Ec 181 KC Border
Convex Analysis and Economic Theory AY 2019–2020

# Topic 5: Topological properties of convex sets

## 5.1 Interior and closure of convex sets

Let $X$ be a vector space. Recall (Definition 0.2.4) the affine combination function function $\kappa\colon X \times X \times \boldsymbol{R} \to X$ is defined by

$$\kappa(x, y, \lambda) = (1-\lambda)x + \lambda y.$$

It is continuous for any topological vector space. Moreover for any set $A$,

$$A \text{ is convex if and only if } \kappa\big(A \times A \times [0,1]\big) \subset A. \tag{$\star$}$$

(Actually we could replace inclusion by equality above, but let's not.)

**5.1.1 Lemma** *In any topological vector space, both the interior and the closure of a convex set are convex.*

*Proof*: Let $C$ be a convex subset of a topological vector space $X$.
Since $\operatorname{int} C \subset C$ and $C$ is convex, for any $0 \leqslant \lambda \leqslant 1$,

$$(1-\lambda)(\operatorname{int} C) + \lambda(\operatorname{int} C) \subset C.$$

But $\operatorname{int} C$ is open, so $(1-\lambda)(\operatorname{int} C) + \lambda(\operatorname{int} C)$ is open. Now the interior of $C$ includes every open subset of $C$, so $(1-\lambda)(\operatorname{int} C) + \lambda(\operatorname{int} C) \subset \operatorname{int} C$. Since $\lambda$ is arbitrary,

$$\kappa\Big((\operatorname{int} C) \times (\operatorname{int} C) \times [0,1]\Big) \subset \operatorname{int} C$$

so ($\star$) implies that $\operatorname{int} C$ is convex. (Note that this works even if $\operatorname{int} C$ is empty.)

To see that $\overline{C}$ is convex, first observe (A.7.10) that

$$\overline{C} \times \overline{C} \times [0,1] = \overline{C \times C \times [0,1]}.$$

Since $\kappa$ is continuous, we have

$$\kappa\Big(\overline{C} \times \overline{C} \times [0,1]\Big) = \kappa\Big(\overline{C \times C \times [0,1]}\Big) \subset \overline{\kappa\Big(C \times C \times [0,1]\Big)} = \overline{C},$$

where the inclusion follows from the continuity of $\kappa$ by Lemma A.7.7, and the final equality follows from $\kappa\Big(C \times C \times [0,1]\Big) = C$ since $C$ is convex.

So ($\star$) implies that $\overline{C}$ is convex. ∎

**5.1.2 Corollary** *The closed convex hull of $A$ is the closure of the convex hull of $A$.*

*Proof*: Let $C$ be any closed convex set that includes $A$. Then, since it is convex, it also includes $\operatorname{co} A$. So the closures satisfy $\overline{\operatorname{co} A} \subset \overline{C} = C$ (A.7.4). But by Lemma 5.1.1, $\overline{\operatorname{co} A}$ is a closed convex set. This shows that $\overline{\operatorname{co} A}$ is the smallest closed convex set that includes $A$, that is, it is equal to $\overline{\operatorname{co}}\, A$. ∎

**5.1.3 Lemma** *If $C$ is a convex set in a topological vector space $X$, and if $x \in \operatorname{int} C$ and $y \in \overline{C}$, then the half-open line segment $[x, y)$ satisfies*

$$[x, y) \subset \operatorname{int} C.$$

*Proof*: This is vacuously true if the interior of $C$ is empty, so assume that the interior of $C$ is nonempty, and let $x$ belong to the interior of $C$ and $y$ belong to the closure of $C$. Fix $0 < \lambda < 1$. We need to show that the point $(1-\lambda)x + \lambda y = x + \lambda(y - x)$ belongs to the interior of $C$. (The case $\lambda = 0$ is automatic.)

