar{y}^1, \dots, \bar{y}^n, \bar{p})$ be a Walrasian equilibrium. Suppose by way of contradiction that the allocation is inefficient. That is, that there exists another allocation

$$(\hat{x}^1, \dots, \hat{x}^m, \hat{y}^1, \dots, \hat{y}^n)$$

such that

$$u_i(\hat{x}^i) \geqslant u_i(\bar{x}^i) \text{ for all } i \quad \text{and} \quad u_i(\hat{x}^i) > u_i(\bar{x}^i) \text{ for some } i.$$

Since every utility is locally nonsatiated, and consumers are maximizing utility, by Lemma 11.7.1 we have

$$u_i(\hat{x}^i) \geqslant u_i(\bar{x}^i) \implies \bar{p} \cdot \hat{x}^i \geqslant \bar{p} \cdot \bar{x}^i \quad \text{and} \quad u_i(\hat{x}^i) > u_i(\bar{x}^i) \implies \bar{p} \cdot \hat{x}^i > \bar{p} \cdot \bar{x}^i.$$

Summing over i gives

$$\bar{p} \cdot \sum_{i=1}^m \hat{x}^i > \bar{p} \cdot \sum_{i=1}^m \bar{x}^i.$$

Since enterprises are maximizing profits, for each j,

$$\bar{p} \cdot \bar{y}^j \geqslant \bar{p} \cdot \hat{y}^j,$$

so summing gives

$$\bar{p} \cdot \sum_{j=1}^n \bar{y}^j \geqslant \bar{p} \cdot \sum_{j=1}^n \hat{y}^j.$$

[1] This terminology may have first appeared in Feldman [15, Chapter 3], who presents both the First Fundamental Theorem of Welfare Economics and the Second Fundamental Theorem of Welfare Economics, but similar terminology was in use earlier. Dorfman, Samuelson, and Solow [13] title their section 14.7 The Basic Theorem of Welfare Economics, and state (p. 410), *"Every competitive equilibrium is a Pareto-optimum; and every Pareto-optimum is a competitive equilibrium."* Arrow [3] titled his paper "An Extension of the Basic Theorems of Classical Welfare Economics."

On the other hand, by definition of allocation we have

$$\sum_{i=1}^{m} \bar{x}^i = \sum_{i=1}^{m} \omega^i + \sum_{j=1}^{n} \bar{y}^j$$

and

$$\sum_{i=1}^{m} \hat{x}^i = \sum_{i=1}^{m} \omega^i + \sum_{j=1}^{n} \hat{y}^j.$$

Stringing these together gives

$$\begin{aligned}
\bar{p} \cdot \left(\sum_{i=1}^{m} \omega^i + \sum_{j=1}^{n} \bar{y}^j \right) &\geqslant \bar{p} \cdot \left(\sum_{i=1}^{m} \omega^i + \sum_{j=1}^{n} \hat{y}^j \right) \\
&= \bar{p} \cdot \sum_{i=1}^{m} \hat{x}^i \\
&> \bar{p} \cdot \sum_{i=1}^{m} \bar{x}^i. \\
&= \bar{p} \cdot \left(\sum_{i=1}^{m} \omega^i + \sum_{j=1}^{n} \bar{y}^j \right),
\end{aligned}$$

a contradiction. ∎

12.2 The Second Welfare Theorem

Consider an Arrow–Debreu–McKenzie model economy

$$E = \Big((X_i, u_i)_{i=1}^m, (Y_j)_{j=1}^n, \omega \Big).$$

Let $Y = \sum_{j=1}^n Y_j$ denote the aggregate production set.

12.2.1 Second Welfare Theorem *Assume the economy E satisfies the following conditions.*

1. *For each consumer $i = 1, \dots, m$*
   - *(a) X_i is nonempty and convex.*
   - *(b) u_i is continuous, monotonic, and explicitly quasiconcave.*
2. *The aggregate production set*
   - *(a) Y is nonempty and convex.*

Let $(\bar{x}^1, \dots, \bar{x}^m, \bar{y}^1, \dots, \bar{y}^n)$ be an efficient allocation. Then there is a nonzero price vector $\bar{p}$ satisfying

1. *For each consumer $i = 1, \dots, m$, $\bar{x}^i$ minimizes $\bar{p} \cdot x$ over the upper contour set $\{x \in X_i : u(x) \geqslant u(\bar{x}^i)\}$.*

   Thus if there is a cheaper point $\tilde{x} \in X_i$ satisfying $\bar{p} \cdot \tilde{x} < \bar{p} \cdot \bar{x}^i$, then $\bar{x}^i$ actually maximizes u_i over the budget set $\{x \in X_i : \bar{p} \cdot x \leqslant \bar{p} \cdot \bar{x}^i\}$.

