Ec 181 KC Border
Convex Analysis and Economic Theory AY 2019–2020

Topic 3: Cones

3.1 Cones

3.1.1 Definition *A **cone** is a nonempty subset of a vector space that is closed under multiplication by nonnegative scalars. That is, it includes the ray through any of its nonzero points. In other words, C is a cone if for all $\alpha \in \boldsymbol{R}$ and $x \in C$*

$$\alpha \geqslant 0 \implies \alpha x \in C.$$

*A cone C is **pointed** if $-C \cap C = \{0\}$, that is, if it includes no lines.*

N.B. *Some authors use the term **wedge** to refer to what I call a cone. They reserve the term cone for what I call a pointed cone.*

By this definition, 0 belongs to every cone, and $\{0\}$ is itself a cone, the **trivial cone**. Note that by definition the empty set is not a cone. Recall the following definition (Definition 0.2.5): A **ray** in a vector space is a set of the form

$$\{\lambda x : \lambda \geqslant 0\}$$

where $x \neq 0$. Clearly every ray is also a cone.

3.1.2 Remark Some authors allow a cone to exclude the origin, and only require closure under multiplication by strictly positive scalars. If the need arises, I shall refer to such a set as a **punctured cone**.

Some authors will also call a set A a cone if it is of the form $C + x$, where C a cone in my sense. If the need should arise, I shall refer to such a set as a **cone with vertex $\boldsymbol{x}$**.

3.1.3 Exercise (Examples of cones) Prove the following.

1. The set of nondecreasing functions on the unit interval is a convex cone.

2. A solution set of the form

$$\{x \in \boldsymbol{R}^{\mathrm{n}} : Ax \geqq 0\}$$

   is a closed convex cone. (In fact, it is a finitely generated convex cone, but we must wait until Theorem 26.2.6 for a proof.)

□

3.1.4 Definition *Every nonempty set A in a vector space derermines a cone, denoted* cone A, *by*

$$\operatorname{cone} A = \{\lambda x : \lambda \geqslant 0,\ x \in A\}.$$

The elements of A are called ***generators*** *of* cone A.

We shall see in Topic 4 that cone A, therein called the conical hull of A, is the intersection of all cones that include A, and it is also the union of all rays that meet A. Or you could easily prove that for yourself.

3.1.5 Definition *A* ***finitely generated convex cone*** *is the convex cone generated by a nonempty finite set.*[1]

3.1.6 Exercise (Properties of cones) Prove the following.

1. Scalar multiples of cones are cones.
2. The sum of two cones is a cone.
3. A cone is convex if and only if it is closed under addition.
4. The cone generated by a convex set is a convex cone.
5. The convex cone generated by the finite set $\{x_1, \dots, x_n\}$ is the set of nonnegative linear combinations of the x_i's. That is,
$$\left\{\sum_{i=1}^{n} \lambda_i x_i : \lambda_i \geqslant 0,\ i = 1, \dots, n\right\}.$$
6. The sum of two finitely generated convex cones is a finitely generated convex cone.

□

The next application of Lemma 2.3.3 is often asserted to be obvious, but is not so easy to prove. It is true in general Hausdorff topological vector spaces, but I'll prove it for the Euclidean space case.[2]

3.1.7 Lemma *Every finitely generated convex cone is closed.*

Proof for the finite dimensional case: Let $A = \{x_1, \dots, x_k\}$ be a finite subset of $\boldsymbol{R}^{\mathrm{m}}$ and let $C = \{\sum_{i=1}^{k} \lambda_i x_i : \lambda_i \geqslant 0,\ i = 1, \dots, k\}$ be the finitely generated convex cone generated by A. Let y be the limit of some sequence y_n in C,

$$y_n \to y.$$

[1] Gale [3, p. 55] calls these finite cones, but since they are not finite sets, I think my terminology is clearer, if more verbose.

[2] The general proof relies on the fact that the span of any finite set in a Hausdorff tvs is a closed subset, and that every m-dimensional subspace of a tvs is linearly homeomorphic to $\boldsymbol{R}^{\mathrm{m}}$. See, e.g., [1, Theorem 5.21 and Corollary 5.22, p. 178].

