# Topic 10: Constrained optima and Lagrangean saddlepoints

## 10.1 An alternative

As an application of the separating hyperplane theorem we present the following result due to Fan, Glicksberg, and Hoffman [1].

**10.1.1 Concave Alternative Theorem** *Let $C$ be a nonempty convex subset of a vector space, and let $f_1, \dots, f_m \colon C \to \boldsymbol{R}$ be concave. Letting $f = (f_1, \dots, f_m) \colon C \to \boldsymbol{R}^{\mathrm{m}}$, exactly one of the following is true.*

$$(\, \exists \bar{x} \in C \,)\ [\, f(\bar{x}) \gg 0 \,]. \tag{1}$$

*Or (exclusive),*

$$(\, \exists p > 0 \,)\ (\, \forall x \in C \,)\ [\, p \cdot f(x) \leqslant 0 \,]. \tag{2}$$

*Proof*: Clearly both cannot be true. Suppose (1) fails. Let $A = \{f(x) : x \in C\}$ be the image of $C$ under the function $f$, and let $\hat{A}$ be its decreasing hull,

$$\hat{A} = \Big\{ y \in \boldsymbol{R}^{\mathrm{m}} : (\, \exists x \in C \,)\ [\, y \leqq f(x) \,] \Big\}.$$

Since (1) fails, we see that $A$ and $\boldsymbol{R}^{\mathrm{m}}_{++}$ are disjoint. Consequently $\hat{A}$ and $\boldsymbol{R}^{\mathrm{m}}_{++}$ are disjoint. Now $\boldsymbol{R}^{\mathrm{m}}_{++}$ is clearly convex, and observe that $\hat{A}$ is also convex. (The set $A$ itself need not be convex—see, for instance, Figure 10.3.1.) To see this, suppose $y_0, y_1 \in \hat{A}$. Then $y_i \leqq f(x_i)$, $i = 0, 1$. Therefore, for any $\lambda \in (0, 1)$,

$$(1 - \lambda) y_0 + \lambda y_1 \leqq (1 - \lambda) f(x_0) + \lambda f(x_1) \leqq f\Big((1 - \lambda) x_0 + \lambda x_1\Big),$$

since each $f_j$ is concave. Therefore $(1 - \lambda) y_0 + \lambda y_1 \in \hat{A}$.

Thus, by the Separating Hyperplane Theorem 8.5.1, there is a nonzero vector $p \in \boldsymbol{R}^{\mathrm{m}}$ properly separating $\hat{A}$ and $\boldsymbol{R}^{\mathrm{m}}_{++}$. We may assume

$$p \cdot \hat{A} \leqslant p \cdot \boldsymbol{R}^{\mathrm{m}}_{++}. \tag{3}$$

Therefore $p > 0$ (Exercise 8.2.4). Let $\varepsilon > 0$, so that $\varepsilon \mathbf{1} \gg 0$, and note that (3) implies for every $y \in \hat{A}$, we have $p \cdot y \leqslant \varepsilon p \cdot \mathbf{1}$. Since $\varepsilon$ may be taken arbitrarily small, we conclude that $p \cdot y \leqslant 0$ for all $y$ in $\hat{A}$. Consequently, $p \cdot f(x) \leqslant 0$ for all $x$ in $C$. ∎

## 10.2 Saddlepoints

In this section we discuss the relation between constrained maxima of concave functions and saddlepoints of the so-called Lagrangean.

**10.2.1 Definition** *Let $f\colon X \times Y \to \boldsymbol{R}$. A point $(x^*; y^*)$ [1] in $X \times Y$ is a **saddlepoint of $f$ (over $X \times Y$)** if it satisfies*

$$f(x; y^*) \leqslant f(x^*; y^*) \leqslant f(x^*; y) \quad \text{for all} \quad x \in X,\ y \in Y.$$

That is, $(x^*; y^*)$ is a saddlepoint of $f$ if $x^*$ maximizes $f(\cdot; y^*)$ over $X$ and $y^*$ minimizes $f(x^*; \cdot)$ over $Y$. Saddlepoints of a function have the following nice interchangeability property.

**10.2.2 Lemma (Interchangeability of saddlepoints)** *Let $f\colon X \times Y \to \boldsymbol{R}$, and let $(x_1; y_1)$ and $(x_2; y_2)$ be saddlepoints of $f$. Then*

$$f(x_1; y_1) = f(x_2; y_1) = f(x_1; y_2) = f(x_2; y_2).$$

*Consequently $(x_1; y_2)$ and $(x_2; y_1)$ are also saddlepoints.*

**10.2.3 Exercise** Prove Lemma 10.2.2. □

## 10.3 Lagrangeans

Saddlepoints play an important rôle in the analysis of constrained maximum problems via Lagrangean functions.

