Ec 181 KC Border
Convex Analysis and Economic Theory AY 2019–2020

Topic 7: Quasiconvex Functions I

7.1 Level sets of functions

Should this be moved?

For an extended real-valued function f on a set X, the **level set**

$$\{f = \alpha\} \text{ is defined to be } \{x \in X : f(x) = \alpha\}$$

Similarly we define the **sublevel set**

$$\{f \leqslant \alpha\} = \{x \in X : f(x) \leqslant \alpha\},$$

the **strict sublevel set**

$$\{f < \alpha\} = \{x \in X : f(x) < \alpha\},$$

and **superlevel sets** and **strict superlevel sets** are defined analogously.

7.2 Quasiconvexity and quasiconcavity

Section 6.2 proved that local extrema of convex and concave functions are global. These results hold for a larger class of functions and the proofs are nearly identical.

7.2.1 Definition *Let C be a convex set in a vector space, and let $f\colon \to \boldsymbol{R}$.*

- *The function f is **quasiconvex** if for all $x, y \in C$ and $0 \leqslant \lambda \leqslant 1$, we have*

$$f\Big((1-\lambda)x + \lambda y\Big) \leqslant \max\{f(x), f(y)\}.$$

- *The function is **quasiconcave** if for all $x, y \in C$ and $0 \leqslant \lambda \leqslant 1$, we have*

$$f\Big((1-\lambda)x + \lambda y\Big) \geqslant \min\{f(x), f(y)\}.$$

7.2.2 Exercise Prove the following.

1. Every convex function is quasiconvex. Every concave function is quasiconcave.

2. Prove that the following statements are equivalent.

   (a) The function $f\colon C \to \boldsymbol{R}$ is quasiconvex.

   (b) For all $\alpha \in \boldsymbol{R}$, the sublevel set $\{x \in C : f(x) \leqslant \alpha\}$ is convex.

(c) For all $\alpha \in \boldsymbol{R}$, the strict sublevel set $\{x \in C : f(x) < \alpha\}$ is convex.

(d) For all $x \in C$ and every $0 \leqslant \lambda \leqslant 1$,

$$f(y) \leqslant f(x) \implies f\Big((1-\lambda)x + \lambda y\Big) \leqslant f(x).$$

3. Repeat the previous exercise for quasiconcave functions, making the appropriate changes.

□

Just as there are strictly convex functions there are strictly quasiconvex functions and the weird intermediate case of explicitly quasiconvex functions.

7.2.3 Definition *Let C be a convex subset of a vector space.*

- *A function $f : C \to \boldsymbol{R}$ is* ***strictly quasiconvex*** *if for every $x, y \in C$ with $x \neq y$. and every $0 < \lambda < 1$,*

$$f(y) \leqslant f(x) \implies f\Big((1-\lambda)x + \lambda y\Big) < f(x).$$

Equivalently, if for every $x \neq y$ and $0 < \lambda < 1$

$$f\Big((1-\lambda)x + \lambda y\Big) < \max\{f(x), f(y)\}.$$

- *A function $f : C \to \boldsymbol{R}$ is* ***explicitly quasiconvex*** *if it is quasiconvex and satisfies*

$$\Big(f(y) < f(x) \text{ and } 0 < \lambda < 1\Big) \implies f\Big((1-\lambda)x + \lambda y\Big) < f(x). \tag{E}$$

- *A function $f : C \to \boldsymbol{R}$ is* ***strictly quasiconcave*** *if for every $x, y \in C$ with $x \neq y$. and every $0 < \lambda < 1$,*

$$f(y) \geqslant f(x) \implies f\Big((1-\lambda)x + \lambda y\Big) > f(x).$$

Equivalently, if for every $x \neq y$ and $0 < \lambda < 1$

$$f\Big((1-\lambda)x + \lambda y\Big) > \min\{f(x), f(y)\}.$$

- *A function $f : C \to \boldsymbol{R}$ is* ***explicitly quasiconcave*** *if it is quasiconcave and*

$$\Big(f(y) > f(x) \text{ and } 0 < \lambda < 1\Big) \implies f\Big((1-\lambda)x + \lambda y\Big) > f(x).$$

Arrow and Hahn [1, p. 87] use term **semi-strict quasiconcavity** instead of explicit quasiconcavity.

7.2.4 Example Note that strict quasiconvexity implies Condition (E). The converse does not hold. In fact, Condition (E) alone does not guarantee even quasiconvexity.