2. *For each producer $j = 1, \dots, n$, $\bar{y}^j$ maximizes profit over Y_j at prices $\bar{p}$. That is,*

$$\bar{p} \cdot \bar{y}^j \geqslant \bar{p} \cdot y \qquad \text{for all } y \in Y_j.$$

That is, $(\bar{x}^1, \dots, \bar{x}^m, \bar{y}^1, \dots, \bar{y}^n, \bar{p})$ is a ***valuation quasi-equilibrium****. If the cheaper point condition holds for each i, then it is a* ***valuation equilibrium****.*

Proof: Since $(\bar{x}^1, \dots, \bar{x}^m, \bar{y}^1, \dots, \bar{y}^n)$ is efficient, it is impossible to make everyone better off. So define the "Scitovsky set" S by

$$S = \sum_{i=1}^{m} P_i(\bar{x}^i),$$

(see Figure 12.2.1) and define the aggregate consumption possibility set A by

$$A = \omega + \sum_{j=1}^{n} Y_j.$$

By efficiency $A \cap S = \varnothing$. (For suppose, $x \in A \cap S$. Since $x \in S$, we can write $x = \sum_{i=1}^m x^i$, where each $x^i \in P(\bar{x}^i)$, or $u(x) > u(\bar{x}^i)$. Since $x \in A$, we can write $x = \omega + \sum_{j=1}^n y^j$. But then $(x^1, \dots, x^m, y^1, \dots, y^n)$ is an allocation, and $u(x^i) > u(\bar{x}^i)$ for each i, contradicting the efficiency of $(\bar{x}^1, \dots, \bar{x}^m, \bar{y}^1, \dots, \bar{y}^n)$.)

Now each $P_i(\bar{x}_i)$ is nonempty, open, and convex since each u_i is continuous, monotonic, and quasiconcave. Therefore the sum S is nonempty, open, and convex. Similarly A is convex. Thus by the Separating Hyperplane Theorem, there is a nonzero price vector $\bar{p}$ satisfying

$$\bar{p} \cdot x \geqslant \bar{p} \cdot y \quad \text{for each } x \in S,\ y \in A.$$

From Lemma 11.6.1, each $\bar{x}^i$ belongs to the closure of $P_i(\bar{x}^i)$, so $\sum_{i=1}^m \bar{x}^i$ belongs to the closure of S. Now $\sum_{i=1}^m \bar{x}^i = \omega + \sum_{j=1}^n \bar{y}^j$ so it also belongs to A. It follows that

$$\bar{p} \cdot x \geqslant \bar{p} \cdot \sum_{i=1}^{m} \bar{x}^i = \bar{p} \cdot \left(\omega + \sum_{j=1}^{n} \bar{y}^j \right) \geqslant \bar{p} \cdot y \quad \text{for each } x \in S,\ y \in A.$$

From the Summation Principle 0.4.1, we then have

$$\bar{p} \cdot \bar{x}^i \leqslant \bar{p} \cdot x \quad \text{for all } x \in P(\bar{x}^i) \qquad \text{and} \qquad \bar{p} \cdot \bar{y}^j \geqslant \bar{p} \cdot y \quad \text{for all } y \in Y_j.$$

Since $U(\bar{x}^i)$ is the closure of $P(\bar{x}^i)$ we also have

$$\bar{p} \cdot \bar{x}^i \leqslant \bar{p} \cdot x \quad \text{for all } x \in U(\bar{x}^i).$$

This proves that we have a valuation quasi-equilibrium. The role of the cheaper point condition is well known (see Lemma 11.7.2). ∎