**10.3.1 Definition** *Given $f, g_1, \ldots, g_m\colon X \to \boldsymbol{R}$, the associated **Lagrangean** $L\colon X \times P \to \boldsymbol{R}$ is defined by*

$$L(x; p) = f(x) + \sum_{j=1}^{m} \pi_j g_j(x) = f(x) + p \cdot g(x),$$

*where $p = (\pi_1, \ldots, \pi_m)$ and $P$ is a subset of $\boldsymbol{R}^{\mathrm{m}}$. (Usually $P = \boldsymbol{R}^{\mathrm{m}}_+$.) The components of $p$ are called **Lagrange multipliers**.*

The first result is that saddlepoints of Lagrangeans are constrained maxima. This result is also called the **easy half of the Saddlepoint Theorem**, and requires no restrictions on the functions.

[1] The use of a semicolon instead of a comma is to visually emphasize the differing roles of $x$ and $y$, but mathematically it plays the same role as a comma. On occasion I may forget this convention and use a comma. Don't fret about it.

**10.3.2 Theorem (Lagrangean saddlepoints are constrained maxima)** *Let $X$ be an arbitrary set, and let $f, g_1, \dots, g_m \colon X \to \boldsymbol{R}$. Suppose that $(x^*, p^*)$ is a saddlepoint of the Lagrangean $L(x;p) = f + p \cdot g$ (over $X \times \boldsymbol{R}^{\mathrm{m}}_{+}$). That is,*

$$L(x;p^*) \underset{(4a)}{\leqslant} L(x^*;p^*) \underset{(4b)}{\leqslant} L(x^*;p) \qquad x \in X,\ p \geqq 0. \tag{4}$$

*Then $x^*$ maximizes $f$ over $X$ subject to the constraints $g_j(x) \geqslant 0$, $j = 1, \dots, m$, and furthermore*

$$\pi_j^* g_j(x^*) = 0, \quad j = 1, \dots, m. \tag{5}$$

*Proof*: Inequality (4*b*) implies $p^* \cdot g(x^*) \leqslant p \cdot g(x^*)$ for all $p \geqq 0$. By the Nonnegativity Test 0.1.1, $g(x^*) \geqq 0$, so $x^*$ satisfies the constraints. Setting $p = 0$, we see that $p^* \cdot g(x^*) \leqslant 0$. This combined with $p \geqq 0$ and $g(x^*) \geqq 0$ implies $p^* \cdot g(x^*) = 0$. Indeed it implies $\pi_j^* g_j(x^*) = 0$ for $j = 1, \dots, m$.

Now note that (4*a*) implies $f(x) + p^* \cdot g(x) \leqslant f(x^*)$ for all $x$. Therefore, if $x$ satisfies the constraints, $g(x) \geqq 0$, we have $f(x) \leqslant f(x^*)$, so $x^*$ is a constrained maximizer. ∎

Condition (5) implies that if the multiplier $\pi_j^*$ is strictly positive, then the corresponding constraint is **binding**, $g_j(x^*) = 0$; and if a constraint is **slack**, $g_j(x^*) > 0$, then the corresponding multiplier satisfies $\pi_j^* = 0$. These conditions are sometimes called the **complementary slackness** conditions.

The converse of Theorem 10.3.2 is not quite true, but almost. To state the correct result we now introduce the notion of a generalized Lagrangean.

**10.3.3 Definition** *A **generalized Lagrangean** $L_\mu \colon X \times P \to \boldsymbol{R}$, where $\mu \geqslant 0$, is defined by*

$$L_\mu(x;p) = \mu f(x) + \sum_{j=1}^{m} \pi_j g_j(x),$$

*where $p = (\pi_1, \dots, \pi_m)$ and $P$ is an appropriate subset of $\boldsymbol{R}^{\mathrm{m}}$.*

Note that each choice of $\mu$ generates a different generalized Lagrangean. However, for $P = \boldsymbol{R}^{\mathrm{m}}_{+}$, as long as $\mu > 0$, a point $(x;p)$ is a saddlepoint of the generalized Lagrangean if and only if $(x;p/\mu)$ is a saddlepoint of the Lagrangean. Thus the only case to worry about is $\mu = 0$.

The next results state that for concave functions, constrained maxima are saddlepoints of some generalized Lagrangean.