For instance, the function $f \colon \boldsymbol{R}^{\mathrm{m}} \to \boldsymbol{R}$ defined by

$$f(x) = \begin{cases} 1 & x = 0, \\ 0 & x \neq 0. \end{cases}$$

is not quasiconvex, since the sublevel set $\{f < 1\} = \boldsymbol{R}^{\mathrm{m}} \setminus \{0\}$ is not convex. But f does satisfy Condition (E). To see this suppose $f(y) < f(x)$ and $0 < \lambda < 1$. The only way this can happen is if $x = 0$ and $y \neq 0$, in which case we have $f\big((1-\lambda)x + \lambda y\big) = 0 < 1 = f(x)$, so Condition (E) is satisfied. □

7.3 Quasi-conXXXity and extrema

Note that the conclusion of Theorem 6.2.2 on local maxima of concave functions does not hold for quasiconcave functions. For instance,

$$f(x) = \begin{cases} 0 & x \leqslant 0 \\ x & x \geqslant 0, \end{cases}$$

has a local maximum at -1, but it is not a global maximum over $\boldsymbol{R}$. However, if f is explicitly quasiconcave, then we have the following.

7.3.1 Theorem (Local maxima of explicitly quasiconcave functions) *Let $f \colon C \to \boldsymbol{R}$ be an explicitly quasiconcave function (C convex). If x^* is a local maximizer of f, then it is a global maximizer of f over C.*

Proof: Let x belong to C and suppose $f(x) > f(x^*)$. Then by the definition of explicit quasiconcavity, for any $1 > \lambda > 0$, $f\big((1-\lambda)x^* + \lambda x\big) > f(x^*)$. Since $(1-\lambda)x^* + \lambda x \to x^*$ as $\lambda \to 0$ this contradicts the fact that f has a local maximum at x^*. ∎

7.3.2 Theorem (Local maxima of strictly quasiconcave functions) *Let $f \colon C \to \boldsymbol{R}$ be a strictly quasiconcave function (C convex). If x^* is a local maximizer of f, then it is the unique global maximizer of f over C.*

Proof: Let x belong to C and suppose $x \neq x^*$ satisfies $f(x) \geqslant f(x^*)$. Then by the definition of strict quasiconcavity, for any $1 > \lambda > 0$, $f\big((1-\lambda)x^* + \lambda x\big) > f(x^*)$. Since $(1-\lambda)x^* + \lambda x \to x^*$ as $\lambda \to 0$ this contradicts the fact that f has a local maximum at x^*. ∎

7.4 Explicit quasiconXXXity and nonextremization

7.4.1 Definition *A function f on a subset of a topological space is **locally nonmaximized** if it has no local maxima. That is, for every point x and every neighborhood V of x, there is a point $y \in \operatorname{dom} D \cap V$ with $f(y) > f(x)$.*

*Similarly, f is **locally nonminimized** if it has no local minima.*[1]

Note that if a continuous function is locally nonmaximized, then its domain cannot be compact, as a continuous function on a compact set always achieves a maximum. Economists may use the term **locally nonsatiated** (or even **locally nonsaturated**) in place of locally nonmaximized.

The next result holds in any tvs, but I'll prove it for $\boldsymbol{R}^{\mathrm{m}}$. It is useful in the characterization of economic quasi-equilibria and will be used in Topic 11. There is, of course, a corresponding result for quasiconvexity.

7.4.2 Proposition *Let C be a convex set in $\boldsymbol{R}^{\mathrm{m}}$. Let f be a lower semicontinuous, locally nonmaximized, quasiconcave function on C. Define the superlevel sets*

$$P(x) = \{y \in C : f(y) > f(x)\} \quad \text{and} \quad U(x) = \{y \in C : f(y) \geqslant f(x)\}.$$

Then for any $x \in C$,

$$P(x) = \operatorname{ri} U(x).$$

Proof: For each x, by local nonmaximization, the set $P(x)$ is nonempty, and by lower semicontinuity, it is a relatively open subset of C, and hence it belongs to $\operatorname{ri} C$. Now obviously

$$P(x) \subset U(x), \text{ and so } P(x) \subset \operatorname{ri} U(x).$$

For the reverse inclusion, suppose by way of contradiction that there exists some $y \in \operatorname{ri} U(x)$, but $y \notin P(x)$, so that $f(y) = f(x)$. Since $y \in \operatorname{ri} U(x)$, there is some $\delta > 0$ so that the open ball $B_\delta(y) \cap C \subset U(x)$.