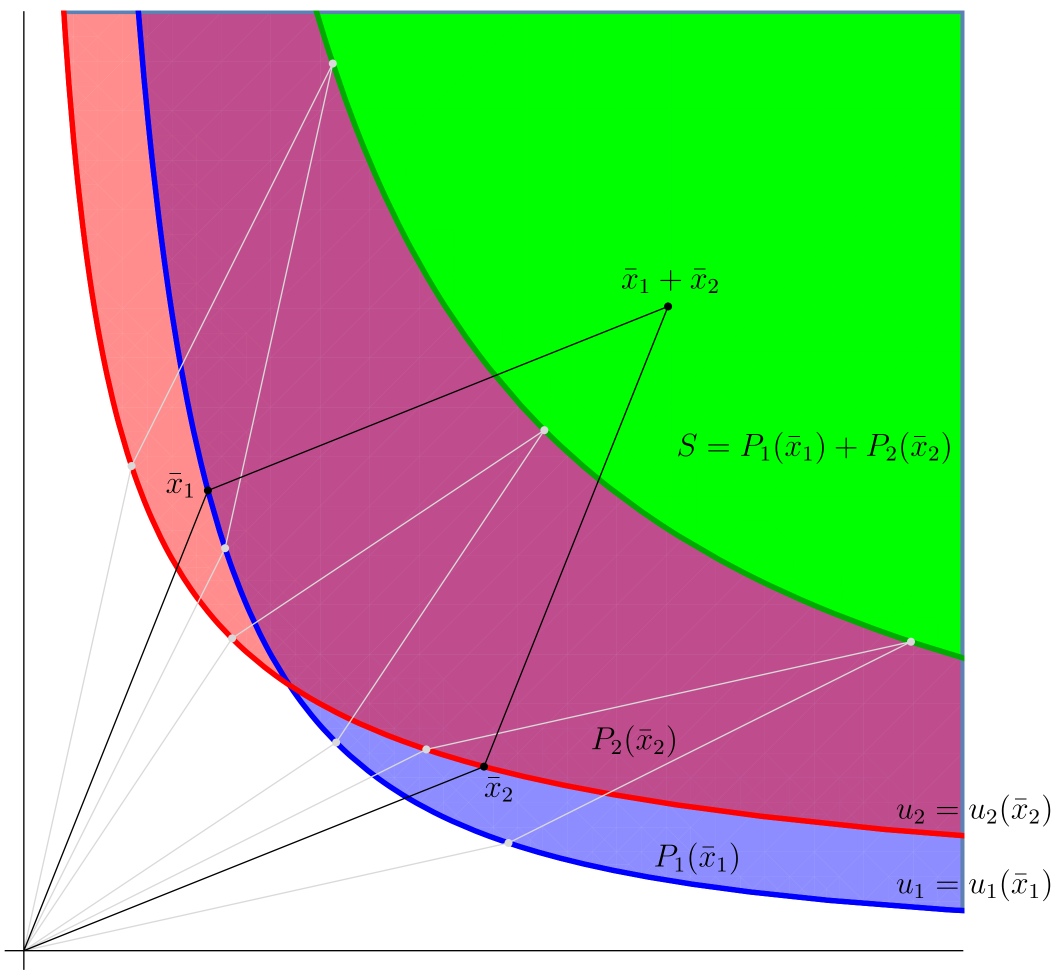

Figure 12.2.1. Construction of the Scitovsky set for 2 consumers and two goods. N.B. In this case, $\bar{x}$ is *not* efficient, and $\bar{x}_1 + \bar{x}_2$ lies in the interior of S, not on its boundary.

12.3 Digression: Drawing the Scitovsky set

Figure 12.2.1 shows the Scitovsky set for two consumers and two commodities. How was it drawn? First pick two utility functions u_1 and u_2, and two utility levels v_1 and v_2. They determine two upper contour sets U_1 and U_2. We want to find the sum $\bar{S} = U_1 + U_2$ of these sets. Now pick a nonnegative vector p and find its minimizer $\hat{x}$ over $\bar{S}$. This will be a point on the lower boundary of $\bar{S}$. Then by the Summation Principle $\hat{x} = \hat{x}_1 + \hat{x}_2$, where $\hat{x}_i$ minimizes p over U_i. That is, p supports U_i at $\hat{x}_i$. So to find the lower boundary of $\bar{S}$, for each p we find the points $\hat{x}_1$ and $\hat{x}_2$ that minimize p over the sets U_1 and U_2 and just add them up.

If you are familiar with demand theory, you will recognize these points as Hicksian compensated demands. Even if you don't recognize them as such, you may still solve for the minimizer $\hat{x}$ as a function of p and v. Then as p ranges over all positive price vectors, the sum $\hat{x}_1(p, v_1) + \hat{x}_2(p, v_2)$ traces out the lower boundary of $\bar{S}$. With two goods only the ratio p_1/p_2 matters, so this is a simple one-parameter parametric plot. At each $\hat{x}_i$, the gradient of u_i is proportional to p (the classical Lagrange Multiplier Theorem), so it's simply a matter of finding the $\hat{x}_i$ where $u_i(\hat{x}_i) = v_i$ and $\nabla u_i(\hat{x}_i) = \lambda_i p$. This is something we learn to do in intermediate microeconomics classes.

12.4 Saddlepoints and the Second Welfare Theorem

This is based on Negishi [18] and Takayama and El-Hodiri [21].