**10.3.4 Theorem (Constrained maxima are not quite saddlepoints)** *Let $C \subset \boldsymbol{R}^{\mathrm{n}}$ be convex, and let $f, g_1, \dots, g_m \colon C \to \boldsymbol{R}$ be concave. Suppose $x^*$ maximizes $f$ subject to the constraints $g_j(x) \geqslant 0$, $j = 1, \dots, m$. Then there exist real numbers $\mu^*, \pi_1^*, \dots, \pi_m^* \geqslant 0$, not all zero, such that $(x^*;p^*)$ is a saddlepoint of the generalized Lagrangean $L_{\mu^*}$. That is,*

$$\mu^* f(x) + \sum_{j=1}^{m} \pi_j^* g_j(x) \underset{(6a)}{\leqslant} \mu^* f(x^*) + \sum_{j=1}^{m} \pi_j^* g_j(x^*) \underset{(6b)}{\leqslant} \mu^* f(x^*) + \sum_{j=1}^{m} \pi_j g_j(x^*) \tag{6}$$

*for all* $x \in C$ *and all* $\pi_1, \dots, \pi_m \geqslant 0$*. Furthermore*

$$\pi_j^* g_j(x^*) = 0, \qquad j = 1, \dots, m. \tag{7}$$

*Proof*: Since $x^*$ is a constrained maximizer there is no $x \in C$ satisfying $f(x) - f(x^*) > 0$ and $g(x) \geqq 0$. Therefore the Concave Alternative 10.1.1 implies the existence of nonnegative $\mu^*, \pi_1^*, \dots, \pi_m^*$, not all zero, satisfying

$$\mu^* f(x) + \sum_{j=1}^{m} \pi_j^* g_j(x) \leqslant \mu^* f(x^*) \quad \text{for every } x \in C.$$

Evaluating this at $x = x^*$ yields $\sum_{j=1}^m \pi_j^* g_j(x^*) \leqslant 0$. But each term in this sum is the product of two nonnegative terms, so (7) holds. This in turn implies (6a). Given that $g_j(x^*) \geqslant 0$ for all $j$, (7) also implies (6b). ∎

**10.3.5 Corollary (When constrained maxima are saddlepoints)** *Under the hypotheses of Theorem 10.3.4 suppose in addition that* ***Slater's Condition****,*

$$(\, \exists \bar{x} \in C \,)\ [\, g(\bar{x}) \gg 0 \,], \tag{S}$$

*is satisfied. Then* $\mu^* > 0$*, and may be taken equal to* $1$*. Consequently the pair* $\big(x^*; (\pi_1^*, \dots, \pi_m^*)\big)$ *is a saddlepoint of the Lagrangean for* $x \in C$*,* $p \geqq 0$*. That is,*

$$L(x; p^*) \leqslant L(x^*; p^*) \leqslant L(x^*; p), \qquad x \in C,\ p \geqq 0, \tag{8}$$

*where* $L(x; p) = f(x) + p \cdot g(x)$*.*

*Proof*: Suppose $\mu^* = 0$. Then evaluating (6) at $x = \bar{x}$ yields $p^* \cdot g(\bar{x}) \leqslant 0$, but $g(\bar{x}) > 0$ implies $\pi_j^* = 0$, $j = 1, \dots, m$. Thus $\mu = 0$ and $\pi_j = 0$, $j = 1, \dots, m$, a contradiction. Therefore $\mu^* > 0$, and by dividing the Lagrangean by $\mu^*$, we may take $\mu^* = 1$. The remainder is then just Theorem 10.3.4. ∎

Combining these results gives us the following.

**10.3.6 The Saddlepoint Theorem** *Let* $f, g_1, \dots, g_m \colon C \to \boldsymbol{R}$ *be concave, where* $C \subset \boldsymbol{R}^{\mathrm{n}}$ *is convex. Assume in addition that* ***Slater's Condition****,*

$$(\, \exists \bar{x} \in C \,)\ [\, g(\bar{x}) \gg 0 \,] \tag{S}$$

*is satisfied.*

*The following are equivalent.*

1. *The point* $x^*$ *maximizes* $f$ *over* $C$ *subject to the constraints* $g_j(x) \geqslant 0$*,* $j = 1, \dots, m$*.*

2. *Then there exist real numbers* $\pi_1^*, \ldots, \pi_m^* \geqslant 0$ *such that* $(x^*; p^*)$ *is a saddlepoint of the Lagrangean* $L$*. That is,*

$$f(x) + \sum_{j=1}^{m} \pi_j^* g_j(x) \leqslant f(x^*) + \sum_{j=1}^{m} \pi_j^* g_j(x^*) \leqslant f(x^*) + \sum_{j=1}^{m} \pi_j g_j(x^*) \tag{9}$$

*for all* $x \in C$ *and all* $\pi_1, \ldots, \pi_m \geqslant 0$*. Furthermore*

$$\pi_j^* g_j(x^*) = 0, \quad j = 1, \ldots, m.$$

**10.3.7 Remark** Sometimes it hard to see the forest for the trees. The most important consequence of the Saddlepoint Theorem is that for the concave case satisfying Slater's condition,

> the constrained maximizer $x^*$ of $f$ is an *unconstrained* maximizer of the Lagrangean $L(\cdot; p^*)$.

