Topic 4: Digression: Hulls

4.1 Some special classes of sets

The notion of a hull is pervasive in mathematics and transcends just the notion of a convex hull, so it is instructive, if perhaps a bit discursive, to consider convex sets and convex hulls as part of a more general phenomenon. The following classes of subsets of a real vector space are all characterized in terms of being closed under particular operations.

4.1.1 Definition *A subset A of a real vector space X is:*

- *a **linear subspace** if it is nonempty and closed under linear combinations. That is, for all $\alpha, \beta \in \boldsymbol{R}$ and $x, y \in A$,*

$$\alpha x + \beta y \in A.$$

  (Thus 0 belongs to every linear subspace, and $\{0\}$ is itself a linear subspace, the **trivial subspace** or **degenerate subspace**.[1] Note that we do not consider the empty set to be a linear subspace.)

- *an **affine subspace** if it includes the line through any two of its points. That is, it is closed under linear combinations where the coefficients sum to unity. More pedantically, A is affine if for all $\alpha, \beta \in \boldsymbol{R}$ and $x, y \in A$*

$$\alpha + \beta = 1 \implies \alpha x + \beta y \in A.$$

  (Note that the empty set is affine.)

- *a **convex set** if it includes the line segment joining any two of its points. That is, it is closed under nonnegative linear combinations where the coefficients sum to unity. That is, A is convex if for all $\alpha, \beta \in \boldsymbol{R}$ and $x, y \in A$*

$$(\alpha, \beta \geqslant 0,\ \alpha + \beta = 1) \implies \alpha x + \beta y \in A.$$

- *a **cone** if it is nonempty and is closed under multiplication by nonnegative scalars, that is, if it includes the ray through any of its nonzero points. That is, A is a cone if for all $\alpha \in \boldsymbol{R}$ and $x \in A$*

$$\alpha \geqslant 0 \implies \alpha x \in A.$$

[1] Mathematicians often use the terms "degenerate" and "trivial" interchangeably. I wonder how they respond when asked to play *Trivial Pursuit*.

(By this definition, 0 belongs to every cone, and $\{0\}$ is itself a cone, the **trivial cone**.[1] Some authors allow a cone to exclude 0 by requiring $\alpha > 0$ in the definition above. Note that the empty set is not a cone.)

*A cone A is **pointed** if $-A \cap A = \{0\}$, that is, if it includes no lines.*

N.B. Some authors use the term **wedge** to refer to what I call a cone. They reserve the term cone for what I call a pointed cone.

- ***closed*** *if it contains the limit of any sequence it includes. That is, if for every sequence $(x_1, x_2, \dots) \subset A$ such that $\lim_{n\to\infty} x_n$ exists,*

$$\lim_{n\to\infty} x_n \in A.$$

(The empty set is closed. Note that this requires a notion of convergence, so we must add the requirement that X be a topological vector space.)

- *A subset A of $X \times \boldsymbol{R}$ is **vertically increasing** if*

$$\Big[(x,\alpha) \in A \text{ and } \beta \geqslant \alpha\Big] \implies (x,\beta) \in A.$$

For instance, the epigraph of a function is vertically increasing. Vertically decreasing sets, such as hypographs, are defined as you should expect.

*A **partially ordered vector space** is a pair $(X, \geqq)$ where $\geqq$ is a partial order on X (transitive, antisymmetric) that satisfies (i) $x \geqq y \implies x+z \geqq y+z$ for all z, and (ii) $x \geqq y \implies \alpha x \geqq \alpha y$ for all $\alpha \geqslant 0$. A vector x is **positive** if $x \geqq 0$. The set of positive vectors is denoted X_+.* (Note that this contradicts my claim on page 0–2 to avoid using the term positive.)

- *The subset $A \subset X$ of a partially ordered vector space is **increasing** if A is closed under addition of nonnegative vectors, that is,*

$$(x \in A \text{ and } y \geqq 0) \implies x + y \in A.$$

Equivalently A is increasing if $A + X_+ \subset A$, or equivalently if $A + X_+ = A$.

*A set is **decreasing** if the above holds for $y \leqq 0$.* (The empty set is both increasing and decreasing, as is X itself.)

4.2 Affine subspaces

The affine subspaces of a vector space are the translates of linear subspaces. The details are worked out in the following exercises.

