Ec 181 KC Border
Convex Analysis and Economic Theory AY 2019–2020

# Topic 1: Convex sets and functions

## 1.1 Convex sets

Recall the affine combination function $\kappa\colon X \times X \times \boldsymbol{R} \to X$ is defined by

$$\kappa(x, y, \alpha) = (1-\alpha)x + \alpha y.$$

**1.1.1 Definition** *A subset $C$ of a real vector space $X$ is a* ***convex set*** *if it includes the line segment joining any two of its points. That is, $C$ is convex if for every real $\alpha$ with $0 \leqslant \alpha \leqslant 1$ and every $x, y \in C$,*

$$(1-\alpha)x + \alpha y \in C.$$

*If $\alpha = 0$ or then $(1-\alpha)x + \alpha y = x$ and if $\alpha = 1$, then $(1-\alpha)x + \alpha y = y$, so $C \subset \kappa\big(C \times C \times [0,1]\big)$. Thus*

$$C \text{ is convex} \iff \kappa\big(C \times C \times [0,1]\big) = C \iff \kappa\big(C \times C \times (0,1)\big) \subset C.$$

*Note that the empty set is convex.*

**1.1.2 Definition** *A* ***convex combination*** *is a linear combination $\alpha x + \beta y$ where $\alpha, \beta \geqslant 0$ and $\alpha + \beta = 1$. More generally, a convex combination is a (finite) linear combination $\alpha_1 x_1 + \cdots + \alpha_k x_k$ where each $\alpha_i \geqslant 0$ and $\sum_{i=1}^k \alpha_i = 1$.*

**1.1.3 Lemma** *If $C$ is convex, then it is closed under general convex combinations. That is, for any $k \geqslant 2$, if $x_i \in C$ and $\alpha_i \geqslant 0$, $i = 1, \dots, k$, and $\sum_{i=1}^k \alpha_i = 1$, then*

$$\alpha_1 x_1 + \cdots + \alpha_k x_k \in C.$$

**1.1.4 Exercise** Prove Lemma 1.1.3. □

**Sample answer:** We prove this by induction on $k$. Let $\mathbb{P}[k]$ denote the following proposition:

> If $C$ is convex, and each $x_i \in C$ and $\alpha_i \geqslant 0$, for $i = 1, \dots, k$, and $\sum_{i=1}^k \alpha_i = 1$, then $\alpha_1 x_1 + \cdots + \alpha_k x_k \in C$.

By definition, $\mathbb{P}[2]$ is true. We now show that $\mathbb{P}[k] \implies \mathbb{P}[k+1]$: Assume $x_i \in C$ and $\alpha_i \geqslant 0$, $i = 1, \dots, k+1$, and $\sum_{i=1}^{k+1} \alpha_i = 1$. We wish to prove that $C$ contains

$$x = \alpha_1 x_1 + \cdots + \alpha_{k+1} x_{k+1}.$$

If $\alpha_{k+1} = 1$, then $x = x_{k+1} \in C$ and we are done. Otherwise we may write

$$\alpha_1 x_1 + \cdots + \alpha_{k+1} x_{k+1} = (1 - \alpha_{k+1}) \underbrace{\left( \frac{\alpha_1}{1 - \alpha_{k+1}} x_1 + \cdots + \frac{\alpha_k}{1 - \alpha_{k+1}} x_k \right)}_{= x'} + \alpha_{k+1} x_{k+1}.$$

The term $x'$ in parentheses belongs to $C$ by $\mathbb{P}[k]$. So $x = (1 - \alpha_{k+1})x' + \alpha_{k+1} x_{k+1}$, where both $x'$ and $x_{k+1}$ belong to $C$. Thus $x \in C$. This proves $\mathbb{P}[k+1]$ and the lemma is proved. ∎