Since $x$ belongs to the interior of $C$, there is an open neighborhood $U$ of zero such that $x + U \subset C$. For any point $z \in C$, the set

$$(1-\lambda)(x+U) + \lambda z = \{(1-\lambda)u + \lambda z : u \in U\} = (1-\lambda)x + \lambda z + (1-\lambda)U$$

of convex combinations of $z$ and points in $x + U$ is an open set that is included in $C$. This set is shaded in Figure 5.1.1. (The figure is drawn for the case where $z \notin x + U$, and $\lambda = 2/3$. For my convenience $U$ is shown as a disk, but it need not be circular.) It is clear from the picture that if $z$ is close enough to $y$, then $(1-\lambda)x + \lambda y$ belongs to the shaded region. We now derive algebraically just how close $z$ must be to $y$.

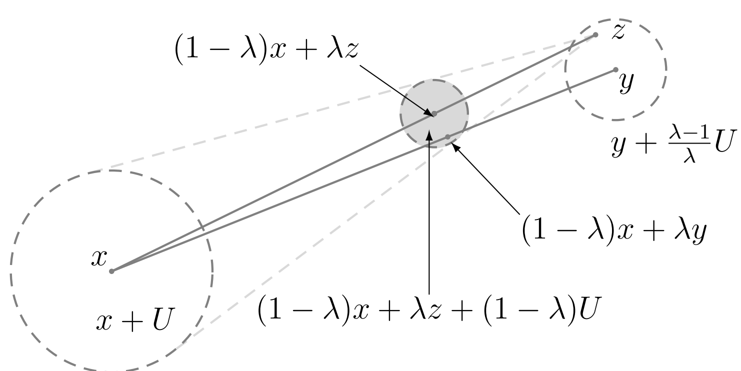



**Figure 5.1.1.** For $z \in C$, the open set $(1-\lambda)x + \lambda z + (1-\lambda)U$ (shaded) is included in $C$, and it contains $(1-\lambda)x + \lambda y$ whenever $z$ is close enough to $y$.

Given $0 < \lambda < 1$, consider the open neighborhood $y + \frac{\lambda-1}{\lambda}U$ of $y$. Since $y \in \overline{C}$, there is some point $z \in C$ that also belongs this neighborhood, say

$$z = y + \frac{\lambda - 1}{\lambda} u \quad \text{where} \quad u \in U.$$

Then

$$\lambda(y - z) = (1 - \lambda)u$$

so

$$\begin{aligned}(1-\lambda)x + \lambda y &= (1-\lambda)x + \lambda z + \lambda(y - z)\\ &= (1-\lambda)(x+u) + \lambda z\\ &\in (1-\lambda)(x+U) + \lambda z \subset \operatorname{int} C.\end{aligned}$$

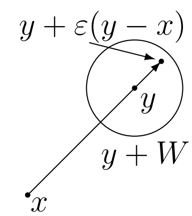
∎

**5.1.4 Corollary** *Let $C$ be a convex subset of a topological vector space. If $C$ has a nonempty interior, then:*

1. $\operatorname{int} C$ *is dense in* $\overline{C}$, *so* $\overline{C} = \overline{\operatorname{int} C}$.
2. $\operatorname{int} C = \operatorname{int} \overline{C}$.

*Proof*: (1) Let $y$ belong to $\overline{C}$. Pick $x \in \operatorname{int} C$. Then $y$ belongs to the closure of $[x, y)$, which is a subset of $\operatorname{int} C$ by Lemma 5.1.3. Thus $\operatorname{int} C$ is dense in $\overline{C}$, that is, $\overline{C} \subset \overline{\operatorname{int} C}$. But $\overline{C} \supset \overline{\operatorname{int} C}$, so we have equality.

(2) Let $y$ belong to $\operatorname{int} \overline{C}$ and let $W$ be a neighborhood of zero satisfying $y + W \subset \overline{C}$. Pick some $x \in \operatorname{int} C$. Then for $0 < \varepsilon$ small enough,




$$x + (1+\varepsilon)(y - x) = y + \varepsilon(y - x) \text{ belongs to } y + W \subset \overline{C}.$$

But $y$ belongs to the half-open line segment $\big[x, y + \varepsilon(y - x)\big)$, so by Lemma 5.1.3 $y$ belongs to $\operatorname{int} C$. Therefore $\operatorname{int} \overline{C} \subset \operatorname{int} C$. The reverse inclusion is trivial, so $\operatorname{int} C = \operatorname{int} \overline{C}$.