4.2.1 Exercise (An affine subspace is a translate of a linear subspace) Let M be a linear subspace of a vector space X, and let $x \in X$. Then $A = M + x$ is an affine subspace of X. □

Sample answer: Let $a, b \in A$. We need to show that $(1-\alpha)a + \alpha b \in A$. Since $a, b \in A = M + x$, we have $a - x \in M$ and $b - x \in M$. Thus $(1-\alpha)(a-x) + \alpha(b-x) = (1-\alpha)a + \alpha b - x \in M$, so $(1-\alpha)a + \alpha b \in A$. ∎

4.2.2 Exercise If M is a linear subspace of X, define the binary relation $\underset{M}{\sim}$ on X by

$$x \underset{M}{\sim} y \quad \text{if} \quad x - y \in M.$$

Show that $\underset{M}{\sim}$ is an equivalence relation (transitive, symmetric, and reflexive).

Show that each equivalence class is an affine subspace of X. □

Sample answer: Symmetry: Since $y - x = -(x - y)$, and M is closed under scalar multiplication, $x - y \in M \implies y - x \in M$. Reflexivity: For any $x \in M$, $x - x = 0 \in M$. Transitivity: Let $x \underset{M}{\sim} y$ and $y \underset{M}{\sim} z$. That is, $x - y \in M$ and $y - z \in M$. Since M is closed under vector addition, we have $(x-y)+(y-z) = y - z \in M$, so $x \underset{M}{\sim} z$.

By definition, the $\underset{M}{\sim}$-equivalence class $[x]$ of a point x in X is

$$[x] = \{y : y - x \in M\} = M + x,$$

which is an affine set by Exercize 4.2.1. ∎

That is, every linear subspace M defines a family of affine subspaces $M + x$, where $x \in X$. For $x, y \in X$, we say the affine subspaces $M + x$ and $M + y$ are **parallel**. Next we show that every affine subspace is of this form.

4.2.3 Exercise (Recovering a linear subspace from an affine subspace) Let A be an affine subspace of X and let $a, b \in A$. Prove the following:

1. The set $A - a = \{x - a : x \in A\}$ is a linear subspace of X.

   Sample answer: Let $M = A - a$. First we show that if $x \in M$, then for any real number α, we have $\alpha x \in M$. To see this observe that if $x \in M$, then $x + a \in A$. Since $a \in A$ and $x + a \in A$, we must have $(1-\alpha)a + \alpha(x+a) \in A$. But $(1-\alpha)a + \alpha(x+a) = \alpha x + a \in A$, so $\alpha x \in M$.

   Now we show that if $x, y \in M$, then $x + y \in M$. Since $x, y \in M$, we have $x + a \in A$ and $y + a \in A$. Since A is affine, $\frac{1}{2}(x+a) + \frac{1}{2}(y+a) = a + \frac{1}{2}(x+y)$ belongs to A. Thus $\frac{1}{2}(x+y) \in M$, so by the first part of the argument with $\alpha = 2$ we have $x + y \in M$.

   That is, M is closed under scalar multiplication and vector addition, so it is a linear subspace. ∎

2. $A - a = A - b$.

Sample answer: By symmetry it suffices to show that $A - a \subset A - b$. So let $x \in A - a$. Then $x + a \in A$ and $b \in A$, so since A is affine $\frac{1}{2}(x+a) + \frac{1}{2}b = a + \frac{1}{2}x + \frac{1}{2}(b-a) \in A$. Thus $\frac{1}{2}x + \frac{1}{2}(b-a) \in A - a$. By the argument above $A - a$ is a linear subspace, so $x + (b - a) \in A - a$. That is, $x + b \in A$, so $x \in A - b$. ∎

3. If M and N are linear subspaces such that $A = M + x = N + y$ for some $x, y \in A$, then $M = N$. This unique subspace is called the **linear subspace parallel to $\boldsymbol{A}$**.

   Sample answer: Since 0 belongs to every linear subspace, we must have $x \in M + x = A$ and $y \in N + y = A$. But then $M = A - x = A - y = N$. ∎

□

Here are a few more simple consequences of the above.

- An affine subspace is a linear subspace if and only if it contains 0.
- Let M denote the unique linear subspace parallel to A. For $x \in A$ and $y \in M$ we have $x + y \in A$.