By local nonmaximization there is some $\bar{z} \in C$ with $f(\bar{z}) > f(y)$. In fact, by lower semicontinuity, there is some $\varepsilon > 0$, such that the open ε-ball $B = B_\varepsilon(\bar{z})$ such that

$$(\forall z \in B)\ [\, f(z) > f(y)\,].$$

(Note that $y \notin B$.) Now consider an open set of the form

$$A_\lambda = (1-\lambda)y + \lambda B, \quad \text{where } \lambda < 0.$$

See Figure 7.4.1. The set A_λ is a ball of radius $|\lambda|\varepsilon$ centered at $\bar{w} = (1-\lambda)y + \lambda\bar{z}$. Moreover, $\|\bar{w} - y\| = |\lambda|\|\bar{z} - y\|$, so

$$w \in A_\lambda \implies \|w - y\| \leqslant \|w - \bar{w}\| + \|\bar{w} - y\| < |\lambda|\Big(\varepsilon + \|\bar{z} - y\|\Big).$$

[1] These terms are not standard. I made them up, because they seemed useful.

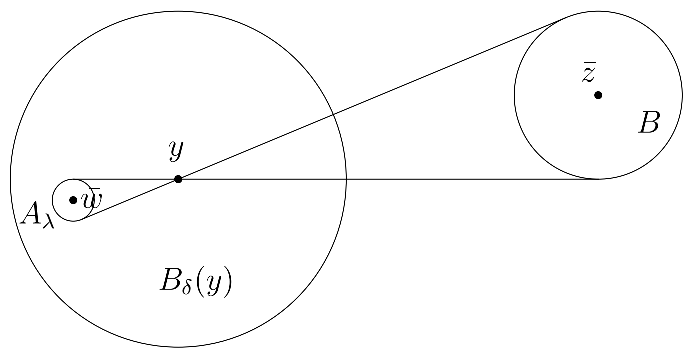

Figure 7.4.1. The set $A_\lambda = (1-\lambda)y + \lambda B$, where $\lambda < 0$.

So for $|\lambda| < \delta / \big(\varepsilon + \|\bar z - y\|\big)$, we have $\|w - y\| < \delta$, so $A_\lambda \subset V$. So fix such a $\lambda < 0$. Now for any point $w \in A_\lambda$ there is by definition some $z \in B$ such that

$$w = (1-\lambda)y + \lambda z, \quad \text{or} \quad y = \frac{1}{1-\lambda} w + \frac{-\lambda}{1-\lambda} z.$$

I claim that this implies that $f(w) = f(y) = f(x)$. To see why, first note that since $A_\lambda \subset B_\delta(y) \subset U(x)$ we have $f(w) \geqslant f(y)$. But if $f(w) > f(y)$, since $f(z) > f(y)$, quasiconcavity implies

$$f(y) \geqslant \min\{f(w), f(z)\} > f(y),$$

a clear contradiction. This means that $f(w) = f(y)$. But w is an arbitrary element of the open (in $\operatorname{aff} C$) set A_λ, so local nonmaximization is violated on A_λ.

This contradiction shows that $\operatorname{ri} U(x) \subset P(x)$, completing the proof that $\operatorname{ri} U(x) = P(x)$. ∎

7.4.3 Corollary *Let C be a convex set in $\boldsymbol{R}^{\mathrm{m}}$. Let f be a lower semicontinuous, locally nonmaximized, quasiconcave function on C. Then f is explicitly quasiconcave.*

Proof: Assume $f(y) > f(x)$. Then $y \in P(x) = \operatorname{ri} U(x)$, so by Lemma 5.1.3, the segment $[y, x) \subset \operatorname{ri} U(x) = P(x)$, so for $1 \geqslant \lambda > 0$, we have $f\big((1-\lambda)x + \lambda y\big) > f(x)$. That is, f is explicitly quasiconcave. ∎

7.4.4 Corollary *Let C be a convex set in $\boldsymbol{R}^{\mathrm{m}}$. Let f be an upper semicontinuous, locally nonminimized, quasiconvex function on C. Then f is explicitly quasiconvex.*

7.4.5 Example Proposition 7.4.2 may fail without quasiconcavity. Let $X = \boldsymbol{R}$ and let $f(x) = x^2$. Then f is locally nonmaximized and continuous, but $P(0) = \boldsymbol{R} \setminus \{0\} \neq \boldsymbol{R} = \operatorname{int} U(0)$. □

References

[1] K. J. Arrow and F. H. Hahn. 1971. *General competitive analysis*. San Francisco: Holden–Day.