∎

Note that in an infinite dimensional space, a convex set with an empty interior may have a closure with a nonempty interior, as the next example shows.

**5.1.5 Example** Consider the vector space $C[0,1]$ of all continuous functions on the unit interval, with the norm given by $\|f\| = \max_x |f(x)|$. The vector space $P[0,1]$ of polynomials on $[0,1]$ is a convex subset of $C[0,1]$ with an empty interior, but it is dense, so

$$\operatorname{int} P[0,1] = \varnothing, \qquad \text{but} \qquad \operatorname{int} \overline{P[0,1]} = C[0,1].$$

□

## 5.2 Internal points, affine hulls, and relative interiors

An important difference between convex analysts and the rest of us is the use of the term "relative interior."

**5.2.1 Definition** *The* ***relative interior*** *of a convex set $C$ in a topological vector space, denoted* $\operatorname{ri} C$*, is defined to be its topological interior relative to its affine hull* $\operatorname{aff} C$*.*

*Similarly, the* ***relative boundary*** *of a convex set is the boundary relative to the affine hull.*

In other words, $x \in \operatorname{ri} C$ if and only if there is some open neighborhood $U$ of $x$ such that $y \in U \cap \operatorname{aff} C$ implies $y \in C$. Even a one point set has a nonempty relative interior in this sense, namely itself. The only convex set with an empty relative interior is the empty set.

You might ask why I haven't defined the relative closure. The reason is that the affine hull of a set in $\boldsymbol{R}^{\mathrm{m}}$ is itself closed, so the closure and relative closure would agree.

**5.2.2 Proposition** *In $\boldsymbol{R}^{\mathrm{m}}$, the relative interior of a nonempty convex set is nonempty.*

*Proof*: If $C$ is a singleton $\{x\}$, then $\{x\}$ is its own relative interior, so assume $C$ has at least two elements. Also, if $x \in C$, then it follows from Exercises 4.2.1 and 4.2.3 that the affine hull $\operatorname{aff} C$ of $C$ is equal to $x + \operatorname{aff}(C - x)$. Therefore $\operatorname{ri} C = x + \operatorname{ri}(C - x)$, so we may assume that $C$ contains 0.

Since $C$ also has a nonzero element, it has a basis (a nonempty maximal linearly independent subset), $b_1, \dots, b_k$. The $k$-dimensional span $M$ of $b_1, \dots, b_k$ is thus the affine hull of $C$. The coordinate mapping $\varphi \colon \sum_{i=1}^k \alpha_i b_i \mapsto (\alpha_1, \dots, \alpha_k)$ is a one-to-one linear homeomorphism between $M$ and $\boldsymbol{R}^{\mathrm{k}}$.

Now any point in $M$ of the form $\sum_{i=1}^k \alpha_i b_i$ with each $\alpha_i \geqslant 0$ and $\sum_{i=1}^k \alpha_i \leqslant 1$ belongs to $C$, as a convex combination 0 and the $b_i$'s. In particular, $y = \sum_{i=1}^k \frac{1}{2k} b_i$ belongs to $C$. In fact, it is an interior point of $C$ relative to $M$. To see this, consider the open ball $B$ in $\boldsymbol{R}^{\mathrm{k}}$ centered at $(1/2k, \dots, 1/2k)$ with radius $1/2k$. For any $(\alpha_1, \dots, \alpha_k)$ in $B$, we have $\alpha_i > 0$, and $\sum_{i=1}^k \alpha_i < 1$, so the set $\varphi^{-1}(B)$ is an open subset of $M$ included in $C$ and containing $y$. Thus $y$ belongs to $\operatorname{ri} C$. ∎

In $\boldsymbol{R}^{\mathrm{m}}$, every linear subspace and so every affine subspace is closed (Corollary 3.1.8). It follows that in $\boldsymbol{R}^{\mathrm{m}}$, a subset $E$ and its closure $\overline{E}$ have the same affine hull. A consequence of this is that in $\boldsymbol{R}^{\mathrm{m}}$, the affine hulls of $\operatorname{ri} C$, $C$ and $\overline{C}$ coincide.