4.2.1 Affine functions

4.2.4 Definition *Let A be an affine subspace of the vector space X. A real function $f \colon X \to \boldsymbol{R}$ is* ***affine*** *if for every $x, y \in A$ and scalar λ,*

$$f\big((1-\lambda)x + \lambda y\big) = (1-\lambda)f(x) + \lambda f(y) = f(x) + \lambda(f(y) - f(x)).$$

Clearly every linear functional is affine.

4.2.5 Exercise (Affine functions) Let A be an affine subspace of the vector space X. A real function f on A is affine if and only if it is of the form

$$f(x) = g(x - a) + \gamma, \tag{A}$$

where a belongs to A and g is linear on the linear subspace $A - a$. Moreover, g is independent of the choice of a in A, and $\gamma = f(a)$.

As a special case, when $A = X$, we may take $a = 0$, so an affine function f on X can be written as $f(x) = g(x) + \gamma$, where g is linear on X and $\gamma = f(0)$. □

Sample answer: First note that if f is defined as in (A), then it is affine. The converse is more involved.

Assume f is affine on A, fix some $a \in A$, and define g on the subspace $M = A - a$ by

$$g(x - a) = f(x) - f(a), \qquad x \in A \tag{1}$$

so that

$$g(y) = f(y+a) - f(a), \qquad y \in M. \tag{2}$$

Then $f(x) = g(x-a) + \gamma$, where $\gamma = f(a)$ as desired. It remains to show that g is linear.

We start by showing that if $y \in M$ and $\alpha \in \boldsymbol{R}$, then $g(\alpha y) = \alpha g(y)$:

$$\begin{aligned}
g(\alpha y) &= f(\alpha y + a) - f(a) && \text{by (2)} \\
&= f\Big((1-\alpha)a + \alpha(y+a)\Big) - f(a) && \text{(simple algebra)} \\
&= (1-\alpha)f(a) + \alpha f(y+a) - f(a) && \text{since } f \text{ is affine} \\
&= \alpha f(y+a) - \alpha f(a) && \text{(simplify)} \\
&= \alpha g(y) && \text{by (2).}
\end{aligned}$$

Next we show that $g(x+y) = g(x) + g(y)$ for $x, y \in M$:

$$\begin{aligned}
g\Big(\tfrac{1}{2}x + \tfrac{1}{2}y\Big) &= f(\tfrac{1}{2}x + \tfrac{1}{2}y + a) - f(a) \\
&= f\Big(\tfrac{1}{2}(x+a) + \tfrac{1}{2}(y+a)\Big) - f(a) \\
&= \tfrac{1}{2}f(x+a) + \tfrac{1}{2}f(y+a) - f(a) \\
&= \tfrac{1}{2}g(x+a) + \tfrac{1}{2}g(y+a).
\end{aligned}$$

Now multiply everything by 2 and use the previous result.

Finally we show that g is independent of the choice of a. So let a, b belong to A and define $g(x) = f(x+a) - f(a)$ and $h(x) = f(y+b) - f(b)$ for $x \in M = A - a = A - b$. We need to show that $g(x) = h(x)$ for all $X \in M$.

First note that since $a, b \in A$ we have $a - b \in M$, and

$$g(b-a) = f(b) - f(a), \text{ and } h(a-b) = f(a) - f(b). \tag{3}$$

Now any $z \in A$ can be written in either of the forms $z = x + a$ or $z = y + b$, where $x, y \in M$. Then

$$f(z) = g(z-a) + f(a) = h(z-b) + f(b),$$

so by (3) and the linearity of h we have

$$g(z-a) = h(z-b) + f(b) - f(a) = h(z-b) + h(b-a) = h(z-a).$$

Since each $x \in M = A - a$ is of the form $x = z - a$ for $x \in A$, we have $g(x) = h(x)$ for all $x \in M$. ∎

4.3 Hulls

For each of the concepts in Section 4.1 there is a related **hull**. In the following definitions when I say that a set E is the **smallest** set having property P, I mean that (i) The set E has property P, and (ii) if F is any set having property P, then $E \subset F$. (A word on notation: When I write $A \subset B$ or $A \supset B$, I allow for $A = B$, otherwise I will write $A \subsetneq B$ or $A \supsetneq B$.)