The rôle of a Lagrange multiplier is to act as a conversion factor between the $g$-values and the $f$-values that transform the constrained maximization problem to an unconstrained maximization problem.

**10.3.8 Example** Concavity is crucial for these results. Even in the classical case of an interior constrained maximizer of a smooth function, even when the Lagrange Multiplier Theorem applies, the constrained maximizer may not be an unconstrained maximizer of the Lagrangean. Sydsaeter [4] points out the following simple example. Let $f(x, y) = xy$ and $g(x, y) = 1 - (x + y)$, so the Lagrangean is

$$L(x, y; \pi) = xy + \pi(1 - x - y).$$

Note that $g$ is linear and so concave, but $f$ is quasiconcave, but not concave, since it exhibits "increasing returns." The point

$$(x^*, y^*) = (1, 1)$$

maximizes $f$ subject to the constraints $g(x, y) \geqslant 0$, $x \geqslant 0$, $y \geqslant 0$. The point $(\bar{x}, \bar{y}) = (0, 0)$ verifies Slater's condition. The multiplier

$$\pi^* = 2$$

satisfies the conclusion of the classical Lagrange multiplier theorem, namely the first order conditions

$$\frac{\partial L(x^*, y^*; \pi^*)}{\partial x} = 2x^*y^* - \pi^* = 0 \quad \text{and} \quad \frac{\partial L(x^*, y^*; \pi^*)}{\partial y} = 2x^*y^* - \pi^* = 0.$$

But $(x^*, y^*)$ does not maximize

$$L(x, y; 2) = xy + 2 - 2x - 2y.$$

Indeed there is no unconstrained maximizer of the Lagrangean for any $\pi \geqslant 0$. □

Karlin [2, vol. 1, Theorem 7.1.1, p. 201] proposed the following alternative to Slater's Condition:

$$(\,\forall p > 0\,)\ (\,\exists \bar{x}(p) \in C\,)\ \Big[\,p \cdot g\big(\bar{x}(p)\big) > 0\,\Big],$$

which we may as well call **Karlin's condition**.

**10.3.9 Theorem** *Let $C \subset \boldsymbol{R}^{\mathrm{n}}$ be convex, and let $g_1, \ldots, g_m \colon C \to \boldsymbol{R}$ be concave. Then $g$ satisfies Slater's Condition if and only it satisfies Karlin's Condition.*

*Proof*: Clearly Slater's Condition implies Karlin's. Now suppose $g$ violates Slater's Condition. Then by the Concave Alternative Theorem 10.1.1, it must also violate Karlin's. ∎

The next example shows what can go wrong when Slater's Condition fails.

**10.3.10 Example** In this example, due to Slater [3], $C = \boldsymbol{R}$, $f(x) = x$, and $g(x) = -(1-x)^2$, so both $f$ and $g$ are concave. Note that Slater's Condition fails because $g \leqslant 0$. The constraint set $\{g \geqslant 0\}$ is the singleton $\{1\}$. Therefore $f$ attains a constrained maximum at $x^* = 1$. There is however no saddlepoint over $\boldsymbol{R} \times \boldsymbol{R}_+$ of the Lagrangean

$$L(x;\pi) = x - \pi(1-x)^2 = -\pi + (1+2\pi)x - \pi x^2.$$

To see that $L$ has no saddlepoint, consider an arbitrary $(\bar{x}, \bar{\pi}) \in \boldsymbol{R} \times \boldsymbol{R}_+$. Since $L(x;0) = x$, if $\bar{\pi} = 0$, then no value for $\bar{x}$ maximizes $L(\,\cdot\,;\bar{\pi})$.