Consider an n-person pure exchange economy with aggregate endowment $\omega \in \boldsymbol{R}^{\mathrm{m}}_{++}$. Let $u_i \colon \boldsymbol{R}^{\mathrm{m}}_{+} \to \boldsymbol{R}$ denote person i's utility function. (This implicitly assumes that preferences are selfish.) Recall that an *allocation* is a vector $x = (x^1, \dots, x^n) \in (\boldsymbol{R}^{\mathrm{m}}_{+})^{\mathrm{n}}$ satisfying $\sum_{i=1}^n x^i = \omega$. An allocation $\bar{x} = (\bar{x}^1, \dots, \bar{x}^n)$ is *Pareto efficient* if there is no allocation $x = (x^1, \dots, x^n)$ satisfying

$$u_i(x^i) \geqslant u_i(\bar{x}^i) \text{ for all } i = 1, \dots, n \quad \text{and} \quad u_i(x^i) > u_i(\bar{x}^i) \text{ for some } i.$$

An allocation $\tilde{x} = (\tilde{x}^1, \dots, \tilde{x}^n)$ is a *valuation equilibrium allocation* if there exists a nonzero price vector $p \in \boldsymbol{R}^{\mathrm{m}}$ such that for every $i = 1, \dots, n$, and every $z \in \boldsymbol{R}^{\mathrm{m}}_{+}$,

$$u^i(z) > u^i(\tilde{x}^i) \quad \implies \quad p \cdot z > p \cdot \tilde{x}^i.$$

That is, everyone is maximizing their utility subject to a budget constraint.

Assume now that each utility function is concave and strictly monotonic. Use the Saddlepoint Theorem to show that every strictly positive Pareto efficient allocation is a valuation equilibrium allocation.

Sample Answer

Let $\bar{x} = (\bar{x}^1, \dots, \bar{x}^n)$ be a strictly positive Pareto efficient allocation. For each i, set $v_i = u^i(\bar{x}^i)$. Then $\bar{x}$ solves the following constrained maximization problem.

$$\underset{(x^1,\dots,x^n)\in(\boldsymbol{R}^{\mathrm{m}}_+)^{\mathrm{n}}}{\text{maximize}}\ u^1(x^1) \text{ subject to } u^i(x^i) \geqslant v_i,\ i = 2,\dots,n, \text{ and } \sum_{i=1}^n x^i = \omega.$$

Since each u^i is monotonic, we may replace the resource constraints with the inequality constraints $\omega_j - \sum_{i=1}^n x^i_j \geqslant 0$, for $j = 1,\dots,m$. Since each $\bar{x}^i > 0$ and each u^i is monotonic, we see that $\tilde{x}$, defined by $\tilde{x}^1 = 0$ and $\tilde{x}^i = \bar{x}^i + \frac{1}{n}\bar{x}^1$ for $i = 2,\dots,n$, satisfies $u^i(\tilde{x}^i) - v_i > 0$ for $i = 2,\dots,n$ and $\omega_j - \sum_{i=1}^n \tilde{x}^i_j = \frac{1}{n}\bar{x}^1_j > 0$, so Slater's Condition is satisfied. Now observe that all the constraints are defined by concave functions.

By the Saddlepoint Theorem there exist nonnegative multipliers $\bar{\mu}_2,\dots,\bar{\mu}_n$ and $\bar{\pi}_1,\dots,\bar{\pi}_m$ such that $(\bar{x};\bar{\mu},\bar{\pi})$ is a saddlepoint of the Lagrangean

$$L(x;\mu,\pi) = u^1(x^1) + \sum_{i=2}^n \mu_i\Big(u^i(x^i) - v_i\Big) + \sum_{j=1}^m \pi_j\Big[\omega_j - \sum_{i=1}^n x^i_j\Big]$$

over $(\boldsymbol{R}^{\mathrm{m}}_+)^{\mathrm{n}} \times \Big[\boldsymbol{R}^{\mathrm{n}}_+ \times \boldsymbol{R}^{\mathrm{m}}_+\Big]$. To make things more symmetric, define $\bar{\mu}^1 = 1$. Then the saddlepoint conditions become

$$\begin{aligned}\sum_{i=1}^n \bar{\mu}_i\Big(u^i(x^i) - v_i\Big) &+ \sum_{j=1}^m \bar{\pi}_j\Big[\omega_j - \sum_{i=1}^n x^i_j\Big] \\ &\leqslant \sum_{i=1}^n \bar{\mu}_i\Big(u^i(\bar{x}^i) - v_i\Big) + \sum_{j=1}^m \bar{\pi}_j\Big[\omega_j - \sum_{i=1}^n \bar{x}^i_j\Big] \qquad (1) \\ &\leqslant \sum_{i=1}^n \mu_i\Big(u^i(\bar{x}^i) - v_i\Big) + \sum_{j=1}^m \pi_j\Big[\omega_j - \sum_{i=1}^n \bar{x}^i_j\Big] \qquad (2)\end{aligned}$$

for all $x \in (\boldsymbol{R}^{\mathrm{m}}_+)^{\mathrm{n}}$ and all $(\mu,\pi) \in \boldsymbol{R}^{\mathrm{n}}_+ \times \boldsymbol{R}^{\mathrm{m}}_+$. Furthermore, the complementary slackness conditions

$$\bar{\mu}_i\Big(u^i(\bar{x}^i) - v_i\Big) = 0 \quad i = 1,\dots,n$$

and

$$\bar{\pi}_j\Big[\omega_j - \sum_{i=1}^n \bar{x}^i_j\Big] \quad j = 1,\dots,m$$

are satisfied.