4.3.1 Definition (The concept of a hull) *For a (nonempty) set A in a vector space X,*

- *the **linear hull**, better known as the **span** of A, denoted $\operatorname{span} A$, is the smallest linear subspace that includes A.*
- *the **affine hull** of A, denoted $\operatorname{aff} A$, is the smallest affine subspace that includes A.*
- *the **convex hull** of A, denoted $\operatorname{co} A$, is the smallest convex set that includes A.*
- *the **conical hull** or **positive hull**, better known as the **cone generated by A**, denoted $\operatorname{cone} A$, is the smallest cone that includes A. See Figure 4.3.1.*

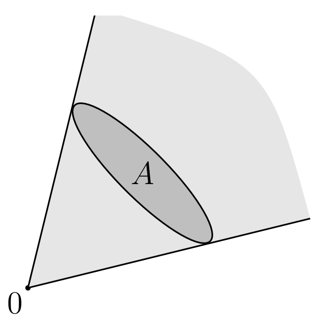

Figure 4.3.1

- *the **closed hull**, known as the **closure** of A, denoted $\overline{A}$ or sometimes $\operatorname{cl} A$, is the smallest closed set that includes A.*
- *the **closed convex hull** of A, denoted $\overline{\operatorname{co}}\, A$, is the smallest closed convex set that includes A.*
- *the **convex conical hull** or the **convex cone generated by A** is the smallest convex cone that includes A.*
- *the **closed convex cone generated by A** is the smallest closed convex cone that includes A.*
- *the **increasing hull** of A is the smallest increasing set that includes A. The decreasing hull is the smallest decreasing set.* (This applies only to partially ordered spaces.)
- *the **vertically increasing hull** of A is the smallest vertically increasing set that includes A. (The vertically decreasing hull is the smallest vertically decreasing set.)*

4.3.2 Remark N.B. In order for these notions to make sense, it is required that there *exists* a *smallest* set with the desired property. That is, we need to make sure we are not looking for something like the smallest number strictly greater than zero. This will be taken up in the next exercise.

The next two exercises state obvious facts that have trivial proofs, and are usually just asserted. Indeed, a mathematically experienced reader can subconsciously put the proof together, and will claim the conclusion is obvious. That may be, but it may not be obvious how many obvious facts need to be strung together to prove something obviously trivial.[2] If you are mathematically experienced, you will find it tedious and pointless to write out a detailed proof. I apologize. I have done two of them for you as examples. It is important to make one such argument in detail, so you will *know* that it is obvious. By the time you do all these exercises, all these assertions will be obvious. If nothing else, the exercises will help you to be more zen.

I know I am belaboring this point, but bear with me. Each of these statements has two parts, the first part is of the form, "If each set in a nonempty family of sets of has property P, then the intersection of the family has property P." Such a statement need not be true for any arbitrary property. For example, consider the property of being infinite. The intersection of a nonempty family of infinite sets need not be an infinite set. (Give an example.) So it is the particular kind of property that makes these results trivial. All the properties in the exercise are of the form, "the set is closed under a particular operation." It is because the property is of this form and the nature of the operations involved that the assertions are true. I hope this convinces the more mathematically inclined among you to at least do these exercises in your head.

4.3.3 Exercise (Mind-numbing exercise 1) Prove the following assertions about subsets of the linear space X.

1. The intersection of a nonempty family of linear subspaces is a linear subspace, and for a nonempty set A,
$$\operatorname{span} A = \bigcap\{L \subset X : L \text{ is a linear subspace and } L \supset A\}.$$
2. The intersection of a nonempty family of affine subspaces is an affine subspace (possibly empty), and
$$\operatorname{aff} A = \bigcap\{L \subset X : L \text{ is an affine subspace and } L \supset A\}.$$
3. The intersection of a nonempty family of convex sets is a convex set (possibly empty), and
$$\operatorname{co} A = \bigcap\{C : C \text{ is convex and } C \supset A\}.$$

[2] This brings to mind the story of the mathematics professor who has just asserted that some proposition is obvious. A puzzled student asks if it is really obvious. The professor paces and thinks hard for fifteen minutes, and then announces, "Yes, it's obvious."