**1.1.5 Definition (The unit *m*-simplex)** *In $\boldsymbol{R}^{\mathrm{m}+1}$, the **unit m-simplex***

$$\Delta_m = \left\{ (\alpha_1, \ldots, \alpha_{m+1}) \in \boldsymbol{R}^{\mathrm{m}+1} : \sum_{i=1}^{m+1} \alpha_i = 1,\ \alpha_i \geqslant 0,\ i = 1, \ldots, m+1 \right\}$$

*It is the set of all convex combinations of the unit coordinate vectors.*

*Sometimes it is convenient to number the indices of a vector in $\boldsymbol{R}^{\mathrm{m}+1}$ using $0, \ldots, m$ instead of $1, \ldots, m+1$, in which case we may write*

$$\Delta_m = \left\{ (\alpha_0, \ldots, \alpha_m) \in \boldsymbol{R}^{\mathrm{m}+1} : \sum_{i=0}^{m} \alpha_i = 1,\ \alpha_i \geqslant 0,\ i = 0, \ldots, m \right\}$$

**1.1.6 Exercise (Examples of convex sets)** Prove the following.

- A subset $A$ of $\boldsymbol{R}$ is an **interval** if $x, y \in A$ and $x < z < y$ imply $z \in A$. A subset of $\boldsymbol{R}$ is convex if and only if its an interval.
- The unit simplex $\Delta_m$ is a compact convex set.
- In a normed vector space, the **unit ball** $\{x : \|x\| \leqslant 1\}$ is convex.
- Let $A$ be an $m \times m$ positive semidefinite matrix, and define the bilinear form $Q(x, y) = x \cdot Ay$ where $x, y \in \boldsymbol{R}^{\mathrm{m}}$. Then for any $\alpha \geqslant 0$, the set $B_\alpha = \{x \in \boldsymbol{R}^{\mathrm{m}} : Q(x, x) \leqslant \alpha\}$ is convex.

  Hint (Fenchel [1, p. 34]): Note that

  $$Q(\lambda x + \mu y, \lambda x + \mu y) = \lambda^2 Q(x, x) + 2\lambda\mu Q(x, y) + \mu^2 Q(y, y) \geqslant 0,$$

  where the inequality follows from positive semidefiniteness.

  For the special case $\lambda = 1$ and $\mu = -1$ we get

  $$Q(x - y, x - y) = Q(x, x) - 2Q(x, y) + Q(y, y) \geqslant 0,$$

  so

  $$Q(x, x) + Q(y, y) \geqslant 2Q(x, y).$$

Thus

$$\begin{aligned}
&Q\big((1-\lambda)x+\lambda y,(1-\lambda)x+\lambda y\big)\\
&\qquad\qquad= (1-\lambda)^2 Q(x,x) + 2(1-\lambda)\lambda Q(x,y) + \lambda^2 Q(y,y)\\
&\qquad\qquad\leqslant (1-\lambda)^2 Q(x,x) + (1-\lambda)\lambda\big[Q(x,x)+Q(y,y)\big] + \lambda^2 Q(y,y)\\
&\qquad\qquad= (1-\lambda)Q(x,x) + \lambda Q(y,y).
\end{aligned}$$

(This shows that $x \mapsto Q(x,x)$ is a convex function, which we shall define shortly.) This shows that $B_\alpha$ is a convex set.

- Let $A$ be an $m \times n$ matrix and let $y \in \boldsymbol{R}^{\mathrm{m}}$. Then a set of the form

$$\{x \in \boldsymbol{R}^{\mathrm{n}} : Ax \leqq y\}$$

is called a **solution set**. Every solution set is convex.

- An $m \times m$ matrix is a **stochastic matrix** if all its entries are nonnegative and each row sums to one. The set of stochastic matrices is a convex set.