**5.2.3 Proposition** *For a convex subset $C$ of $\boldsymbol{R}^{\mathrm{m}}$,*

$$\overline{\operatorname{ri} C} = \overline{C}, \qquad \textit{and} \qquad \operatorname{ri}(\operatorname{ri} C) = \operatorname{ri} C.$$

*Proof*: By Proposition 5.2.2, the interior of $C$ relative to its affine hull $M$ is nonempty. By Corollary 5.1.4, the closure of this interior (in $M$) is the closure of $C$ in $M$. But in $\boldsymbol{R}^{\mathrm{m}}$ every affine set is closed, so the closure relative to $M$ is the same as the closure relative to $\boldsymbol{R}^{\mathrm{m}}$. This proves $\overline{\operatorname{ri} C} = \overline{C}$.

For the second statement, by definition $\operatorname{ri}(\operatorname{ri} C)$ is the topological interior of $\operatorname{ri} C$ relative to the affine hull of $\operatorname{ri} C$. But since the affine hull is closed (in $\boldsymbol{R}^{\mathrm{m}}$), the affine hull of $\operatorname{ri} C$ is the same as the affine hull of $\overline{C}$, and so the same as the affine hull of $C$. But $\operatorname{ri} C$ is open in $\operatorname{aff} C$, so it equals its interior in $\operatorname{aff} C$, namely $\operatorname{ri}(\operatorname{ri} C)$. ∎

**5.2.4 Corollary** *The relative boundary of a convex set $C$ is equal to $\overline{C} \setminus \operatorname{ri} C$.*

This need not be true in an infinite dimensional topological vector space.

There is another characterization of the relative interior that relies on the useful notion of intrinsic core.

**5.2.5 Definition** *Let $A$ be a subset of the vector space $X$, and let $L$ be an affine subspace of $X$ that includes $A$. A point $x$ is an* ***internal point of A relative to L*** *if for each point $y \in L$ distinct from $x$, there is an $\varepsilon > 0$ such that the line segment $\big(x - \varepsilon(y-x)\big), x + \varepsilon(y-x)\big)$ is included in $A$.*

*When $L = X$, we simply say that $x$ is an* ***internal point*** *of $A$.*

*The set of internal points of $A$ is called the* ***core*** *of $A$ or the* ***algebraic interior*** *of $A$, denoted $\operatorname{cor} A$.*

*The set of internal points of $A$ relative to its affine hull $\operatorname{aff} A$ is called the* ***intrinsic core*** *of $A$, denoted $\operatorname{icr} A$.*

This whole section needs to be polished.

Add a picture.

**5.2.6 Proposition** *Let $C$ be a convex subset of a topological vector space. Then every point in $\operatorname{ri} C$ is an intrinsic core point of $C$.*

*Proof*: Let $A$ be the affine hull of $C$. Then $\operatorname{ri} C$ is a relatively open subset of $A$ in the topological sense. Define the continuous function $h \colon \boldsymbol{R} \to A$ by $h(\lambda) = x + \lambda(x-y)$. Since $h(0) = x \in \operatorname{ri} C$. So by continuity, the inverse image $B = h^{-1}(\operatorname{ri} C)$ of the open subset $\operatorname{ri} C$ of $A$ is an open set in $\boldsymbol{R}$ that contains 0. That is, there is some $\eta > 0$ such that $|\varepsilon| < \eta$ implies that $\varepsilon \in B$, so $h(\varepsilon) \in \operatorname{ri} C \subset C$. ∎

In the finite dimensional case, we have the converse.

**5.2.7 Proposition** *In a finite dimensional topological vector space, the relative interior and the intrinsic core of a convex set coincide.*

*Proof*: Let $C$ be a convex subset of a finite dimensional vector space. Proposition 5.2.6 shows that $\operatorname{ri} C \subset \operatorname{icr} C$.