We now show that no $\bar{\pi}_j$ is zero. For suppose $\bar{\pi}_k = 0$. Let e^k denote the k^{th} unit coordinate vector in $\boldsymbol{R}^{\mathrm{m}}$. Let us now evaluate (1) for x given by $x^1 = \bar{x}^1 + e^k$ and $x^i = \bar{x}^i$ for $i = 2,\dots,n$. This yields

$$\begin{aligned}u^1(\bar{x}^1 + e^k) - v_1 + \sum_{i=2}^n \bar{\mu}_i\Big(\bar{u}^i(\bar{x}^i) - v_i\Big) + \sum_{j=1}^m \bar{\pi}_j\Big[\omega_j - \sum_{i=1}^n \bar{x}^i_j\Big] - \bar{\pi}_k \\ \leqslant u^1(\bar{x}^1) - v_1 + \sum_{i=2}^n \bar{\mu}_i\Big(u^i(\bar{x}^i) - v_i\Big) + \sum_{j=1}^m \bar{\pi}_j\Big[\omega_j - \sum_{i=1}^n \bar{x}^i_j\Big],\end{aligned}$$

which in light of the assumption that $\bar\pi_k = 0$ boils down to

$$u^1(\bar x^1 + e^k) \leqslant u^1(\bar x^1),$$

which contradicts the strict monotonicity of u^1.

Moreover, no $\bar\mu_i = 0$ either. For suppose $\bar\mu_k = 0$ for some $k > 1$. Consider equation (1) for x given by $x^1 = \bar x^1 + \bar x^k$, $x^k = 0$, and $x^i = \bar x^i$ for $i \neq 1, k$. Then we get

$$u^1(\bar x^1 + \bar x^k) \leqslant u^1(\bar x^1),$$

which again contradicts the strict monotonicity of u^1.

We now show that for each $i = 1, \dots, n$, the point $(\bar x^i; \frac{1}{\bar\mu_i})$ is a saddlepoint of the function

$$L_i(z;\nu) = u^i(z) + \nu\Big(\sum_{j=1}^m \bar\pi_j(\bar x^i_j - z_j)\Big) \tag{3}$$

over $\boldsymbol{R}^{\mathrm{m}}_+ \times \boldsymbol{R}_+$. That is, we need to show that

$$u^i(z) + \frac{1}{\bar\mu_i}\Big(\sum_{j=1}^m \bar\pi_j(\bar x^i_j - z_j)\Big) \leqslant u^i(\bar x^i) + \frac{1}{\bar\mu_i}\Big(\sum_{j=1}^m \bar\pi_j(\bar x^i_j - \bar x^i_j)\Big) \tag{4}$$

$$\leqslant u^i(\bar x^i) + \nu\Big(\sum_{j=1}^m \bar\pi_j(\bar x^i_j - \bar x^i_j)\Big) \tag{5}$$

for all $z \in \boldsymbol{R}^{\mathrm{m}}_+$ and all $\nu \in \boldsymbol{R}_+$. Clearly (5) is true. Suppose by way of contradiction that for some k and some $z^k \in \boldsymbol{R}^{\mathrm{m}}$, inequality (4) is violated. That is,

$$u^k(z^k) + \frac{1}{\bar\mu_k}\Big(\sum_{j=1}^m \bar\pi_j(\bar x^k_j - z^k_j)\Big) > u^k(\bar x^k). \tag{6}$$

Then subtracting v_k from each side and multiplying by the positive scalar $\bar\mu_k$ yields

$$\bar\mu_k\Big(u^k(z^k) - v_k\Big) + \sum_{j=1}^m \bar\pi_j(\bar x^k_j - z^k_j) > \bar\mu_k\Big(u^k(\bar x^k) - v_k\Big).$$