□

**1.1.7 Exercise (Elementary properties of convex sets)** Prove the following.

1. The intersection of a family of convex sets is convex.
2. The sum of convex sets is convex.
3. Scalar multiples of convex sets are convex.
4. More generally, if $T\colon X \to Y$ is a linear transformation between vector spaces and $A$ is convex subset of $X$, then the image $T(A)$ is a convex subset of $Y$.
5. As a special case of the previous result, if $C$ is a convex subset of $X \times Y$, the projections

$$\Big\{x \in X : (\,\exists y \in Y\,)\ [\,(x,y) \in C\,]\Big\} \text{ and } \Big\{y \in Y : (\,\exists x \in X\,)\ [\,(x,y) \in C\,]\Big\}$$

are convex subsets of $X$ and $Y$ respectively.

6. A set $C$ is convex if and only if

$$(\,\forall \alpha,\beta > 0\,)\ [\,\alpha C + \beta C = (\alpha+\beta)C\,].$$

□

While the sum of two convex sets is necessarily convex, the sum of two nonconvex sets may also be convex. For example, let $A$ be the set of rationals in $\boldsymbol{R}$ and let $B$ be the union of 0 and the irrationals. Neither set is convex, but their sum is the set of all real numbers, which is of course convex.

## 1.2 Convex functions

Geometrically, a function on a subset of a vector space is convex if the line segment joining any two points on its graph lies above the graph. Given points $x$ and $y$ in the domain, a typical point on the segment joining $\big(x, f(x)\big)$ and $\big(y, f(y)\big)$ is of the form $\Big((1-\alpha)x + \alpha y, (1-\alpha)f(x) + \alpha f(y)\Big)$. Note that this requires that $f$ be defined at $(1-\alpha)x + \alpha y$, so the geometry can be expressed as follows.

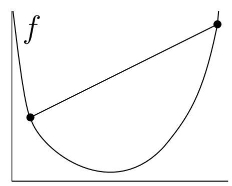


**Figure 1.2.1.** A (strictly) convex function.

**1.2.1 Definition** *A function $f \colon C \to \boldsymbol{R}$ on a convex subset $C$ of a vector space is **convex** if*

$$f\Big((1-\alpha)x + \alpha y\Big) \leqslant (1-\alpha)f(x) + \alpha f(y) \tag{\textbf{C}}$$

*for all $x, y$ in $C$ with $x \neq y$ and all $0 < \alpha < 1$.*

Note that if $x = y$ or $\alpha = 0$ or $\alpha = 1$, then the inequality above is automatically true. This definition is one that you would typically find in a real analysis text, such as Royden [8].

**1.2.2 Exercise** Prove by induction on $n$ that a function $f$ is convex if and only if

$$f\Big(\sum_{i=1}^{n} \lambda_i x_i\Big) \leqslant \sum_{i=1}^{n} \lambda_i f(x_i)$$

for every convex combination $\sum_{i=1}^{n} \lambda_i x_i$. □

**1.2.3 Exercise** A function $f \colon C \to \boldsymbol{R}$ on a subset $C$ of a vector space is **convex** if and only if its epigraph is a convex set. □

**Sample answer:** ( $\Longrightarrow$ ) Let $f$ satisfy (**C**), let $(x, \beta)$ and $(y, \gamma)$ belong to epi $f$, and let $0 < \alpha < 1$. We must show that the point $\Big((1-\alpha)x + \alpha y, (1-\alpha)\beta + \alpha\gamma\Big)$ belongs to epi $f$.

Now by the definition of epigraph we have

$$\beta \geqslant f(x) \quad \text{and} \quad \gamma \geqslant f(y),$$

so multiplying by the positive numbers $1-\alpha$ and $\alpha$ we have

$$(1-\alpha)\beta \geqslant (1-\alpha)f(x) \quad \text{and} \quad \alpha\gamma \geqslant \alpha f(y).$$

Thus

$$(1-\alpha)\beta + \alpha\gamma \geqslant (1-\alpha)f(x) + \alpha f(y) \geqslant f\big((1-\alpha)x + \alpha y\big),$$

the first inequality is the sum of the previous two, and the second inequality is just (**C**) (unless $x = y$, in which case it is trivial). But this asserts that $\big((1-\alpha)x + \alpha y, (1-\alpha)\beta + \alpha\gamma\big)$ belongs to epi $f$.