By Lemma 2.3.3 we can write each y_n as a nonnegative linear combination of an independent subset of the x_i's. Since there are only finitely many such subsets, by passing to a subsequence we may assume without loss of generality that each y_n depends on the same independent subset $\{x_1, \dots, x_p\}$. We can find vectors $z_1, \dots, z_{m-p}$ so that $\{x_1, \dots, x_p, z_1, \dots, z_{m-p}\}$ is a basis for $\boldsymbol{R}^{\mathrm{m}}$. We can then write

$$y_n = \sum_{i=1}^{p} \lambda_{n,i} x_i + \sum_{j=1}^{m-p} 0 z_j$$

for each n where each $\lambda_{n,i} \geqslant 0$, and

$$y = \sum_{i=1}^{p} \lambda_i x_i + \sum_{j=1}^{m-p} \alpha_j z_j.$$

Since $y_n \to y$, and the coordinate map is continuous (Lemma A.12.1), we have $\lambda_{n,i} \to \lambda_i \geqslant 0$, for $i = 1, \dots, p$, and $0 \to \alpha_j = 0$, so that y belongs to C. ∎

3.1.8 Corollary *Every linear subspace of $\boldsymbol{R}^{\mathrm{m}}$ is closed.*

Proof: Let $v_1, \dots, v_k$ be a basis for the subspace M. Then M is the finitely generated convex cone generated by $\pm v_1, \dots, \pm v_k$, and so closed. ∎

3.2 Dual cones

The following definition follows the usage by Gale [3, p. 53]. Draw a picture!!

3.2.1 Definition *For a nonempty set A in $\boldsymbol{R}^{\mathrm{m}}$, its* ***dual cone*** *$A^*$ is defined by*

$$A^* = \{p \in \boldsymbol{R}^{\mathrm{m}} : (\forall x \in A)\ [\, p \cdot x \leqslant 0\,]\}.$$

The double dual cone A^{**} is simply the dual cone of A^*.

3.2.2 Exercise

1. Prove that the dual cone of a nonempty set is a nonempty (perhaps trivial) convex cone.

2. Let A be a nonempty subset of $\boldsymbol{R}^{\mathrm{m}}$. Let C be its closed convex conical hull. Prove that $A^* = C^*$ and that A^* is a closed convex cone, and that $C \subset A^{**}$.

□

In fact, in $\boldsymbol{R}^{\mathrm{m}}$ the double dual A^{**} is the closed convex cone generated by A. You don't yet have the machinery to prove that—wait for Corollary 8.3.3. Moreover we will eventually show that the dual cone of a finitely generated convex cone is also a finitely generated convex cone (Corollary 26.2.7).

3.3 A word on terminology

In Definition 9.4.1 below, we define the polar A° of a set A to be

$$A^{\circ} = \{p : (\forall x \in A) [p \cdot x \leqslant 1]\}.$$

This is not generally a cone. But if C is itself a cone, then $C^{\circ} = C^*$. Consequently man authors use the term **polar cone** for the dual cone. This includes Hiriart-Urruty and Le Marechal [4, p. 49], Rockafellar [7, p. 121], and Stoer and Witzgall [9, p. 71].

Fenchel [2, pp. 9-10] restricts attention to the case where A is itself a cone, and uses the term **normal cone** for what I call the dual cone. He reserves the term polar cone to refer to the dual cone of a closed convex cone (p. 10).

Some authors reverse the inequality and define $A^* = \{p : (\forall x \in A) [p \cdot x \geqslant 0]\}$. This list includes Nikaidô [6, p. 33] and Simon [8, p. 73] (who also explains why his definition is the "right" one).

Valentine [10, p. 61] uses the term dual cone to mean the epigraph of the profit function (see Section 9.1 below). Lay [5, p. 147] defines the dual cone of a set $A \subset \boldsymbol{R}^{\mathrm{m}}$ to be the set $\{(p, \alpha) \in \boldsymbol{R}^{\mathrm{m}+1} : (\forall x \in A) [(p, \alpha) \cdot (x, -1) \leqslant 0]\}$.

The takeaway is that you need to check each author's definition, because there does not seem to be a well-established standard.

References

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[3] D. Gale. 1960. *Theory of linear economic models*. New York: McGraw-Hill.

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[9] J. Stoer and C. Witzgall. 1970. *Convexity and optimization in finite dimensions I*. Number 163 in Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berüsichtigung der Anwendungsgebiete. Berlin: Springer–Verlag.

[10] F. Valentine. 1964. *Convex sets*. New York: McGraw-Hill.