On the other hand if $\bar{\pi} > 0$, the first order condition for a maximizer at $\bar{x}$ is $\frac{\partial L(\bar{x},\bar{\pi})}{\partial x} = 0$, or $1 + 2\bar{\pi} - 2\bar{\pi}\bar{x} = 0$, which implies $\bar{x} = 1 + 1/(2\bar{\pi}) > 1$. But for $\bar{x} > 1$, $L(\bar{x};\pi)$ is strictly decreasing in $\pi$, so no $\bar{\pi}$ is a minimizer. □

### 10.3.1 The role of Slater's Condition

In this section we present a geometric argument that illuminates the role of Slater's Condition in the saddlepoint theorem. The saddlepoint theorem was proved by invoking the Concave Alternative Theorem 10.1.1, so let us return to the underlying argument used in its proof. In the framework of Theorem 10.3.4, define the function $h \colon C \to \boldsymbol{R}^{\mathrm{m}+1}$ by

$$h(x) = \big(g_1(x), \ldots, g_m(x), f(x) - f(x^*)\big)$$

and set

$$A = \{h(x) \in \boldsymbol{R}^{m+1} : x \in C\} \quad \text{and} \quad \hat{A} = \Big\{y \in \boldsymbol{R}^{m+1} : (\,\exists x \in C\,)\ [\,y \leqq h(x)\,]\Big\}.$$

Then $\hat{A}$ is a convex set bounded in part by $A$. Figure 10.3.1 depicts the sets $A$ and $\hat{A}$ for Slater's example 10.3.10, where $f(x) - f(x^*)$ is plotted on the vertical

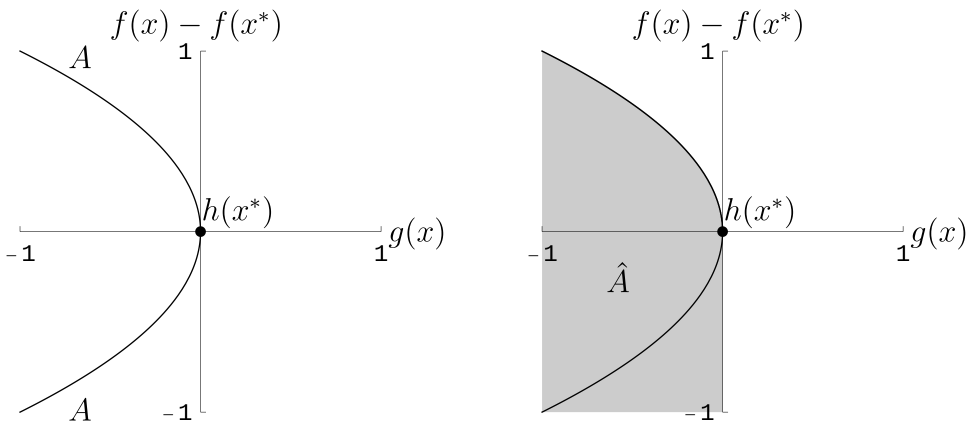


**Figure 10.3.1.** The sets $A$ and $\hat{A}$ for Slater's example.

axis and $g(x)$ is plotted on the horizontal axis. Now if $x^*$ maximizes $f$ over the convex set $C$ subject to the constraints $g_j(x) \geqslant 0$, $j = 1, \ldots, m$, then $h(x^*)$ has the largest vertical coordinate among all the points in $H$ whose horizontal coordinates are nonnegative.

The semipositive $m + 1$-vector $\hat{p}^* = (\pi_1^*, \ldots, \pi_m^*, \mu^*)$ from Theorem 10.3.4 is obtained by separating the convex set $\hat{A}$ and $\boldsymbol{R}^{\mathrm{m}+1}_{++}$. It has the property that

$$\hat{p}^* \cdot h(x) \leqslant \hat{p}^* h(x^*)$$

for all $x \in C$. That is, the vector $\hat{p}^*$ defines a hyperplane through $h(x^*)$ that supports $\hat{A}$ at $h(x^*)$. It is clear in the case of Slater's example that the supporting hyperplane is a vertical line. The fact that the hyperplane is vertical means that $\mu^*$ (the multiplier on $f$) must be zero.

If there is a non-vertical supporting hyperplane through $h(x^*)$, then $\mu^*$ is nonzero, so we can divide by it and obtain a full saddlepoint of the true Lagrangean. This is where Slater's condition comes in.

In the one dimensional, one constraint case, Slater's Condition reduces to the existence of $\bar{x}$ satisfying $g(\bar{x}) > 0$. This rules out having a vertical supporting line through $h(x^*)$. To see this, note that the vertical component of $h(x^*)$ is $f(x^*) - f(x^*) = 0$. If $g(x^*) = 0$, then the vertical line through $h(x^*)$ is simply the vertical axis, which cannot support $\hat{A}$, since $h(\bar{x}) \in A$ lies to the right of the axis. See Figure 10.3.2.

In Figure 10.3.2, the shaded area is included in $\hat{A}$. For instance, let $C = (-\infty, 0]$, $f(x) = x$, and $g(x) = x + 1$. Then the set $\hat{A}$ is just $\{y \in \boldsymbol{R}^2 : y \leqq (0, 1)\}$.