( $\Longleftarrow$ ) Since $\big(x, f(x)\big)$ and $\big(y, f(y)\big)$ belong to epi $f$, the convexity of epi $f$ implies that $\big((1-\lambda)x + \lambda y, (1-\lambda)f(x) + \lambda f(y)\big) \in \operatorname{epi} f$ for $0 \leqslant \lambda \leqslant 1$. By definition of epi $f$, we have $(1-\lambda)f(x) + \lambda f(y) \geqslant f\big((1-\lambda)x + \lambda y\big)$. That is, $f$ satisfies (**C**). ∎

This result suggests an alternative definition of a convex function, which is the one preferred by convex analysts. (For instance, Rockafellar [7].) It applies equally well to extended-real valued functions.

**1.2.4 Definition** *A function $f\colon X \to \boldsymbol{R}^\sharp$ on a vector space is **convex** if its epigraph is a convex set.*

One of the virtues of the convex analysts' definition is that we can extend a conventional real-valued convex function $f$ defined on a subset $C$ of a vector space $X$ to an extended real-valued function $\tilde f$ defined on all of $X$ by setting

$$\tilde f(x) = \begin{cases} f(x) & x \in C \\ \infty & x \notin C. \end{cases}$$

Note that epi $\tilde f$ = epi $f$, so $\tilde f$ is convex in the convex analyst's sense if and only if $f$ is convex in the conventional sense.

**1.2.5 Definition** *The **effective domain** of an extended real-valued convex function $f$ is defined to be*

$$\operatorname{dom} f = \{x \in X : f(x) < \infty\},$$

*which is the projection of the epigraph onto $X$.*

When $\tilde f$ is defined by extending a convex function $f$ as above, then dom $\tilde f = C$. There is a caveat in this regard. What happens when an extended real-valued function takes on both infinite values, say $f(x) = \infty$ and also $f(y) = -\infty$. Then $(1-\alpha)f(x) + \alpha f(y)$ is undefined, so condition (**C**) makes no sense. The epigraph is still well defined, so Definition 1.2.1 can still be used to define convexity of $f$. Some authors, e.g., Rockafellar [7], are happy to allow such functions, but call them **improper**. Other authors, such as Hiriart-Urruty and Lemaréchal [5] simply rule out by definition functions that assume the value $-\infty$ as being convex. There are arguments to be made on both sides, but I will allow consideration of improper convex functions.

A further qualification is this: The function that is identically $\infty$ is also considered to be improper.

**1.2.6 Definition** *A convex function is* ***proper*** *if it never assumes the value* $-\infty$ *and its effective domain is nonempty.*

For a proper convex function, the usual rules for operating with extended real numbers (Section A.1) allow (**C**) to be used as an equivalent definition of convexity.

**1.2.7 Proposition** *Let* $A$ *be a convex subset of* $X \times \boldsymbol{R}$ *and let* $C$ *be the projection of* $A$ *on* $X$*. Then* $C$ *is convex and the function* $f$ *defined on* $C$ *by*

$$f(x) = \inf\{\alpha : (x, \alpha) \in A\}$$

*is a (possibly extended real-valued) convex function on* $C$*.*

Draw a picture

.