Evaluating (1) evaluated at $x \in (\boldsymbol{R}^{\mathrm{m}}_+)^{\mathrm{n}}$ defined by $x^i = \bar x^i$ for $i \neq k$ and $x^k = z^k$, we get

$$\sum_{i\neq k} \bar\mu_i\Big(u^i(\bar x^i) - v_i\Big) + \sum_{j=1}^m \bar\pi_j\Big[\omega_j - \sum_{i\neq k}\bar x^i_j\Big] + \bar\mu_k\Big(u^k(z^k) - v_k\Big) - \sum_{j=1}^m \bar\pi_j z^k_j$$
$$\leqslant \sum_{i=1}^n \bar\mu_i\Big(u^i(\bar x^i) - v_i\Big) + \sum_{j=1}^m \bar\pi_j\Big[\omega_j - \sum_{i=1}^n \bar x^i_j\Big],$$

which implies

$$u^k(z^k) + \frac{1}{\bar\mu_k}\Big(\sum_{j=1}^m \bar\pi_j(\bar x^k_j - z^k_j)\Big) \leqslant u^k(\bar x^k),$$

which in turn contradicts (6). This contradiction shows that $(\bar x^k; \frac{1}{\bar\mu_k})$ is a saddlepoint of (3). But now by the easy half of the Saddlepoint Theorem, we see that $\bar x^k$ maximizes $u^k(z)$ over $\boldsymbol{R}^{\mathrm{m}}_+$ subject to $\bar\pi \cdot z \leqslant \bar\pi \cdot \bar x^k$ for each k. That is, $\bar x$ is a valuation equilibrium allocation at the prices $\bar\pi$.

12.5 Core of a pure exchange economy

Consider a pure exchange economy $\mathcal{E}$ with m consumers and ℓ goods. (Each consumption set is $\boldsymbol{R}^\ell_+$.) The endowment of consumer i is ω^i and his utility function is u_i.

12.5.1 Definition *A* ***coalition*** *is a nonempty subset of consumers. An allocation* $(x^1, \dots, x^m)$ *is* ***blocked by coalition S*** *if there is a partial allocation* $(\tilde{x}^i)_{i \in S}$ *such that*

1. $\sum_{i \in S} \tilde{x}^i = \sum_{i \in S} \omega^i$.
2. *For each* $i \in S$, $u(\tilde{x}^i) > u_i(x^i)$.

The allocation is ***weakly blocked*** *if (2) is replaced by*

2′. *For each* $i \in S$, $u_i(\tilde{x}^i) \geqslant u_i(x^i)$, *and for some* $k \in S$, $u_k(\tilde{x}^k) > u_k(x^k)$.

The ***core*** *of the economy is the set of unblocked allocations.*

The core is a generalization of the *contract curve* that was introduced by Francis Y. Edgeworth [14]. The term core goes back to Gillies [16] in his 1963 dissertation on cooperative games. Its use in economics goes back to Shubik [20] in 1959. Scarf [19], Debreu [10], and Debreu and Scarf [11] proved the first "limit theorem" for the core, and Aumann [5] applied the concept to "nonatomic" economies. An excellent monograph on the relation of the core to the set of Walrasian equilibria is Kirman and Hildenbrand [17].

12.5.2 Lemma *If each utility is continuous and strictly monotonic, then an allocation is blocked if and only if it is weakly blocked.*

12.5.3 Theorem *Assume each utility is monotonic. Then every Walrasian equilibrium allocation is in the core.*

Proof: This is the same as the proof of the First Welfare Theorem 12.1.1. Let $(\bar{x}^1, \dots, \bar{x}^m, p)$ be a Walrasian equilibrium, and suppose by way of contradiction that the allocation $(\bar{x}^1, \dots, \bar{x}^m)$ is blocked. Then there is a coalition S and $(\tilde{x}^i)_{i \in S}$ satisfying

$$u(\tilde{x}^i) \geqslant u_i(\bar{x}^i)$$

for each $i \in S$ and

$$\sum_{i \in S} \tilde{x}^i = \sum_{i \in S} \omega^i. \tag{7}$$

Since utilities are monotonic, in equilibrium all income is spent (Lemma 11.7.1) so $p \cdot \bar{x}^i = p \cdot \omega^i$. Also, by utility maximization subject to the budget constraint, we have

$$u_i(\tilde{x}^i) > u_i(\bar{x}^i) \implies p \cdot \tilde{x}^i > p \cdot \bar{x}^i = p \cdot \omega^i$$

for each $i \in S$. Summing over S yields

$$p \cdot \sum_{i \in S} \tilde{x}^i > p \cdot \sum_{i \in S} \bar{x}^i = p \cdot \sum_{i \in S} \omega^i,$$

which contradicts (7). ∎

12.6 Core of a replica economy

12.6.1 Definition *The n^{th} replica $\mathcal{E}_n$ of $\mathcal{E}$ has $n \times m$ consumers, n of each of m* ***types****. Consumers of type i have the same endowment ω^i and the same utility u_i.*

12.6.2 Lemma (Equal treatment property) *Assume that the consumers' utilities are strictly monotonic, strictly quasiconcave, and continuous. Then in the core of a replica economy, consumers of the same type receive the same consumption.*