4. The intersection of a nonempty family of cones is a cone, and for a nonempty set A, the
$$\text{cone generated by } A = \bigcap\{C \subset X : C \text{ is a cone and } C \supset A\}.$$

5. The intersection of a nonempty family of closed sets is a closed set, and
$$\overline{A} = \bigcap\{C \subset X : C \text{ is closed and } C \supset A\}.$$

6. The intersection of a nonempty family of increasing sets is an increasing set, and
$$\text{increasing hull } A = \bigcap\{C \subset X : C \text{ is increasing and } C \supset A\}.$$

7. The intersection of a nonempty family of vertically increasing sets is a vertically increasing set, and
$$\text{vertically increasing hull } A = \bigcap\{C \subset X : C \text{ is vertically increasing and } C \supset A\}.$$

□

Hint: Let me prove the first assertion. Let $\mathcal{L}$ be a nonempty family of linear subspaces of the vector space X, and let $M = \bigcap\{L \subset X : L \in \mathcal{L}\}$. Our job is to prove that M is a linear subspace of X. So let $\alpha, \beta \in \boldsymbol{R}$ and assume $x, y \in M$. Then $x, y \in L$ for each $L \in \mathcal{L}$, as $M \subset L$. But each L is a linear subspace, so $\alpha x + \beta y \in L$ for each $L \in \mathcal{L}$. Therefore $\alpha x + \beta y \in \bigcap\{L \subset X : L \in \mathcal{L}\} = M$. This proves that M is a linear subspace.

Recall that $\operatorname{span} A$ is the smallest linear subspace that includes A. What exactly does this mean? It means that (i) $\operatorname{span} A$ is a linear subspace that includes A, and (ii) if B is any linear subspace that includes A, then $B \supset \operatorname{span} A$. (This makes it smallest with respect to the partial order $\subset$ on the subsets of X.) So let $\mathcal{L} = \{L : L \text{ is a linear subspace and } L \supset A\}$. Since X is itself a linear space that includes A, we see that $\mathcal{L}$ is nonempty. We have just seen that $\bigcap\{L : L \text{ is a linear subspace and } L \supset A\} = \bigcap\{L : L \in \mathcal{L}\}$ is indeed a linear subspace, and moreover it is clear that it includes A. Now let B be a linear subspace that includes A. Then $B \in \mathcal{L}$, so $\bigcap\{L : L \in \mathcal{L}\} \subset B$. In other words, $\bigcap\{L : L \in \mathcal{L}\}$ is the smallest linear subspace that includes A, and so by definition is $\operatorname{span} A$. ∎

There are other useful ways to characterize these sets.

4.3.4 Exercise (Mind-numbing exercise 2) Prove the following assertions.

1. If L is a linear subspace and $\alpha_1, \dots, \alpha_n \in \boldsymbol{R}$ and $x_1, \dots, x_n \in L$, then
$$\alpha_1 x_1 + \cdots + \alpha_n x_n \in L,$$
and moreover
$$\operatorname{span} A = \{\alpha_1 x_1 + \cdots + \alpha_n x_n : n \geqslant 1;\ \alpha_1, \dots, \alpha_n \in \boldsymbol{R};\ x_1, \dots, x_n \in A\}.$$
In other words, span A consists of all **linear combinations** of points in A.

2. If L is an affine subspace and $\alpha_1, \dots, \alpha_n \in \boldsymbol{R}$ and $x_1, \dots, x_n \in L$, then
$$\alpha_1 + \cdots + \alpha_n = 1 \implies \alpha_1 x_1 + \cdots + \alpha_n x_n \in L,$$
and moreover
$$\operatorname{aff} A = \left\{ \sum_{i=1}^n \alpha_i x_i : n \geqslant 1;\ \alpha_1, \dots, \alpha_n \in \boldsymbol{R};\ x_1, \dots, x_n \in A;\ \sum_{i=1}^n \alpha_i = 1 \right\}.$$
In other words, the affine hull aff A of A consists of all **affine combinations** of points in A.

3. If C is a convex set and $\alpha_1, \dots, \alpha_n \in \boldsymbol{R}$ and $x_1, \dots, x_n \in C$, then
$$(\alpha_1, \dots, \alpha_n \geqslant 0, \text{ and } \alpha_1 + \cdots + \alpha_n = 1) \implies \alpha_1 x_1 + \cdots + \alpha_n x_n \in C,$$
and moreover
$$\operatorname{co} A = \left\{ \sum_{i=1}^n \alpha_i x_i : n \geqslant 1;\ \alpha_1, \dots, \alpha_n \geqslant 0,\ x_1, \dots, x_n \in A;\ \sum_{i=1}^n \alpha_i = 1 \right\}.$$
In other words, co A consists of all **convex combinations** of points in A. (This was Exercise 2.1.3.)