**1.2.8 Exercise** Prove Proposition 1.2.7 □

## 1.3 Related concepts

**1.3.1 Definition** *A function* $f\colon C \to \boldsymbol{R}$ *on a convex subset* $C$ *of a vector space is:*

$$\textbf{\textit{convex}} \text{ if } f\Big((1-\lambda)x + \lambda y\Big) \leqslant (1-\lambda)f(x) + \lambda f(y)$$

$$\textbf{\textit{strictly convex}} \text{ if } f\Big((1-\lambda)x + \lambda y\Big) < (1-\lambda)f(x) + \lambda f(y)$$

$$\textbf{\textit{concave}} \text{ if } f\Big((1-\lambda)x + \lambda y\Big) \geqslant (1-\lambda)f(x) + \lambda f(y)$$

$$\textbf{\textit{strictly concave}} \text{ if } f\Big((1-\lambda)x + \lambda y\Big) > (1-\lambda)f(x) + \lambda f(y)$$

*for all* $x, y$ *in* $C$ *with* $x \neq y$ *and all* $0 < \lambda < 1$*.*

If $f$ is improper and assumes both values $\infty$ and $-\infty$, then this definition of strict convexity is unusable, so we would have to define a strictly convex function as one with no line segment included in its graph. Just as convex analysts define convex functions in terms of their epigraphs, they define concave functions in terms of their hypographs.

**1.3.2 Definition** *An extended real-valued function is* ***concave*** *if its hypograph is a convex subset of* $X \times \boldsymbol{R}$*. Given a concave function* $f\colon X \to \boldsymbol{R}^\sharp$*, its* ***effective domain****,* dom $f$*, is the projection of its hypograph on* $X$*, that is,*

$$\operatorname{dom} f = \{x \in X : f(x) > -\infty\}.$$

*The effective domain of a concave function is a (possibly empty) convex set.*

**1.3.3 Exercise** Prove the following.

1. The sum of convex functions is convex. The sum of concave functions is concave.

2. A nonnegative multiple of a convex function is convex. A nonnegative multiple of a concave function is concave.

3. Consequently, the collection of convex functions is a convex cone (see Definition 3.1.1 below) in the vector space of functions. Also, the collection of concave functions is a convex cone in the vector space of functions.

4. The pointwise limit of a sequence of convex functions is convex. The pointwise limit of a sequence of concave functions is concave.

5. The set of points where the pointwise supremum of a family of convex functions is finite is a convex set. On this set the supremum is a convex function. Similarly, the pointwise infimum of a family of concave functions is concave.

6. Every affine function is both convex and concave.

□

## 1.4 Complements

**1.4.1 Exercise (Alternate criteria for convexity)**

1. A function $f$ on a convex set $C$ is **midpoint-convex** if for all $x, y \in C$, it satisfies
$$f\left(\tfrac{1}{2}x + \tfrac{1}{2}y\right) \leqslant \tfrac{1}{2}f(x) + \tfrac{1}{2}f(y).$$
Prove that a continuous midpoint-convex function is convex. (This is true in any topological vector space.)

2. A stronger result was pointed out to me by Tomasz Strzalecki. He attributes it to Hardy, Littlewood, and Polya. But they [4, § 3.7, p. 73] in turn attribute it to both Reisz [6][1] and to Jessen.[2] It is a clever way of using continuity to replace the "for every $\lambda$" clause in the definition of convexity with a "there exists some $\lambda$" clause.

[1] The point is made in the last paragraph of section 8, on p. 471.

[2] Hardy, *et al.*, give the Jessen reference simply as: Om Uligheder imellem Potensmiddelværdier, *Mat. Tidsskr. B*, 1931 (no volume or page numbers). The Jessen archives at http://www.math.ku.dk/arkivet/jessen/bjpap3.htm list a different date: Om Uligheder imellem Potensmiddelværdier, *Matematisk Tidsskrift, B*, 1933, pp. 1–19. The archives also list Bemærkninger om konvekse Funktioner og Uligheder imellem Middelværdier, I and II, *Matematisk Tidsskrift, B*, 1931, pp. 17–28 and 84–95. When I get the time to learn Danish and find a library that has the journal, I may try to track down the proper citation. Or perhaps I should just ask John Chipman.