That is, let $(x^{1,1}, \dots, x^{1,n}, \dots, x^{m,1}, \dots, x^{m,n})$ belong to the core of $\mathcal{E}_n$. Then for each type i, and each $j, k = 1, \dots, n$ we have

$$x^{i,j} = x^{i,k}.$$

Proof: Let $(x^{1,1}, \dots, x^{1,n}, \dots, x^{m,1}, \dots, x^{m,n})$ belong to the core of $\mathcal{E}_n$. Since every consumer of type i has the same utility, they can all agree on which of them, say (i, j_i), has the worst consumption allocation $x^{i,j}$. (They may be indifferent, in which case any of them qualifies as having the worst allocation.) Form a coalition S that has one consumer of each type, that consumer having the worst allocation for their type. Consider the partial allocation $(\tilde{x}^i)_{i \in S}$ (here we are indexing members of S solely by their type) defined by

$$\tilde{x}^i = \frac{\sum_{j=1}^n x^{i,j}}{n}$$

Now by definition of an allocation

$$\sum_{i=1}^m \sum_{j=1}^n x^{i,j} = \sum_{i=1}^m \sum_{j=1}^n \omega^{i,j} = n \sum_{i=1}^m \omega^i.$$

Dividing by n we get

$$\sum_{i=1}^m \tilde{x}^i = \sum_{i=1}^m \frac{\sum_{j=1}^n x^{i,j}}{n} = \sum_{i=1}^m \omega^i.$$

Now suppose by way of contradiction that for some type i, we have unequal treatment. Then by strict quasiconcavity of utility, $\tilde{x}^i = \frac{1}{n} \sum_{j=1}^n x^{i,j}$ satisfies $u_i(\tilde{x}^i) > u_i(x^{i,j_i})$, where (i, j_i) is the worst off of type i. Then S weakly blocks via $(\tilde{x}^1, \dots, \tilde{x}^m)$, a contradiction. Thus we must have equal treatment. ∎

Given equal treatment, we can treat every core allocation in a replica economy as if it were an allocation the original economy. (This is not true of general allocations, since an allocation in $\mathcal{E}_n$ actually belongs to $\boldsymbol{R}^{mn\ell}$, not $\boldsymbol{R}^{m\ell}$.)

12.6.3 Theorem (Limit of the core) *Assume utilities are strictly monotonic, continuous, and strictly quasiconcave. Suppose the allocation $(\bar{x}^1, \dots, \bar{x}^m)$ belongs to the core of $\mathcal{E}_n$ for each n. Then there exists a nonzero price vector $p \in \boldsymbol{R}^\ell$ such that $(\bar{x}^1, \dots, \bar{x}^m, p)$ is a Walrasian quasi-equilibrium.*

Proof: (This treatment is based on Debreu [10] and lectures by Ket Richter.) The proof is similar to the proof of the second welfare theorem, but involves the initial endowment. For each $i = 1, \dots, m$ define

$$P_i = \{z \in \boldsymbol{R}^\ell : u_i(\omega^i + z) > u_i(\bar{x}^i)\}.$$

That is, P_i is the set of net trades from ω^i that make a consumer of type i better off than his core allocation $\bar{x}^i$. Define

$$P = \text{convex hull} \bigcup_{i=1}^m P_i.$$

That is, P is the set of all vectors of the form $\sum_{i=1}^m \alpha_i z^i$ where each $z^i \in P_i$, $\alpha_i \geqslant 0$, and $\sum_{i=1}^m \alpha_i = 1$. (See Lemma 2.1.6.)

I claim that $0 \notin P$. To see why, note that the continuity of utility implies that each P_i is open, so that their union is open, which in turn implies that the convex hull is open (Proposition 5.3.1). So assume by way of contradiction that 0 belongs to P. Then there is some strictly negative vector $\hat{v} \ll 0$ that also belongs to P. We can thus write $\hat{v} = \sum_{i=1}^m \hat{\alpha}_i z^i$ where each $z^i \in P_i$, $\hat{\alpha}_i \geqslant 0$, and $\sum_{i=1}^m \hat{\alpha}_i = 1$. Moreover, since the mapping $(\beta_1, \dots, \beta_m) \to \sum_{i=1}^m \beta_i z^i$ is continuous, we can find α_i close enough to $\hat{\alpha}_i$ such that each α_i is rational, $\sum_{i=1}^m \alpha_i = 1$, and

$$v = \sum_{i=1}^m \alpha_i z^i \ll 0.$$

Putting all the coefficients α_i over a common denominator n we get

$$0 \gg v = \sum_{i=1}^m \frac{k_i}{n} z^i, \tag{8}$$

where $\sum_{i=1}^m k_i = n$. Consider now a coalition S that has n members, k_i members of each type i, and consider the partial equal treatment allocation where each consumer in S of type i receives