Later we shall see that if $f$ and the $g_j$'s are linear, then Slater's Condition is not needed to guarantee a non-vertical supporting line. Intuitively, the reason for this is that for the linear programming problems considered, the set $\hat{A}$ is polyhedral, so even if $g(x^*) = 0$, there is still a non-vertical line separating $\hat{A}$ and $\boldsymbol{R}^{\mathrm{m}+1}_{++}$. The proof of this fact relies on some results on linear inequalities that will be proven

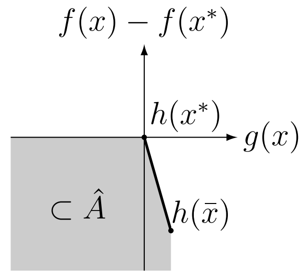


**Figure 10.3.2.** Slater's condition rules out a vertical supporting line.

later. It is subtle because Slater's condition rules out a vertical supporting line. In the linear case, there may be a vertical supporting line, but if there is, there is also a non-vertical supporting line that yields a Lagrangean saddlepoint. As a case in point, consider $C = (-\infty, 0]$, $f(x) = x$, and $g(x) = x$. Then the set $\hat{A}$ is just $\{y \in \boldsymbol{R}^2 : y \leqq 0\}$, which is separated from $\boldsymbol{R}^2_{++}$ by every semipositive vector.

## 10.4 Lagrangean Saddlepoints and Minimization

Minimizing $f$ is the same as maximizing $-f$ so we can consider a constrained minimization problem for convex functions as maximization problem for concave functions. So assume $f, g_1, \dots, g_m$ are convex functions and we wish to

$$\begin{aligned} &\underset{x}{\text{minimize}}\, f(x) \text{ subject to} \\ &g_i(x) \leqslant 0, \quad \text{i} \quad = 1, \dots, m. \end{aligned}$$

Note that since we are assuming that each $g_i$ is convex rather than concave, and we are writing the constraints as $g_i(x) \leqslant 0$, the constraint set is still convex.

This problem is the same as the concave maximization problem

$$\begin{aligned} &\underset{x}{\text{maximize}}\, -f(x) \text{ subject to} \\ &-g_i(x) \geqslant 0, \quad \text{i} \quad = 1, \dots, m. \end{aligned}$$

The Lagrangean for this is

$$L(x; p) = -f(x) - \sum_{i=1}^{m} \pi_i g_i(x).$$

So if $(\bar{x}, \bar{p})$ is a saddlepoint of this Lagrangean, then $\bar{x}$ solves the constrained minimization problem. Now $(\bar{x}, \bar{p})$ is a saddlepoint of this Lagrangean over $X \times \boldsymbol{R}^{\text{m}}_{+}$ if

$$-f(x) - \bar{p} \cdot g(x) \leqslant -f(\bar{x}) - \bar{p} \cdot g(\bar{x}) \leqslant -f(\bar{x}) - p \cdot g(\bar{x})$$

for all $x \in X$ and all $p \in \boldsymbol{R}^{\mathrm{m}}_{+}$. Multiplying by $-1$ will flip the inequalities so

$$f(x) + \bar{p} \cdot g(x) \geqslant f(\bar{x}) + \bar{p} \cdot g(\bar{x}) \geqslant f(\bar{x}) + p \cdot g(\bar{x}).$$

At this point it might be useful to introduce the notion of a **reverse saddlepoint**, which is not a standard term.

**10.4.1 Definition** *Let $f \colon X \times Y \to \boldsymbol{R}$. A point $(x^*; y^*)$ in $X \times Y$ is a **reverse saddlepoint of $f$ (over $X \times Y$)** if it satisfies*

$$f(x; y^*) \geqslant f(x^*; y^*) \geqslant f(x^*; y) \quad \textit{for all} \quad x \in X,\ y \in Y.$$

So if $(\bar{x}, \bar{p})$ is a reverse saddlepoint of

$$L(x; p) = f(x) + p \cdot g(x)$$

then $\bar{x}$ solves the constrained minimization problem. A useful consequence of this approach is that if $\bar{x}, \bar{p}$ is reverse saddlepoint of $L$, then $\bar{x}$ is an *unconstrained* minimizer of $L(\cdot, \bar{p})$.