4. If C is a convex cone and $\alpha_1, \dots, \alpha_n \in \boldsymbol{R}$ and $x_1, \dots, x_n \in C$, then
$$\alpha_1, \dots, \alpha_n \geqslant 0 \implies \alpha_1 x_1 + \cdots + \alpha_n x_n \in C,$$
and moreover
$$\text{convex conical hull } A = \left\{ \sum_{i=1}^n \alpha_i x_i : n \geqslant 1;\ \alpha_1, \dots, \alpha_n \geqslant 0;\ x_1, \dots, x_n \in A \right\}.$$
In other words, the convex cone generated by A consists of all **nonnegative linear combinations** of points in A.

5. The increasing hull of A is the set
$$\{x+y : x \in A \text{ and } y \geqq 0\} = A + X_+.$$

I need to define the notation X_+.

6. The vertically increasing hull of A is the set
$$\{(x, \alpha+\beta) : (x,\alpha) \in A \text{ and } \beta \geqslant 0\} = A + \Big(\{0\} \times \boldsymbol{R}_+\Big).$$ □

Hint: You might ask, what is there to prove here? Well the definitions of linear subspace, affine subspace, and convex set were stated in terms of a set being closed under various linear combinations of two elements ($n = 2$). You need to show that this implies closure under corresponding linear combinations of any (finite) n points. Clearly this calls for a proof by induction, which, while trivial, is instructive to write down. Here is the affine case:

Let L be an affine set and let $\mathbb{P}(n)$ denote the proposition:

If $x_1, \dots, x_n \in L$, and $\alpha_1 + \cdots + \alpha_n = 1$, then $\alpha_1 x_1 + \cdots + \alpha_n x_n \in L$. $\quad\mathbb{P}(n)$

Clearly $\mathbb{P}(1)$ is true, since $x \in L \implies 1x \in L$.

We now show that for all $n \geqslant 1$, $\mathbb{P}(n) \implies \mathbb{P}(n+1)$:

Assume $\mathbb{P}(n)$, and let $x_1, \dots, x_{n+1}$ belong to L, and let $\alpha_1, \dots, \alpha_{n+1} = 1$. I claim that for some i we must have $1 - \alpha_i \neq 0$. (Why? Well suppose $1 - \alpha_i = 0$ for all i. Then $\alpha_i = 1$ for all i, so $\alpha_1 + \cdots + \alpha_{n+1} = n+1 \neq 1$, a contradiction.) By renumbering if necessary, we may assume $\gamma = 1 - \alpha_{n+1} \neq 0$. Then $\sum_{i=1}^n \alpha_i/\gamma = 1$, so by the induction hypothesis $\mathbb{P}(n)$, the point $y = \sum_{i=1}^n \frac{\alpha_i}{\gamma} x_i$ belongs to L. Therefore by definition, the affine combination
$$\gamma y + \alpha_{n+1} x_{n+1} = \sum_{i=1}^{n+1} \alpha_i x_i$$
belongs to L. This completes the proof by induction.

For the second assertion (following "moreover"), let
$$B = \left\{ \sum_{i=1}^n \alpha_i x_i : n \geqslant 1;\ \alpha_1, \dots, \alpha_n \in \boldsymbol{R};\ x_1, \dots, x_n \in A;\ \sum_{i=1}^n \alpha_i = 1 \right\}.$$
You also need to show B is indeed the smallest affine set that includes A. From the argument just made we know that if L is an affine set that includes A, it must include B. Therefore it is enough to show that B itself is an affine set. So assume $x, y \in B$ and $\alpha + \beta = 1$. Then we may write $x = \sum_{i=1}^m \gamma_i x_i$ where $\sum_{i=1}^m \gamma_i = 1$ and each $x_i \in A$, and $y = \sum_{j=1}^\ell \delta_j y_j$ where $\sum_{j=1}^\ell \delta_j = 1$ and each $y_j \in B$. So
$$\alpha x + \beta y = \alpha \sum_{i=1}^m \gamma_i x_i + \beta \sum_{j=1}^\ell \delta_j y_j = \sum_{i=1}^m (\alpha\gamma_i) x_i + \sum_{j=1}^\ell (\beta\delta_j) y_j.$$
Since $\sum_{i=1}^m \alpha\gamma_i + \sum_{j=1}^\ell \beta\delta_j = \alpha + \beta = 1$, it follows that $\alpha x + \beta y$ belongs to B, so B is affine. ∎