Suppose that $f$ is defined on a convex set $C$ of a tvs and has the following existential property:

$$(\forall\, x, y \in C\,)\ \ (\,\exists\, 0 < \lambda < 1\,)\ \Big[\, f\Big((1-\lambda)x + \lambda y\Big) \leqslant (1-\lambda)f(x) + \lambda f(y) \Big]. \tag{$\mathbf{E}$}$$

Prove that if $f$ is also continuous, then $f$ is convex.

3. Show that the results above need not hold for discontinuous $f$. □

For even more general notions along these lines, see the paper by Green and Gustin [2].

**Sample answer:** Clearly part (2) implies part (1), so I shall prove part (2), using a proof by contradiction. So assume by way of contradiction that $f$ is continuous and satisfies ($\mathbf{E}$), but is not convex. Then there exist $x, y$, and $0 < \bar\lambda < 1$ such that $f\Big((1-\bar\lambda)x + \bar\lambda y\Big) > (1-\bar\lambda)f(x) + \bar\lambda f(y)$. By continuity, the set $A = \{\lambda : f\Big((1-\lambda)x + \lambda y\Big) > (1-\lambda)f(x) + \lambda f(y)\}$ is open, and by hypothesis contains $\bar\lambda$. But $0, 1 \notin A$, so $\bar\lambda$ is contained in a maximal open interval included in $A$. That is, there exist some $0 \leqslant \alpha < \bar\lambda < \beta \leqslant 1$ such that for all $\lambda$ in $(\alpha, \beta)$, we have $f\Big((1-\lambda)x+\lambda y\Big) > (1-\lambda)f(x)+\lambda f(y)$, but $f\Big((1-\alpha)x+\alpha y\Big) = (1-\alpha)f(x)+\alpha f(y)$ and $f\Big((1-\beta)x + \beta y\Big) = (1-\beta)f(x) + \beta f(y)$. (This argument works in any tvs). Now consider $x' = (1-\alpha)x + \alpha y$ and $y' = (1-\beta)x + \beta y$. By construction, for every $0 < \lambda < 1$, the point $(1-\lambda)x' + \lambda y'$ strictly between $x'$ and $y'$ satisfies $f\Big((1-\lambda)x' + \lambda y'\Big) > (1-\lambda)f(x') + \lambda f(y')$. But this violates ($\mathbf{E}$) applied to the points $x'$ and $y'$, a contradiction. Therefore $f$ must be convex.

(3) This has got to be a pretty weird function, and its description takes us pretty far afield, but here goes.

Recall (or just take my word for it) that the real numbers $\boldsymbol{R}$ can be thought of as a vector space $X$ over the field $\mathbb{Q}$ of rational numbers. (After all, a rational linear combination of real numbers is another real number.) As such, it has a Hamel basis, that is, a set $B$ of vectors (real numbers) such that every vector (real number) is a unique (finite) rational linear combination of elements of $B$. (This makes $X$ an infinite dimensional vector space over $\mathbb{Q}$.) Moreover we may find such a basis that contains the vector $1 \in X$. Uncountably many of these basis elements are irrational. Fix some irrational $\alpha$ in $B$ and consider the set of rational linear combinations of basis vectors of the form $n1 + m\alpha$ where $n, m$ are integers. Such sets are discussed in section 6 of my notes on the Kolmogorov Extension Theorem, or see Halmos [3, Theorem 16.C, p. 69], and it follows from Proposition 11 in my notes that in every interval $A$, for every $M$, there is a point $n + m\alpha \in A$ with $m > M$.

Now define $f\colon \boldsymbol{R} \to \mathbb{Q}$ so that $f(x)$ is the coefficient on the vector $\alpha$ in the unique representation of $x$ as a linear combination of basis vectors. It follows from the remark above that $f$ is unbounded on every interval, and consequently not continuous. Moreover $f$ is linear with respect to linear combinations with coefficients in $\mathbb{Q}$, and since $1/2$ is rational, $f$ is mid-point convex, but it is not convex! It is also not Lebesgue measurable, but that's a different story. ■

## References

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