### A note on writing Lagrangeans

Suppose we wish to minimize a convex function, but the constraints are given as $g_i(x) \geqslant 0$, where each $g_i$ is concave. Then I argue it is useful to write the Lagrangean as

$$\tilde{L}(x, p) = f(x) - \sum_i \pi_i g_i(x) = f(x) - p \cdot g(x)$$

for then the constrained minimizer corresponds to a reverse saddlepoint of this function over $X \times \boldsymbol{R}^{\mathrm{m}}_{+}$. Again, a useful consequence of this approach is that if $\bar{x}, \bar{p}$ is reverse saddlepoint of $\tilde{L}$, then $\bar{x}$ is an *unconstrained* minimizer of $\tilde{L}(\cdot, \bar{p})$.

Similarly if we wish to maximize a concave $f$ subject to constraints of the form $g_i(x) \leqslant 0$ where each $g_i$ is convex, then it also makes sense to write the Lagrangean this way too as a constrained maximizer corresponds to a saddlepoint of this function over $X \times \boldsymbol{R}^{\mathrm{m}}_{+}$. That is, the Lagrange multipliers will still turn out nonnegative. If the constraint functions are affine, they are both concave and convex, so you have some leeway as to how to write things.

I leave it as an exercise to reformulate the Saddlepoint Theorem in each of these contexts. We shall use this observation later in Section 28.1 on dual linear programs.

## 10.5 Decentralization and Lagrange Multipliers

This section is inspired by Uzawa [6]. Consider the case of a firm that has $n$ departments, each of which can produces a single product, according to a one-product concave production function $f_j$ of $\ell$ **inputs** or **factors**. This means that if department $j$ uses $\xi_k$ units of input $k$, $k = 1, \dots, \ell$, it can produce

$$y_j = f_j(\xi_1, \dots, \xi_\ell)$$

units of product $j$.

The firm has already contracted to have a stock $\sigma_k > 0$ of factor $k$ available for use. (Alternatively, consider a country with $n$ industries that sell their outputs on world markets and each factor is immobile and fixed in supply.) If the sale price of the $j^{\text{th}}$ output is $\pi_j > 0$, then the firm (or country) faces the constrained maximization problem of maximizing the value of output, that is, revenue, subject to the resource constraint on each factor.

To keep track of all these variables, use an $n \times \ell$ matrix $X$, and let $x_j = (\xi_{j1}, \dots, \xi_{j\ell})$ be its $i^{\text{th}}$ row. That is,

$\xi_{jk}$ is the quantity of factor $k$ devoted to the production output $j$.

The firm's objective is to

$$\underset{X}{\text{maximize}} \sum_{j=1}^{n} \pi_j f_j(x_j) \quad \text{subject to}$$

$$\begin{aligned} \sum_{j=1}^{n} \xi_{jk} &\leqslant \sigma_k, \qquad & k &= 1, \dots, \ell \\ \xi_{jk} &\geqslant 0, & & \begin{matrix} j = 1, \dots, n \\ k = 1, \dots, \ell \end{matrix}. \end{aligned}$$

Let $w = (\omega_1, \dots, \omega_\ell)$ be vector of Lagrange multipliers for the resource constraints. The Lagrangean for this problem can be written as:

$$L\Big(X; w\Big) = \sum_{j=1}^{n} \pi_j f_j(x_j) + \sum_{k=1}^{\ell} \omega_k \Big(\sigma_k - \sum_{j=1}^{n} \xi_{jk}\Big). \tag{10}$$

By assumption, each $f_j$ is concave, so the objective function $\sum_{j=1}^{n} \pi_j f_j(x_j)$ is concave. Moreover, the constraint function $g_k(X) := \sigma_k - \sum_{j=1}^{n} \xi_{jk}$ is affine in $X$, and so concave. Note that as long as each $\sigma_k > 0$, then Slater's Condition is satisfied.

Therefore by the Saddlepoint Theorem 10.3.6, a point $\hat{x}$ solves the constrained maximization problem if and only if the there is a vector $\hat{w}$ such that $(\hat{x}; \hat{w})$ is a saddlepoint of the Lagrangean over $\Big(\boldsymbol{R}_+^\ell\Big)^n \times \boldsymbol{R}_+^\ell$. Now regroup the Lagrangean as

$$L(X; w) = \Big(\pi_1 f_1(x_1) - \sum_{k=1}^{\ell} \omega_k \xi_{1k}\Big) + \dots + \Big(\pi_n f_n(x_n) - \sum_{k=1}^{\ell} \omega_k \xi_{nk}\Big) + \sum_{k=1}^{\ell} \omega_k \sigma_k. \tag{11}$$

By the Saddlepoint Theorem the constrained maximizer $\hat{x}$ is an unconstrained maximizer of the Lagrangean evaluated at $w = \hat{w}$. But note that this implies that for each $j = 1, \ldots, n$,

$$\hat{x}_j \text{ maximizes } \pi_j f_j(x_j) - \sum_{k=1}^{\ell} \hat{\omega}_k \xi_{jk}.$$

In other words, the saddlepoint values of the Lagrange multipliers are *factor wages* such that each optimal $\hat{x}_j$ unconstrainedly maximizes the profit at the price $\pi_j$ and the vector of factor wages $\hat{w}$. These multipliers, sometimes known as **shadow wages**, allow the constrained problem to be **decentralized** as $n$ independent profit maximization decisions.