$$\tilde{x}^i = \omega^i + z^i - v.$$

By monotonicity, since $v \ll 0$ we have

$$u_i(\tilde{x}^i) > u_i(\omega^i + z^i).$$

By construction, z^i belongs to P_i, so

$$u_i(\omega^i + z^i) > u_i(\bar{x}^i),$$

so

$$u_i(\tilde{x}^i) > u_i(\bar{x}^i).$$

I now need to show that this partial allocation $\tilde{x}$ is feasible for the coalition S. But

$$\sum_{i\in S} k_i \tilde{x}^i = \sum_{i\in S} k_i(\omega^i + z^i - v) = \sum_{i\in S} k_i\omega^i + \sum_{i\in S} k_i z^i - \sum_{i\in S} k_i v = \sum_{i\in S} k_i\omega^i,$$

where the last equality follows from (8). The upshot is that $(\tilde{x}^i)$ blocks the allocation $(\bar{x}^i)$ in the n-replica economy $\mathcal{E}_n$, a contradiction. Therefore

$$0 \notin P.$$

We now use the separating hyperplane theorem to find the existence of a nonzero $p \in \boldsymbol{R}^\ell$ such that $p \cdot z \geqslant 0$ for all $z \in P$. Since each $P_i \subset P$, for each i,

$$z \in P_i \implies p \cdot z \geqslant 0. \tag{9}$$

Now suppose $u_i(x) > u_i(\bar{x}^i)$. Setting $z = x - \omega^i$ we have $u_i(\omega^i + z) = u_i(x) > u_i(\bar{x}^i)$, so $z \in P_i$. Thus (9) implies $p \cdot (x - \omega^i) = p \cdot z \geqslant 0$. Thus

$$u_i(x) > u_i(\bar{x}^i) \quad\implies\quad p \cdot x \geqslant p \cdot \omega^i.$$

Since utilities are monotonic, if $u_i(x) \geqslant u_i(\bar{x}^i)$ there is a sequence $x_n \to x$ with $u(x_n) > u_i(x) \geqslant u_i(\bar{x}^i)$. Thus $p \cdot x_n \geqslant p \cdot \omega^i$, so by continuity

$$u_i(x) \geqslant u_i(\bar{x}^i) \implies p \cdot x \geqslant p \cdot \omega^i.$$

In particular, $p \cdot \bar{x}^i \geqslant p \cdot \omega^i$ for each i, and since $\sum_{i=1}^m \bar{x}^i = \sum_{i=1}^m \omega^i$, we conclude that for each i,

$$p \cdot \bar{x}^i = p \cdot \omega^i.$$

Thus we have shown that $p \cdot \bar{x}^i = p \cdot \omega^i$ and $u_i(x) \geqslant u_i(\bar{x}^i)$ implies $p \cdot x \geqslant p \cdot \bar{x}^i$, which proves that we have a Walrasian quasi-equilibrium. ∎

12.7 Edgeworth equilibria

12.7.1 Definition *An* ***Edgeworth equilibrium*** *for the economy $\mathcal{E}$ is an allocation $(x^1, \dots, x^m)$ such that for every $n \geqslant 1$, the n^{th} replica*

$$(x^{1,1}, \dots, x^{1,n}, \dots, x^{m,1}, \dots, x^{m,n})$$

of the allocation belongs to the core of the n^{th} replica economy $\mathcal{E}_n$.

You can show that under the assumptions of the previous section, every Edgeworth equilibrium is a Walrasian quasi-equilibrium.

I believe the term was coined by Aliprantis, Brown, and Burkinshaw [1].

12.8 Approximate equilibria

Relevant cites include: Arrow and Hahn [4], Bewley [6], Anderson [2].

12.9 Complements

12.9.1 Exercise: Monotonicity vs. local nonsatiation

The following proposition shows that while locally nonsatiated utilities are more general than monotonic utilities, the assumption of monotonicity is in a way harmless.

12.9.1 Proposition *If u is an upper semicontinuous locally nonsatiated utility on $\boldsymbol{R}^\ell_+$, then the function v defined by $v(x) = \max\{u(y) : 0 \leqq y \leqq x\}$ is monotonic, upper semicontinuous, and generates the same demand as u. Moreover, if u is quasiconcave, then v is quasiconcave.*

When I say that v generates the same demand as u, I mean that for each set of the form

$$\beta(p, w) = \{x \in \boldsymbol{R}^\ell_+ : p \cdot x \leqslant w\}, \qquad (p \gg 0,\ w > 0),$$

the set of maximizers for u and v coincide.

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