For the reverse inclusion, let $x$ be an intrinsic core point of $C$. If $C$ is a singleton, the conclusion is immediate, so it suffices to consider the case where $x = 0$ and $C$ has at least two points. In this case, the affine hull $L$ of $C$ is actually an $m$-dimensional linear subspace. As such it has a basis $v_1, \dots, v_m$. By hypothesis, 0 is an intrinsic core point, so by scaling the basis we may assume that $\pm v_1, \dots, \pm v_m$ all belong to $C$. Then

$$\left\{ \sum_{i=1}^{m} \alpha_i v_i : \sum_{i=1}^{m} |\alpha_i| < 1 \right\}$$

is an open convex subset of $C$ that contains 0. ∎

There are significant differences between a topologist's relative interior and convex analyst's relative interior.

> For instance, it is *not* true that $A \subset B$ implies $\operatorname{ri} A \subset \operatorname{ri} B$.

For instance, consider a closed interval $B = [a, b]$ and one of its endpoints, $A = \{a\}$. Then $\operatorname{ri} B = (a, b)$ and $\operatorname{ri} A = \{a\}$ itself, so $\operatorname{ri} A \cap \operatorname{ri} B = \varnothing$.

I'll leave the proofs of the next two theorems as exercises.

Prove these

**5.2.8 Proposition (Rockafellar [1, Theorem 6.5, p. 47])** *Let $\{C_i\}_{i \in I}$ be a family of convex subsets of $\boldsymbol{R}^{\mathrm{m}}$, and assume*

$$\bigcap_{i \in I} \operatorname{ri} C_i \neq \varnothing.$$

*Then*

1. $$\overline{\bigcap_{i \in I} C_i} = \bigcap_{i \in I} \overline{C_i},$$

   *and,*

2. *for finite $I$,*
   $$\operatorname{ri} \bigcap_{i \in I} C_i = \bigcap_{i \in I} \operatorname{ri} C_i.$$

**Sample answer:** ∎

**5.2.9 Proposition (Rockafellar [1, Corollaries 6.6.1, 6.6.2, pp. 48–49])** *For convex subsets $C$, $C_1$, $C_2$ of $\boldsymbol{R}^{\mathrm{m}}$, and $\lambda \in \boldsymbol{R}$,*

$$\operatorname{ri}(\lambda C) = \lambda \operatorname{ri} C, \quad \operatorname{ri}(C_1 + C_2) = \operatorname{ri} C_1 + \operatorname{ri} C_2, \quad \text{and} \quad \overline{C_1 + C_2} \supset \overline{C_1} + \overline{C_2}.$$

## 5.3 Topological properties of convex hulls

**5.3.1 Proposition** *The convex hull of an open set is open.*

*Proof*: By Exercise 4.3.4(3) any point $x$ in $\operatorname{co} A$ is a convex combination $\sum_{i=1}^{n} \alpha_i x_i$ from $A$. If $A$ is open, then for each $x_i$ there is an open neighborhood $U_i$ of zero such that $x_i + U_i \subset A$. Then $\sum_{i=1}^{n} \alpha_i (x_i + U_i)$ is an open neighborhood of $x$ included in the convex hull $\operatorname{co} A$. ∎

We have shown in Corollary 2.4.4 that the convex hull of a compact set is compact. The same cannot be said of closed sets.

**5.3.2 Example (The convex hull of a closed set need not be closed)** Consider the closed subset of the plane

$$A = \{(x, y) \in \boldsymbol{R}^2 : y = 1/|x|,\ x \neq 0\}.$$

Then

$$coA = \{(x, y) \in \boldsymbol{R}^2 : y > 0\}.$$

By the way, this answers Exercise 2.2.2. You should also compare it to Example 0.2.2. □

## References

[1] R. T. Rockafellar. 1970. *Convex analysis*. Number 28 in Princeton Mathematical Series. Princeton: Princeton University Press.