We now verify that the converse is true, namely, if

1. each $\hat{x}_j$ unconstrainedly maximizes the profit at price $\pi_j$ and factor wages $\hat{w}$, and
2. the market for inputs clears, that is, if $\sum_{j=1}^{n} \hat{x}_{jk} \leqslant \sigma_k$ for $k = 1, \ldots, \ell$, and each $\hat{\omega}_k \Big(\sigma_k - \sum_{k=1}^{\ell} \hat{x}_{jk}\Big) = 0$,

then $\hat{X}$ solves the constrained maximization problem.

To see this, write the profit maximization condition as

$$\pi_i f_j(x_j) - \sum_{k=1}^{\ell} \hat{\omega}_k \xi_{jk} \leqslant \pi_i f_j(\hat{x}_j) - \sum_{k=1}^{\ell} \hat{\omega}_k \hat{\xi}_{jk}, \quad \text{for all } x_j \geqq 0 \ .$$

Sum over $j = 1, \ldots, n$ and add $\sum_{k=1}^{\ell} \hat{\omega}_k \sigma_k$ to both sides to get

$$\begin{aligned} \Big(\pi_1 f_1(x_1) - \sum_{k=1}^{\ell} \hat{\omega}_k \xi_{1k}\Big) + \cdots + \Big(\pi_n f_n(x_n) - \sum_{k=1}^{\ell} \hat{\omega}_k \xi_{nk}\Big) + \sum_{k=1}^{\ell} \hat{\omega}_k \sigma_k \\ \leqslant \Big(\pi_1 f_1(\hat{x}_1) - \sum_{k=1}^{\ell} \hat{\omega}_k \hat{\xi}_{1k}\Big) + \cdots + \Big(\pi_n f_n(\hat{x}_n) - \sum_{k=1}^{\ell} \hat{\omega}_k \hat{\xi}_{nk}\Big) + \sum_{k=1}^{\ell} \hat{\omega}_k \sigma_k \end{aligned}$$

for all $X$. Comparing this the Lagrangean as written in (11), we see that this is the first inequality of the saddlepoint condition for the Lagrangean. The market clearing condition, (2.) above implies that for any vector $w = (\omega_1, \ldots, \omega_\ell) \geqq 0$, we have $\sum_{k=1}^{\ell} \omega_k \Big(\sigma_k - \sum_{k=1}^{\ell} \hat{x}_{jk}\Big) \geqslant 0$, so comparing this to the Lagrangean as written in (10), the second inequality of the saddlepoint condition holds. Thus by the easy half of the Saddlepoint Theorem, $\hat{X}$ solves the firm's revenue maximization problem

Thus the prices of the outputs determine the wages of the factors as saddlepoint Lagrange multipliers.

## References

[1] K. Fan, I. Glicksberg, and A. J. Hoffman. 1957. Systems of inequalities involving convex functions. *Proceedings of the American Mathematical Society* 8:617–622. http://www.jstor.org/stable/2033529

[2] S. Karlin. 1987. *Mathematical methods and theory in games, programming, and economics*. New York: Dover. Reprint of the 1959 two-volume edition published by Addison–Wesley.

[3] M. L. Slater. 1950. Lagrange multipliers revisited: A contribution to non-linear programming. Discussion Paper Math. 403, Cowles Commission. Reissued as Cowles Foundation Discussion Paper #80 in 1959. http://cowles.econ.yale.edu/P/cd/d00b/d0080.pdf

[4] K. Sydsaeter. 1974. Letter to the editor on some frequently occurring errors in the economic literature concerning problems of maxima and minima. *Journal of Economic Theory* 9(4):464–466. DOI: 10.1016/0022-0531(74)90046-5

[5] H. Uzawa. 1958. The Kuhn–Tucker conditions in concave programming. In K. J. Arrow, L. Hurwicz, and H. Uzawa, eds., *Studies in Linear and Nonlinear Programming*, number 2 in Stanford Mathematical Studies in the Social Sciences, chapter 3, pages 32–37. Stanford, California: Stanford University Press.

[6] ——— . 1958. A note on the Menger–Wieser theory of imputation. *Zeitschrift für Nationalökonomie* 18(3):318–334. DOI: 10.1007/BF01317023